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L^p-Estimates for quasilinear subelliptic equations with VMO coefficients under the controllable growth

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Abstract

We prove an interior L^p-estimate of X-gradient of weak solutions to a class of quasilinear subelliptic equations with VMO coefficients under controllable growth. Here, we use a reverse Hölder inequality and De Giorgi's iteration to establish the boundedness of their weak solutions. Then a local L^p-estimate of the X-gradient of the weak solutions is derived by way of the bootstrap argument.

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Keywords: controllable growth; subelliptic equation; L^p-estimate; reverse Hölder inequality; bootstrap argument

1 Introduction

Given a family of smooth vector fields $X = (X_1, X_2, \ldots, X_m)$ defined on $\mathbb{R}^n$ satisfying the Hörmander finite rank condition, we assume that each component $b_{ki}(x)$ with $1 \leq k \leq m$ and $1 \leq i \leq n$ of vector field $X_k = \sum_{i=1}^n b_{ki}(x)\frac{\partial}{\partial x_i}$, for $k = 1, 2, \ldots, m$, is a smooth function defined on $\mathbb{R}^n$. By $X^* = (X_1^*, X_2^*, \ldots, X_m^*)$ we denote the formal adjoint operator to X. Let Ω be a bounded open set of $\mathbb{R}^n$ for $n \geq 2$. In this paper, we consider the following quasilinear subelliptic equations:

$$\sum_{i,j}^m X_i^*\big(A_{ij}(x,u)X_j u + a_i(x,u)\big) = b(x,u,Xu), \qquad \text{a.e. } x \in \Omega, \tag{1.1}$$

where $A_{ij}(x,u)$ satisfies an uniformly sub-ellipticity, and $b(x,u,Xu)$ is supposed under the controllable growth; for details see the assumptions H1-H3 below.

Before imposing some structural assumptions on $A_{ij}(x,u)$ and $b(x,u,Xu)$; and stating our main result, let us first recall a few of notations and basic facts involving the Carnot-Carathéodory metric induced by the smooth vector fields X on $\mathbb{R}^n$.

Definition 1.1 An absolutely continuous path $\gamma : [0,T] \to \mathbb{R}^n$ is called an X-subunit, if there exist functions c_j: $[0,T] \to \mathbb{R}$ for $j = 1, 2, \ldots, m$; such that

$$\dot{\gamma}(t) = \sum_{j=1}^m c_j(t)X_j\big(\gamma(t)\big)$$

with

$$\sum_{j=1}^{m} c_j(t)^2 \le 1$$

for almost every $t \in [0, T]$; for details see [1, 2].

In the context, we assume that $X_1, \dots, X_m$ satisfy the Hörmander finite rank condition

$$\operatorname{Rank}\bigl(\operatorname{Lie}[X_1, X_2, \dots, X_m]\bigr) = r, \quad \forall x \in \mathbb{R}^n. \tag{1.2}$$

It is well known that if the vector fields satisfy the Hörmander condition (1.2) at every point of $\mathbb{R}^n$, there are subunitary curves connecting any two given points $x, y \in \mathbb{R}^n$. Therefore, we can introduce a distance function induced by the smooth vector fields X as follows:

$$\rho(x,y) = \inf\bigl\{T > 0 : \exists \gamma : [0,T] \to \mathbb{R}^n X\text{-subunit with } \gamma(0) = x, \gamma(T) = y\bigr\}.$$

This is the most natural metric associated to the stratification of the Lie algebra, which have been studied in a celebrated paper [3]. Here $(\mathbb{R}^n, \rho)$ is called the Carnot-Carathéodory space with a C.-C. distance; see [1, 4, 5]. Note that these vector fields $(X_1, \dots, X_m)$ are free up to the order r, then there exists a positive constant $C > 0$ satisfying the following relation between the C.-C. distance and the Euclidean metric (*cf.* [6]):

$$C^{-1}|x-y| \le d_X(x,y) \le C|x-y|^{\frac{1}{r}}.$$

In the sequel, all distances which we use except a special explanation will be regarded as the C.-C. distance. In particular, for any fixed $x \in \Omega$ let $B_R(x)$ denote the ball $\{y \in \mathbb{R}^n : \rho(x,y) < R\}$ with $R \le \rho(x, \partial\Omega)$. In fact, the distance function $\rho(\cdot,\cdot)$ satisfies the local doubling property: namely, for $B_{2R}(x) \subset \Omega$ there exists a positive constant R_0 depending only on vector fields X and Ω such that for all $0 < R \le R_0$, we have

$$\bigl|B_{2R}(x)\bigr| \le 2^Q \bigl|B_R(x)\bigr|, \tag{1.3}$$

where the least integer Q is the homogeneous dimension of X in $\mathbb{R}^n$; see [3, 7, 8].

Next, let us recall the following horizontal Sobolev spaces with respect to the horizontal vector fields X (*cf.* [1, 4, 5]). For any $1 \le p < \infty$ and $k \in \mathbb{N}$, we set

$$HW^{1,p}(\Omega, X) := \bigl\{u \in L^p(\Omega) : X_j u \in L^p(\Omega), j = 1, \dots, m\bigr\} \tag{1.4}$$

with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)},$$

where $X_j u$ denotes the X_j-gradient of u in the sense of distribution. Additionally, the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$ is denoted by $HW_0^{1,p}(\Omega)$. In order to impose some structure assumptions on $A(x,u)$ and $b(x,u,Xu)$, we need to recall two useful notations (see [2, 9–11]).

Definition 1.2 (BMO functions) Let $\Omega(x,r) = \Omega \cap B_r(x)$. For any $0 < s < +\infty$, we say that $u \in L^1_{\mathrm{loc}}(\Omega)$ belongs to $\mathrm{BMO}(\Omega)$ if

$$M_u(s) := \sup_{x\in\Omega, 0<r<s} \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} \bigl|u(y) - \bar{u}_{\Omega(x,r)}\bigr|\,dy < +\infty$$

with

$$\bar{u}_{\Omega(x,r)} = \fint_{\Omega(x,r)} u(y)\,dy = \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} u(y)\,dy.$$

Definition 1.3 (VMO functions) Let $M_u(s)$ be defined as above. For $u \in \mathrm{BMO}(\Omega)$, we say $u \in \mathrm{VMO}(\Omega)$ if

$$\lim_{s\to 0} M_u(s) = 0,$$

where $M_u(r)$ is called the VMO modulus of u.

On account of these notations above, we are now in a position to impose some structure assumptions on the leading coefficients $A_{ij} = A_{ij}(x,u)$ and the lower terms $a_i(x,u)$, $b(x,u,Xu)$:

H1. (Uniform ellipticity) There exist constants $L \geq \mu > 0$ such that $\mu|\xi|^2 \leq \sum_{i,j}^m A_{ij}\xi_i\xi_j \leq L|\xi|^2$ for any $\xi \in \mathbb{R}^n$.

H2. (VMO_x in x and continuity in u to $A_{ij}(x,u)$) $A_{ij}(x,u)$ is VMO in x uniformly to any $u \in \mathbb{R}$; and $A_{ij}(x,u)$ is continuous in u for any $x \in \mathbb{R}^n$; namely, there exist a positive constant C and a nonnegative continuous function $\omega(r) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\omega(0) = 0$, such that

$$\bigl|A_{ij}(x,u) - A_{ij}(x,v)\bigr| \leq C\omega\bigl(|u-v|\bigr), \quad \forall u,v \in \mathbb{R}. \tag{1.5}$$

H3. (Controllable growth) There exist constants $\mu_1, \mu_2 > 0$ such that

$$\bigl|a_i(x,u)\bigr| \leq \mu_1\bigl(|u|^{\frac{\gamma}{2}} + f_i(x)\bigr), \tag{1.6}$$

$$\bigl|b(x,u,Xu)\bigr| \leq \mu_2\bigl(|Xu|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + g(x)\bigr), \tag{1.7}$$

where $\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_m(x)) \in [L^p(\Omega)]^m$ with $p > Q$, $g(x) \in L^q(\Omega)$ with $q > \frac{pQ}{Q+p}$, and

$$\gamma = \begin{cases} \frac{2Q}{Q-2}, & Q > 2, \\ \gamma > 2, & Q = 2. \end{cases}$$

As we know, the weak solutions $u \in HW_0^{1,2}(\Omega)$ of quasilinear subelliptic equations (1.1) are understood in the distributional sense:

$$\int_\Omega \bigl(A_{ij}(x,u)X_j u + a_i(x,u)\bigr)X_i\varphi\,dx = \int_\Omega b(x,u,Xu)\varphi\,dx, \quad \forall \varphi \in HW_0^{1,2}(\Omega). \tag{1.8}$$

Let us now review some recent studies on the subelliptic topic. Recent tremendous studies on subelliptic PDEs arising from non-commuting vector fields have been well developed; for details see [3, 6, 7, 12–19] and references therein. The regularity of subelliptic operators was first introduced by Hörmander in [20], which stimulated people's interest in subelliptic problems to a large degree. Since then, many important results about the fundamental solution to subelliptic operators and harmonic analysis theory on stratified nilpotent Lie groups have been obtained by Folland [21], Rothschild and Stein [13], and Nagel,Stein and Wainger [3]. These results laid a solid foundation for further investigation of subelliptic Partial Differential Equations theory. Up to the 1990s, the function theory and harmonic analysis tools on Carnot groups, such as the Sobolev embedding inequality of X-gradient and the isoperimetric inequality, became increasingly mature; for more details see [1, 3, 5, 13, 16, 21–24] and references therein. From this, there was a large amount of literature as regards the problems of subelliptic PDEs on the Carnot-Carathéodory metric space. For instance, Capogna, Danielli and Garofalo [1] in 1993 studied an embedding theorem and the Harnack inequality for nonlinear subelliptic equations. Meanwhile, the Harnack inequality for solutions to quasilinear subelliptic differential equations concerning (1.1), a class of degenerate subelliptic equations and a class of strongly degenerate Schrödinger operators were already established by Franchi, Lu and Wheeden in [14, 22, 25], respectively. In addition, Lu [15] also obtained the existence and size estimates for the Green's functions of differential operators constructed from degenerate vector fields. Lately, Xu and Zuily in [6] obtained Schauder estimates of quasilinear subelliptic equations with smooth coefficients under natural growth, and further proved that their weak solutions are smooth if all given datum are smooth. Recently, Bramanti, Brandolini and Fanciullo [9, 26–28] studied L^p-estimates for nonvariational hypoelliptic operators with VMO coefficients, Schauder estimates for parabolic nondivergence operators, $C^{k,\alpha}$-regularity to quasilinear equations, and BMO estimates for nonvariational operators with discontinuous coefficients structured on Hörmander's vector fields, respectively. Closely related to this article, we would like to mention that Di Fazio and Fanciullo established L^p-theory of nonlinear operators with coefficients under the Cordes conditions in Heisenberg group [29] and Carnot groups [30]. As for the problems of geometric subelliptic equations, Jost and Xu [31] and Wang [8] got partial regularities of subelliptic harmonic mappings, respectively. Moreover, a similar result for subelliptic p-harmonic mappings was obtained by Hajłasz and Strzelecki in [32]. Zheng and Feng [18] very recently gave various estimates of the Green's function of quasilinear subelliptic equations and their applications to regularity problems.

On the other hand, there has been tremendous work on the L^p-theory of elliptic equations with discontinuous coefficients. It was remarkable and unpredictable that Chiarenza *et al.* [33] first derived results as regards $W^{2,p}$-estimates for elliptic equations with VMO coefficients based on the Calderón-Zygmund theorem and estimates of commutators. Later, L^p-estimates were extended to divergence form elliptic equations with discontinuous coefficients by Di Fazio in [34]. In recent years, two different approaches to elliptic and parabolic equations with VMO coefficients were developed in Byun and Wang [35] and Dong, Kim, and Krylov [36], respectively. Zhu, Bramanti and Niu [37] recently attained interior L^p-estimates for divergence degenerate elliptic systems in Carnot groups by way of rather geometric arguments from Byun and Wang's work on series, which include the Hardy-Littlewood maximal functions and modified versions of the Vitali covering lemma.

In general, when quasilinear or nonlinear equations are considered, the regularity of weak solutions has been investigated under various growth conditions on lower order terms. For the quasilinear setting, Zheng and Feng [38] recently obtained an optimal interior Hölder continuity by way of an interior reverse Hölder's inequality for quasilinear elliptic equations with the controlled growth conditions under the assumption that the leading coefficients are in the class of VMO functions with respect to x variables. Later, Dong and Kim [2] used a unified approach to get global Hölder continuity by the interior and boundary reverse Hölder's inequalities and bootstrap argument for quasilinear divergence form elliptic and parabolic equations over Lipschitz domains with controlled growth conditions on low order terms. Yu and Zheng [39] also derived the same Hölder continuity to quasilinear elliptic equations with the controlled growth based on a modified A-harmonic approximation and the Caccioppoli inequality, also see [40, 41].

Motivated by these recent papers, we are here devoted to the study of the L^p-theory of weak solutions to quasilinear divergence form subelliptic equations, which originates from these papers [2, 38, 39] with a controllable growth in the case of usual Euclidean metric with standard gradient. Another reason is partially inspired by interior L^p-estimates of divergence degenerate elliptic equations in Carnot groups [37]. Therefore, our aim of this paper is to attain the L^p-theory for quasilinear subelliptic equations, whose conclusions and approach are different from the Hölder estimates of weak solutions derived in [2, 38, 39]. Now, let us summarize our main result as follows.

Theorem 1.4 *Let $u \in HW^{1,2}(\Omega)$ be any weak solution to quasilinear subelliptic equations (1.1) satisfying the structural conditions $H1$-$H3$. Suppose that $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \in [L^p(\Omega)]^m$ and $g(x) \in L^q(\Omega)$ with p, q satisfying $p > Q$ and $q > \frac{pQ}{Q+p}$. Then, for any $\Omega' \subset\subset \Omega$, we have $u \in HW^{1,r}(\Omega')$ with $r = \min\{p, q^*\}$; moreover, for any $\Omega' \subset\subset \Omega$ we have*

$$\|Xu\|_{L^r(\Omega')} \leq C\big(\|\mathbf{f}(x)\|_{L^p(\Omega)} + \|g(x)\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\big), \tag{1.9}$$

where

$$q^* = \begin{cases} \frac{qQ}{Q-q}, & \frac{pQ}{Q+p} < q < Q, \\ q^* > q, & q \geq Q, \end{cases}$$

is the Sobolev conjugate index of q.

The main difficulty of our proof is that the bootstrap argument in [2] cannot be applied directly to weak solutions under the controlled growth. To this end, we first establish an interior reverse Hölder's inequality. Additionally, the relaxation of the regularity assumptions on the leading coefficients from uniform continuity to VMO relies on the L^p-theory of linear equations with VMO coefficients. So, it is another challenging role how to make use of L^p-theory of linear subelliptic equations with VMO coefficients, and here we do it by De Giorgi's iteration argument. In the context, we do not consider boundary-value problems for the reasons that it also makes the presentation clearer.

The rest of the paper is organized as follows. In Section 2, we will recall some basic facts and improve an integrable index of equation (1.1) to one larger than 2 by the reverse Hölder inequality. In Section 3, we prove our main theorem, Theorem 1.4. Our argument is first to establish the boundedness of the weak solutions of equation (1.1) by using De Giorgi's iteration, and then improve gradually the integrable index of X-gradient via the L^p-theory of linear subelliptic equations, perturbation argument and a bootstrap argument.

2 Preliminaries

In the context, we adopt the usual convention of denoting by C a general constant, which may vary from line to line in the same chain of inequalities. This section is devoted to establishing the reverse Hölder inequality to equation (1.1) and introducing some useful lemmas. Now let us first recall the Sobolev embedding inequality with respect to the horizontal vector fields (*cf.* [1, 4, 5]). Indeed, one of our main techniques to do the Moser iteration is to use the Poincaré and Sobolev embedding theorems for vector fields satisfying Hörmander condition.

Lemma 2.1 *Let $1 \le p < Q$ and $1 \le q \le \frac{Qp}{Q-p}$, where Q is the homogeneous dimension of X in $\mathbb{R}^n$.*

(1) *If $u(x) \in HW^{1,p}(B_{R_0}, X)$, then there exists a positive constant $C = C(p, q, Q, X)$ such that for any $0 < R < R_0$, we have*

$$\|u - \bar{u}_R\|_{L^q(B_R)} \le CR^{1+Q(\frac{1}{q}-\frac{1}{p})}\|Xu\|_{L^p(B_R)}, \tag{2.1}$$

where $\bar{u}_R = \frac{1}{|B_R|}\int_{B_R} u\,dx$;

(2) *If $u \in HW_0^{1,q}(B_{R_0}, X)$, then there exists a positive constant $C = C(p, q, Q, X)$ such that for any $0 < R < R_0$, we have*

$$\left(⨍_{B_R} |u|^q\right)^{\frac{1}{q}} \le CR\left(⨍_{B_R} |Xu|^p\right)^{\frac{1}{p}}. \tag{2.2}$$

We would like to remark that the Poincaré and Sobolev embedding theorems for vector fields satisfying Hörmander condition are very important to study subelliptic PDE, which was first established due to Jerison's work [42]. Later, the optimal result for $q = \frac{pQ}{Q-p}$ and $p > 1$ was first proved by Lu in his two very earlier papers [23, 25]. As their applications, Lu also established the Harnack inequality for a class of degenerate subelliptic equations. In the case $q = \frac{pQ}{Q-p}$ and $p \ge 1$ (including $p = 1$), the optimal results above and isoperimetric inequalities were proved by Franchi, Lu and Wheeden in [22].

Next, we recall the following reverse Hölder inequality from Theorem 2.3 of Chapter 5 in [11].

Lemma 2.2 *Suppose that $h(x)$ and $u(x)$ are nonnegative measurable functions satisfying $h(x) \in L^t(\Omega)$ and $u(x) \in L^s(\Omega)$ with $t > s > 1$. If for $\forall x_0 \in \Omega$ and $\forall R : 0 < R < R_0 \le \operatorname{dist}(x_0, \partial\Omega)$ we have*

$$⨍_{B_{\frac{R}{2}}(x_0)} u^s\,dx \le b\left(\left\{⨍_{B_R(x_0)} u\,dx\right\}^s + ⨍_{B_R(x_0)} h^s\,dx\right) + \theta ⨍_{B_R(x_0)} u^s\,dx, \tag{2.3}$$

with constants $b > 1$ and $0 \le \theta < 1$, then there exist positive constants $\delta = \delta(b, Q, q, s)$ and $C = C(b, Q, q, r)$ such that $u \in L^t_{\mathrm{loc}}(\Omega)$ for any $t \in [s, s + \delta)$ and

$$\left\{⨍_{B_{\frac{R}{2}}(x_0)} u^t\,dx\right\}^{\frac{1}{t}} \le C\left\{⨍_{B_R(x_0)} u^s\,dx\right\}^{\frac{1}{s}} + C\left\{⨍_{B_R(x_0)} h^t\,dx\right\}^{\frac{1}{t}}. \tag{2.4}$$

On the basis of the reverse Hölder inequality above, we can obtain a self-improving integrability of X-gradient to equation (1.1) by a standard approach. For our papers to be self-contained we give its complete proof as follows.

Lemma 2.3 *Let $u \in HW^{1,2}(\Omega)$ be a weak solution to equation* (1.1) *under the controllable growth. Suppose that the coefficients and the lower terms satisfy the structural assumptions* H1 *and* H3, *with $\mathbf{f}(x) \in [L^p(\Omega)]^m$ for $p > 2$ and $g(x) \in L^q(\Omega)$ for $q \geq \frac{2Q}{Q+2}$. Then there exists an integrable index $t \in (2, \min\{p, \frac{2(\gamma-1)}{\gamma}q\})$ such that for any $\Omega' \subset\subset \Omega$ we have $u \in HW^{1,t}(\Omega')$. Moreover, for any open ball $B_R(x) \subset \Omega$ we have*

$$\begin{aligned}
&\left(\fint_{B_{\frac{R}{2}}} \left(|u|^\gamma + |Xu|^2\right)^{\frac{t}{2}} dx\right)^{\frac{1}{t}} \\
&\quad \leq C\left\{\left(\fint_{B_R} \left(|u|^\gamma + |Xu|^2\right) dx\right)^{\frac{1}{2}} + \left(\fint_{B_R} \left|\mathbf{f}(x)\right|^p dx\right)^{\frac{1}{p}} + R\left(\fint_{B_R} |g|^q dx\right)^{\frac{1}{q}}\right\}, \qquad (2.5)
\end{aligned}$$

where $C = C(\mu, L, Q, p, q) > 0$.

Proof For any given $x_0 \in \Omega$ and $B_R := B_R(x_0) \subset \Omega$, we suppose that $\eta \in C_0^\infty(B_R)$ is a cut-off function satisfying

$$0 \leq \eta(x) \leq 1, \qquad \eta(x) = 1 \quad \text{on } B_{R/2}, \qquad |X\eta| \leq \frac{C}{R}. \qquad (2.6)$$

Let us take $\varphi = \eta^2(u - \bar{u}_R)$ as the test function in equation (1.8) to get

$$\begin{aligned}
&\int_\Omega A_{ij}(x,u) X_i\left(\eta^2(u - \bar{u}_R)\right) X_j u \, dx + \int_\Omega a_i(x,u) X_i\left(\eta^2(u - \bar{u}_R)\right) dx \\
&\quad = \int_\Omega b(x,u,Xu)\eta^2(u - \bar{u}_R)\, dx.
\end{aligned}$$

By uniformly ellipticity H1 and the controllable growth H3, we have

$$\begin{aligned}
\mu \int_\Omega \eta^2 |Xu|^2 \, dx &\leq \int_\Omega \eta^2 A_{ij}(x,u) X_i u X_j u \, dx \\
&= -2\int_\Omega A_{ij}(x,u)\left((u - \bar{u}_R)X_i\eta\right)(\eta X_j u)\, dx \\
&\quad - \int_\Omega a_i(x,u) X_i\left((u - \bar{u}_R)\eta^2\right) dx \\
&\quad + \int_\Omega b(x,u,Xu)\eta^2(u - \bar{u}_R)\, dx \\
&\leq 2L\int_\Omega \left|(u - \bar{u}_R)X\eta\right| \cdot |\eta Xu|\, dx \\
&\quad + \mu_1 \int_\Omega \left||u|^{\frac{\gamma}{2}} + f_i(x)\right| \cdot \left|X\left((u - \bar{u}_R)\eta^2\right)\right| dx \\
&\quad + \mu_2 \int_\Omega \left||Xu|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + g(x)\right| \cdot \left|(u - \bar{u}_R)\eta^2\right| dx \\
&:= 2LI_1 + \mu_1 I_2 + \mu_2 I_3. \qquad (2.7)
\end{aligned}$$

In the sequel, we focus on the estimates of the integral expressions I_1, I_2, I_3, respectively.

Estimate of I_1: by Young's inequality it follows that

$$I_1 \le \frac{\mu}{16L}\int_\Omega |\eta Xu|^2\,dx + \frac{16L}{\mu}\int_\Omega \bigl|(u-\bar{u}_R)X\eta\bigr|^2\,dx.$$

Estimate of I_2: by using Young's inequality again it yields

$$\begin{aligned} I_2 &= \int_\Omega \bigl(\eta|u|^{\frac{\gamma}{2}}\bigr)\cdot|\eta Xu|\,dx + \int_\Omega \bigl|\eta\mathbf{f}(x)\bigr|\cdot|\eta Xu|\,dx \\ &\quad + 2\int_\Omega \bigl(\eta|u|^{\frac{\gamma}{2}}\bigr)\cdot\bigl|(u-\bar{u}_R)X\eta\bigr|\,dx + 2\int_\Omega \bigl|\eta\mathbf{f}(x)\bigr|\cdot\bigl|(u-\bar{u}_R)X\eta\bigr|\,dx \\ &\le \frac{\mu}{16\mu_1}\int_\Omega |\eta Xu|^2\,dx \\ &\quad + C(\mu,\mu_1)\biggl\{\int_{B_R}|u|^\gamma\,dx + \int_{B_R}\bigl|\mathbf{f}(x)\bigr|^2\,dx + \frac{1}{R^2}\int_{B_R}\bigl|(u-\bar{u}_R)\bigr|^2\,dx\biggr\}. \end{aligned}$$

Estimate of I_3: we further divide it in three parts as follows:

$$I_3 \le \int_{B_R}\bigl(|Xu|^{2(1-\frac{1}{\gamma})}|u-\bar{u}_R| + |u|^{\gamma-1}|u-\bar{u}_R| + \bigl|g(x)(u-\bar{u}_R)\bigr|\bigr)\,dx := J_1 + J_2 + J_3.$$

To estimate J_1, we employ Hölder inequality, the Sobolev embedding inequality in Lemma 2.1 and (2.6), and obtain

$$\begin{aligned} J_1 &\le \biggl(\int_{B_R}|Xu|^2\,dx\biggr)^{1-\frac{1}{\gamma}}\biggl(\int_{B_R}|u-\bar{u}_R|^\gamma\,dx\biggr)^{\frac{1}{\gamma}} \\ &\le CR^{Q(\frac{1}{\gamma}-\frac{1}{2})+1}\biggl(\int_{B_R}|Xu|^2\,dx\biggr)^{\frac{1}{2}-\frac{1}{\gamma}}\biggl(\int_{B_R}|Xu|^2\,dx\biggr). \end{aligned}$$

Applying Young's inequality to the estimates of J_2 and J_3 yields

$$\begin{aligned} J_2 &\le \frac{1}{16\mu_2}\int_{B_R}|u-\bar{u}_R|^\gamma\,dx + C(\gamma,\mu_2)\int_{B_R}|u|^\gamma\,dx, \\ J_3 &\le \frac{1}{16\mu_2}\int_{B_R}|u-\bar{u}_R|^\gamma\,dx + C(\gamma,\mu_2)\int_{B_R}\bigl|g(x)\bigr|^{\frac{\gamma}{\gamma-1}}\,dx. \end{aligned}$$

As to the second term on the right-hand side of J_2 above, by the Hölder inequality, for any $s>1$ we have

$$\begin{aligned} \int_{B_R}|u|^\gamma\,dx &\le 2^{\gamma-1}\biggl(\int_{B_R}|u-\bar{u}_R|^\gamma\,dx + |B_R|\biggl(\fint_{B_R}|u|\,dx\biggr)^\gamma\biggr) \\ &\le 2^{\gamma-1}\biggl(\int_{B_R}|u-\bar{u}_R|^\gamma\,dx + |B_R|\biggl(-\int_{B_R}|u|^{\frac{\gamma s}{2}}\,dx\biggr)^{\frac{2}{s}}\biggr). \end{aligned} \tag{2.8}$$

Note that the Sobolev inequality implies

$$\int_{B_R} |u-\bar{u}_R|^\gamma\,dx \le CR^{\gamma[Q(\frac{1}{\gamma}-\frac{1}{2})+1]}\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{\gamma}{2}-1}\int_{B_R}|Xu|^2\,dx. \tag{2.9}$$

Let us put the estimates of J_1, J_2, J_3 and (2.8), (2.9) together, which yields

$$\begin{aligned} I_3 &\le C\left\{R^\kappa\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{1}{2}-\frac{1}{\gamma}}+R^{\gamma\kappa}\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{\gamma}{2}-1}\right\}\int_{B_R}|Xu|^2\,dx \\ &\quad + C|B_R|\left(\fint_{B_R}|u|^{\frac{\gamma s}{2}}\,dx\right)^{\frac{2}{s}}+C\int_{B_R}\left|g(x)\right|^{\frac{\gamma}{\gamma-1}}\,dx, \end{aligned}$$

where $\kappa = 1+Q(\frac{1}{\gamma}-\frac{1}{2})\ge 0$, and $\min\{\frac{1}{2}-\frac{1}{\gamma},\frac{\gamma}{2}-1\}>0$ due to $\gamma>2$.

We are now in a position to put the estimates of I_1, I_2, I_3 and (2.8) into (2.7), then we can conclude that there exists a positive constant $C=C(\mu,\mu_1,\mu_2,\gamma)$ such that

$$\begin{aligned} &\int_{\frac{R}{2}}\left(|Xu|^2+|u|^\gamma\right)dx \\ &\quad\le \frac{C}{R^2}\int_{B_R}|u-\bar{u}_R|^2+C|B_R|\left(\fint_{B_R}|u|^{\gamma\frac{s}{2}}\,dx\right)^{\frac{2}{s}}+C\int_{B_R}\left(\left|g(x)\right|^{\frac{\gamma}{\gamma-1}}+\left|\mathbf{f}(x)\right|^2\right)dx \\ &\qquad + C\left[R^\kappa\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{1}{2}-\frac{1}{\gamma}}+R^{\gamma\kappa}\left(\int_{B_R}|Xu|^2\,dx\right)^{\frac{\gamma}{2}-1}\right]\int_{B_R}|Xu|^2\,dx. \end{aligned}\tag{2.10}$$

Setting $\vartheta = C[R^\kappa(\int_{B_R}|Xu|^2\,dx)^{\frac{1}{2}-\frac{1}{\gamma}}+R^{\gamma\kappa}(\int_{B_R}|Xu|^2\,dx)^{\frac{\gamma}{2}-1}]$. By virtue of Lebesgue's absolute continuity with respect to the integral domain, we know that $\vartheta\to 0$ as $R\to 0$. Therefore, we can obtain $0<\vartheta<1$ by only choosing R to be a suitable small positive constant. Let us apply the Sobolev embedding inequality in Lemma 2.1 to $\int_{B_R}|u-\bar{u}_R|^2dx$ with $s=\frac{2Q}{Q+2}>1$ in (2.10), then we get

$$\begin{aligned} &\fint_{B_{\frac{R}{2}}}\left(|Xu|^2+|u|^\gamma\right)dx \\ &\quad\le C\left\{\left(\fint_{B_R}\left(|Xu|^2+|u|^\gamma\right)^{\frac{s}{2}}\right)^{\frac{2}{s}}dx\right. \\ &\qquad\left.+\fint_{B_R}\left(\left|H(x)\right|^2+\left|\mathbf{f}(x)\right|^2\right)dx+\vartheta\fint_{B_R}|Xu|^2\,dx\right\}, \end{aligned}\tag{2.11}$$

where $H(x)=|g(x)|^{\frac{\gamma}{2(\gamma-1)}}$. By the reverse Hölder inequality in Lemma 2.2, we know there exists a $t\in(2,\min\{p,\frac{2(\gamma-1)}{\gamma}q\})$ such that

$$\begin{aligned} &\left(\fint_{B_{\frac{R}{2}}}\left(|u|^\gamma+|Xu|^2\right)^{\frac{t}{2}}\right)^{\frac{1}{t}} \\ &\quad\le C\left(\fint_{B_R}\left(|u|^\gamma+|Xu|^2\right)dx\right)^{\frac{1}{2}}+C\left(\fint_{B_R}\left(\left|H(x)\right|^2+\left|\mathbf{f}(x)\right|^2\right)^{\frac{t}{2}}dx\right)^{\frac{1}{t}} \end{aligned}$$

$$
\begin{aligned}
&\leq C\left\{\left(\fint_{B_R}\left(|u|^\gamma + |Xu|^2\right)dx\right)^{\frac{1}{2}} + \left(\fint_{B_R}\left|\mathbf{f}(x)\right|^t dx\right)^{\frac{1}{t}} + R\left(\fint_{B_R}|g|^{\frac{t\gamma}{2(\gamma-1)}}dx\right)^{\frac{2(\gamma-1)}{t\gamma}}\right\} \\
&\leq C\left\{\left(\fint_{B_R}\left(|u|^\gamma + |Xu|^2\right)dx\right)^{\frac{1}{2}} + \left(\fint_{B_R}\left|\mathbf{f}(x)\right|^p dx\right)^{\frac{1}{p}} + R\left(\fint_{B_R}|g|^q dx\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where we employed the monotone increasing of $(\fint_{B_R}|f(x)|^t\,dx)^{\frac{1}{t}}$ with respect to t in the last inequality,which is due to $2 < t < p$ and $\frac{t\gamma}{2(\gamma-1)} < q$ when $t \in (2, \min\{p, \frac{2(\gamma-1)}{\gamma}q\})$. This lemma is proved. □

3 The proof of main theorem

In this section, we are devoted to enhancing the integrable index of X-gradient of their weak solutions based on L^p-estimates of linear subelliptic equations, perturbation argument and the bootstrap argument. First of all, we prove the boundedness of the weak solutions to equation (1.1) by way of the idea from De Giorgi's iteration; also see [25]. To this end, let us consider in $HW^{1,2}(\Omega)$ the following linear subelliptic equations in divergence form:

$$
\sum_{i,j}^{m} X_i^*\left(a_{ij}(x)X_j u\right) = \sum_{i}^{m} X_i^* f_i, \quad \text{a.e. } x \in \Omega, \tag{3.1}
$$

where $a_{ij} \in \mathrm{VMO}(\Omega)$ and satisfy uniformly ellipticity H1, and $\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_m(x)) \in [L^p(\Omega)]^m$ with $p > 2$. We recall an interior L^p-estimate of X-gradient to equation (3.1), which can be referred to Theorem 2.8 in [37].

Lemma 3.1 *Let $u \in HW^{1,2}_{\mathrm{loc}}(\Omega)$ be any weak solution to linear subelliptic equations* (3.1). *Suppose that the leading coefficients $a_{ij} \in \mathrm{VMO}(\Omega)$ and satisfy uniformly ellipticity* H1, *and $\mathbf{f}(x) \in [L^p(\Omega)]^m$ with $2 < p < +\infty$. Then $u \in HW^{1,p}(\Omega')$ for any $\Omega' \subset\subset \Omega$. Moreover, there exists a positive constant $C = C(\mu, L, Q, p, R)$ such that for any $B_R \subset \Omega$ we have*

$$
\|Xu\|_{L^p(B_{R/2})} \leq C\left(\left\|\mathbf{f}(x)\right\|_{L^p(B_R)} + \|u\|_{L^2(B_R)}\right). \tag{3.2}
$$

For the convenience of studying quasilinear subelliptic equations (1.1), we here give the following conclusion, which can be found a similar conclusion from [43] in the case of Euclidean metric and usual gradient.

Lemma 3.2 *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^n$. For any $g(x) \in L^q(\Omega)$ with $q > 1$, there exists a vector-valued function $G(x) : \Omega \to \mathbb{R}^m$ with $G(x) = \{G^1(x), G^2(x), \dots, G^m(x)\} \in [L^{q^*}(\Omega)]^m$ such that $g(x) = \sum_i^m X_i^*(G^i(x))$; and we have*

$$
\|G\|_{(L^{q^*}(\Omega))^m} \leq C(Q, q, \partial\Omega)\|g\|_{L^q(\Omega)}, \tag{3.3}
$$

where

$$
q^* = \begin{cases} \frac{Qq}{Q-q}, & 1 \leq q < Q, \\ \text{any } q^* \geq q, & q \geq Q, \end{cases}
$$

is the Sobolev conjugate index of q.

Proof Given any fixed point $y \in \mathbb{R}^n$, let $\Gamma(x,y)$ be the fundamental solution to sub-Laplacian equations $\sum_{i=1}^k X_i^* X_i u = 0$ in $\mathbb{R}^n$. By Theorem 2.2 in [4], its fundamental solution $\Gamma(x,y)$ deserves the following local properties:

$$\Gamma(x,y) \simeq \frac{\rho(x,y)^2}{|B(x,\rho(x,y))|} \simeq \rho(x,y)^{2-Q}, \tag{3.4}$$

and there exists a positive constant $C = C(Q)$ such that

$$\left|X^s \Gamma(x,y)\right| \le C \frac{\rho(x,y)^{2-s}}{|B(x,\rho(x,y))|} \le C\rho(x,y)^{2-s-Q}, \quad s = 1,2,\ldots. \tag{3.5}$$

Note that $\partial\Omega$ is Lipschitz continuous, there exists an extending function $\tilde{g}(x)$ defined in $\mathbb{R}^n$ such that $\tilde{g}(x) = g(x)$ on Ω and $\|\tilde{g}\|_{L^q(\mathbb{R}^n)} \le C(\Omega)\|g\|_{L^q(\Omega)}$; moreover, $\tilde{g}(x)$ has compact support in $\mathbb{R}^n$, namely, there exists $\Omega \subset\subset V \subset \mathbb{R}^n$ such that $\operatorname{supp}(\tilde{g}) \subset V$. Therefore, it is easy to see that $N\tilde{g}(x) = \int_{\mathbb{R}^n} \Gamma(x,y)\tilde{g}(y)\,dy$ satisfies $\sum_{i=1}^k X_i^* X_i(N\tilde{g}(x)) = g(x)$, a.e. $x \in \Omega$.

Thanks to the Calderon-Zygmund theory of a singular integral operator, it follows that

$$\left\|X^2\left(N\tilde{g}(x)\right)\right\|_{L^q(V)} \le C(Q,q,\Omega)\|g\|_{L^q(\Omega)}.$$

Now let us restrict $G(x) := \{X_1(N\tilde{g}(x)),\ldots,X_m(N\tilde{g}(x))\}$ to Ω and employ the Sobolev embedding inequality of X-gradient of Theorem 2.1, then it yields

$$\left\|G(x)\right\|_{L^{q^*}(\Omega)} = \left\|X\left(N\tilde{g}(x)\right)\right\|_{L^{q^*}(\Omega)} \le C\left\|X^2\left(N\tilde{g}(x)\right)\right\|_{L^q(V)} \le C\|g\|_{L^q(\Omega)}.$$

This lemma is proved. □

Further, we consider the following linear subelliptic equations in divergence form:

$$-X_i^*\left(a_{ij}(x)X_j u\right) = -X_i^* f_i(x) + g(x), \quad \text{a.e. } x \in \Omega, \tag{3.6}$$

where $a_{ij} \in \mathrm{VMO}(\Omega)$ satisfies the uniform ellipticity H1, and $\mathbf{f}(x) \in [L^p(\Omega)]^m$, $g(x) \in L^q(\Omega)$ with p, q satisfying $p > 2$ and $q > \frac{pQ}{Q+p}$. By a simply computation we have the following.

Lemma 3.3 *Let $u \in HW^{1,2}_{\mathrm{loc}}(\Omega)$ be any weak solution to equation* (3.6). *Suppose that the leading coefficients $a_{ij} \in \mathrm{VMO}(\Omega)$ satisfy the uniform ellipticity* H1 *and* H2, *and $\mathbf{f}(x) \in [L^p(\Omega)]^m$, $g(x) \in L^q(\Omega)$ with p,q satisfying $p > 2$ and $q > \frac{2Q}{Q+2}$. Then, for any $\Omega' \subset\subset \Omega$, we have $u \in HW^{1,r}(\Omega')$ with $r = \min\{p,q^*\}$. Moreover, there exists a positive constant $C = C(\mu,L,Q,p,q,R)$ such that for any $B_R \subset \Omega$, we have*

$$\|Xu\|_{L^r(B_{R/2})} \le C\left(\left\|\mathbf{f}(x)\right\|_{L^r(B_R)} + \|g\|_{L^q(B_R)} + \|u\|_{L^2(B_R)}\right) \tag{3.7}$$

with $r = \min\{p,q^*\}$.

Proof On the basis of Lemma 3.2, we know that for $g(x) \in L^q(\Omega)$ there exists a vectorial-valued function $G(x) = (G_1(x),\ldots,G_m(x)) \in [L^{q^*}(\Omega)]^m$ such that $g(x) = -\sum_{i=1}^m X_i^* G_i(x)$ for a.e. $x \in \Omega$, and

$$\|G\|_{L^{q^*}(B_R)} \le C\|g\|_{L^q(B_R)}.$$

In this way, equation (3.6) can be rewritten as

$$-X_i^*\big(a_{ij}(x)X_j u\big) = -X_i^*(\mathbf{f} + G), \quad \text{a.e. } x \in B_R, \tag{3.8}$$

where $\mathbf{f} + G \in [L^r(B_R)]^m$ and $r = \min\{p, q^*\}$. Using Lemma 3.1 it yields

$$\begin{aligned}\|Xu\|_{L^r(B_{R/2})} &\le C\big(\|\mathbf{f} + G\|_{L^r(B_R)} + \|u\|_{L^2(B_R)}\big)\\ &\le C\big(\|\mathbf{f}\|_{L^r(B_R)} + \|G\|_{L^{q^*}(B_R)} + \|u\|_{L^2(B_R)}\big)\\ &\le C\big(\|\mathbf{f}\|_{L^r(B_R)} + \|g\|_{L^q(B_R)} + \|u\|_{L^2(B_R)}\big),\end{aligned}$$

where $C = C(\mu, Q, p, q, R)$. This completes the proof of Lemma 3.3. □

In order to get the boundedness of weak solutions to equation (1.1) under the controllable growth, we will use so-called De Giorgi's iteration argument (*cf.* Lemma 5.1 in Chapter 5 of [44]). We denote by $A_k = \{x \in \Omega : u(x) > k\}$ the distributional function of u on Ω, and by $|A_k|$ denote the measure of A_k with the C.-C. metric.

Lemma 3.4 *Let $u(x)$ be a measurable function defined on Ω. If, for any $k \ge k_0 > 0$ there exist constants γ, α, and ε satisfying $\gamma, \varepsilon > 0$, $0 \le \alpha \le 1 + \varepsilon$ such that*

$$\int_{A_k} (u - k)\, dx \le N_0 k^{\alpha} |A_k|^{1+\varepsilon}. \tag{3.9}$$

Then $u(x)$ is essentially bounded on Ω; namely, there exists a positive constant $N = N(\gamma, \alpha, \varepsilon, k_0, \|u\|_{L^1_{(A_{k_0})}})$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) \le N.$$

Based on Lemma 3.4 above and Lemma 2.1 (Sobolev inequality of X-gradient), we obtain the following useful conclusion (*cf.* Lemma 5.2 in Chapter 2 of [44]).

Lemma 3.5 *Let $u(x) \in HW^{1,2}(\Omega)$ and $Q \ge 2$. If, for any $k \ge k_0 > 0$, there exist constants γ, ε, α satisfying $\gamma, \varepsilon > 0$ and $0 \le \alpha \le 2 + \varepsilon$ such that*

$$\int_{A_k} |Xu|^2\, dx \le N_0 k^{\alpha} |A_k|^{1-\frac{2}{Q}+\varepsilon}. \tag{3.10}$$

Then $u(x)$ is essentially bounded on Ω, and there exists a positive constant $N = N(\gamma, \alpha, \varepsilon, k_0, \|u\|_{L^1_{(A_{k_0})}})$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} \big|u(x)\big| \le N.$$

Hence, in order to establish the boundedness of weak solutions to equation (1.1) we only need to prove that the weak solutions of equation (1.1) satisfy inequality (3.10).

Lemma 3.6 (Boundedness of weak solutions) *Let $u(x) \in HW^{1,2}(\Omega)$ be any weak solution to quasilinear subelliptic equations* (1.1). *Suppose that the leading coefficients and lower terms satisfy the structural assumptions H*1 *and H*3. *Then $u(x)$ is essentially bounded, and*

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}\bigl|u(x)\bigr| \le M,$$

where $M = M(Q, \mu, \mu_1, \mu_2, \|\mathbf{f}\|_{L^p}, \|g\|_{L^q}) > 0$.

Proof Notice that the assumptions of H1 and H3 on $a_i(x,u)$, by Young's inequality, yield

$$A_{ij}(x,u)X_iuX_ju \ge \mu|Xu|^2,$$
$$\left|\sum_{i=1}^m a_i(x,u)X_iu\right| \le \mu_1\bigl(|u|^{\frac{\gamma}{2}}|Xu| + \bigl|\mathbf{f}(x)\bigr||Xu|\bigr) \le \frac{\mu}{4}|Xu|^2 + \frac{8\mu_1}{\mu}|u|^\gamma + \mu_1\bigl|\mathbf{f}(x)\bigr||Xu|,$$

which implies

$$\sum_{i,j=1}^m \bigl(A_{ij}(x,u)X_ju + a_i(x,u)\bigr)X_iu \ge \frac{3}{4}\mu|Xu|^2 - C(\mu,\mu_1)|u|^\gamma - \mu_1\bigl|\mathbf{f}(x)\bigr||Xu|. \tag{3.11}$$

Using the controllable growth of $b(x,u,Xu)$ and Young's inequality, it follows that

$$\begin{aligned}\bigl|b(x,u,Xu)u\bigr| &\le \mu_2|u|\bigl(|Xu|^{2(1-\frac{1}{\gamma})} + |u|^{\gamma-1} + \bigl|g(x)\bigr|\bigr)\\ &\le \frac{\mu}{4}|Xu|^2 + C(\mu,\mu_2)|u|^\gamma + \mu_2|u|\bigl|g(x)\bigr|.\end{aligned} \tag{3.12}$$

Let us combine (3.11) and (3.12) and take $\varphi = (u-k)_+$ with $k > 0$ determined later as the test function. By integrating on the distributional function A_k we have

$$\begin{aligned}\int_{A_k}|Xu|^2\,dx &\le C(\mu,\mu_1,\mu_2)\int_{A_k}|u|^\gamma\,dx\\ &\quad + \mu_1\int_{A_k}\bigl|\mathbf{f}(x)\bigr||Xu|\,dx + \mu_2\int_{A_k}|u|\bigl|g(x)\bigr|\,dx\\ &:= C(\mu,\mu_1,\mu_2)K_1 + \mu_1K_2 + \mu_2K_3.\end{aligned} \tag{3.13}$$

Now let us estimate K_1, K_2, K_3, respectively, as follows.

To estimate K_2, we have

$$\begin{aligned}K_2 &\le |A_k|^{1-\frac{1}{p}-\frac{1}{2}}\left(\int_{A_k}\bigl|\mathbf{f}(x)\bigr|^p\,dx\right)^{\frac{1}{p}}\left(\int_{A_k}|Xu|^2\,dx\right)^{\frac{1}{2}}\\ &\le \frac{1}{4\mu_1}\int_{A_k}|Xu|^2\,dx + C|A_k|^{\frac{p-2}{p}}\left(\int_{A_k}\bigl|\mathbf{f}(x)\bigr|^p\,dx\right)^{\frac{2}{p}}.\end{aligned} \tag{3.14}$$

To estimate K_3, we get

$$\begin{aligned} K_3 &\le |A_k|^{1-\frac{1}{2^*}-\frac{1}{q}} \left(\int_{A_k} |u|^{2^*}\,dx \right)^{\frac{1}{2^*}} \left(\int_{A_k} \big|g(x)\big|^q\,dx \right)^{\frac{1}{q}} \\ &\le C|A_k|^{\frac{Q+2}{2Q}-\frac{1}{q}} \left(\int_{A_k} |Xu|^2\,dx \right)^{\frac{1}{2}} \left(\int_{A_k} \big|g(x)\big|^q\,dx \right)^{\frac{1}{q}} \\ &\le \frac{1}{4\mu_2} \int_{A_k} |Xu|^2\,dx + C|A_k|^{\frac{Q+2}{Q}-\frac{2}{q}} \left(\int_{A_k} \big|g(x)\big|^q\,dx \right)^{\frac{2}{q}}. \end{aligned} \tag{3.15}$$

To estimate K_1, by the Sobolev inequality in Lemma 2.1 we deduce

$$K_1 \le C|A_k|^{\gamma\kappa_1} \left(\int_{A_k} |Xu|^2\,dx \right)^{\frac{\gamma}{2}-1} \int_{A_k} |Xu|^2\,dx \tag{3.16}$$

with $\kappa_1 = (\frac{1}{\gamma}-\frac{1}{2})+\frac{1}{Q} \ge 0$ and $\frac{\gamma}{2}-1>0$. Now we put (3.14), (3.15), and (3.16) into (3.13), then by using Lebesgue's absolute continuity on the integral domain and choosing a suitable large $k>0$ we derive

$$C|A_k|^{\gamma\kappa} \left(\int_{B_R} |Xu|^2\,dx \right)^{\frac{\gamma}{2}-1} \le \frac{1}{4},$$

which implies

$$\int_{A_k} |Xu|^2\,dx \le C(\mu,\mu_1,\mu_2)M|A_k|^{\varrho},$$

where $\varrho = \min\{\frac{p-2}{p}, \frac{Q+2}{Q}-\frac{2}{q}\}$ and $M = (\|\mathbf{f}(x)\|_{L^p} + \|g(x)\|_{L^q})^2$.

Considering $p>Q$ and $q>\frac{pQ}{Q+p}$, it yields $\frac{Q+2}{Q}-\frac{2}{q} > \frac{Q+2}{Q} - \frac{2}{\frac{Qp}{Q+p}} = \frac{p-2}{p}$. Then we have

$$\varrho = \min\left\{ \frac{p-2}{p}, \frac{Q+2}{Q}-\frac{2}{q} \right\} = \frac{p-2}{p} = 1-\frac{2}{Q}+\varepsilon_1,$$

where $\varepsilon_1 = \frac{2}{Q}-\frac{2}{p} > 0$. Hence, the boundedness of u on Ω is obtained due to Lemma 3.5. This lemma is proved. □

Proof of Theorem 1.4 Let us prove it in two steps by semilinear setting and quasilinear setting.

Step 1. First let us consider the following semilinear subelliptic equations:

$$-\sum_{i,j}^{m} X_i^*\big(A_{ij}(x)X_j u + a_i(x,u)\big) = b(x,u,Xu), \quad \text{a.e. } x \in \Omega, \tag{3.17}$$

where $b(x,u,Xu)$ is under the controllable growth. Our idea is to use the bootstrap argument to improve the integrable index of X-gradient of weak solutions. It is easily seen that $\sup_{x\in\Omega}|u| \le M$ due to Lemma 3.6. Considering $b(x,u,Xu)$ satisfies the controllable growth H3, we derive

$$\begin{cases} |a_i(x,u)| \le \mu_1(M^{\frac{\gamma}{2}} + f_i(x)) \in L^p(\Omega), \\ |b(x,u,Xu)| \le \mu_2(|Xu|^{2(1-\frac{1}{\gamma})} + M^{\gamma-1} + g(x)). \end{cases} \tag{3.18}$$

Setting $\chi = 2(1-\frac{1}{\gamma})$, by Lemma 2.3 it follows that there exists an integrable index $p_0 > 2$ such that $Xu \in L^{p_0}_{\mathrm{loc}}(\Omega)$, which implies

$$b(x,u,Xu) \in L^{q_1}_{\mathrm{loc}}(\Omega), \qquad q_1 = \min\left\{\frac{p_0}{\chi}, q\right\}.$$

Thanks to Lemma 3.3, it yields

$$Xu \in L^{r_1}(\Omega'), \qquad r_1 = \min\{p, q_1^*\}, \quad \text{for any } \Omega' \subset\subset \Omega. \tag{3.19}$$

(i) If $q \le \frac{p_0}{\chi}$, then $q_1 = q$ and $r_1 = r = \min\{p, q^*\}$. Thus, Theorem 1.4 is proved.
(ii) If $q > \frac{p_0}{\chi}$, then $q_1 = \frac{p_0}{\chi}$, and

$$q_1^* = \begin{cases} \frac{Qp_0}{Q\chi - p_0}, & \frac{p_0}{\chi} < Q, \\ q_1^* > q_1, & \frac{p_0}{\chi} \ge Q. \end{cases} \tag{3.20}$$

If now $\frac{p_0}{\chi} \ge Q$, then $r_1 = \min\{p, q^*\}$, Thus, Theorem 1.4 holds again.

If instead $\frac{p_0}{\chi} < Q$, then $q_1^* = \frac{Qp_0}{Q\chi - p_0} < q^*$, so $r_1 = \min\{p, \frac{Qp_0}{Q\chi - p_0}\}$. For the case of $p \le \frac{Qp_0}{Q\chi - p_0}$, we can also obtain Theorem 1.4. For the other case with $p > \frac{Qp_0}{Q\chi - p_0}$, we have $Xu \in L^{\frac{Qp_0}{Q\chi - p_0}}_{\mathrm{loc}}(\Omega')$, namely, $|Xu|^\chi \in L^{\frac{Qp_0}{(Q\chi - p_0)\chi}}_{\mathrm{loc}}(\Omega')$. Again using the controllable growth (3.18), we have

$$b(x,u,Xu) \in L^{q_2}_{\mathrm{loc}}(\Omega), \qquad q_2 = \min\left\{\frac{Qp_0}{(Q\chi - p_0)\chi}, q\right\} \ge q_1.$$

Thus, by Lemma 3.3 it follows that

$$Xu \in L^{r_2}_{\mathrm{loc}}(\Omega'), \qquad r_2 = \min\{p, q_2^*\} \ge r_1.$$

Iterating the above procedure we can arrive at $Xu \in L^r(\Omega')$ with $r = \min\{p, q^*\}$ after finite steps. This is because the integral index of Xu is improved by a fixed step length χ. This completes Step 1.

Step 2. We now consider quasilinear subelliptic equation (1.1) under the controllable growth H3. For any $x_0 \in \Omega$, let $B_R = B_R(x_0) \subset \Omega$ be a ball centered at x_0 with radii R in Ω. We set $\bar{u}_R = \fint_{B_R} u\,dx$, then equation (1.1) can be rewritten as

$$-X_i^*\bigl(A_{ij}(x,\bar{u}_R)X_j u\bigr) = X_i^*\bigl(a_i(x,u) - \bigl(A_{ij}(x,\bar{u}_R) + A_{ij}(x,u)\bigr)X_j u\bigr) + b(x,u,Xu), \quad x \in B_R.$$

By Lemma 3.6 it implies that $\sup_{x\in\Omega}|u| \le M$. Combining the controllable growth (3.18) and the L^p-estimates of semilinear subelliptic equations in the step 1 above, we conclude that for $r = \min\{p, q^*\}$ we have

$$\begin{aligned} \|Xu\|_{L^r(B_{\frac{R}{2}})} &\le C\bigl(\bigl\|a_i(x,u) - \bigl(A_{ij}(x,\bar{u}_R) - A_{ij}(x,u)\bigr)Xu\bigr\|_{L^r(B_R)} \\ &\quad + \bigl\|b(x,u,Xu)\bigr\|_{L^q(B_R)} + \|u\|_{L^2(B_R)}\bigr) \\ &\le C\bigl(\bigl\|\mathbf{f}(x)\bigr\|_{L^p(B_R)} + \bigl\|g(x)\bigr\|_{L^q(B_R)} + \|u\|_{L^2(B_R)}\bigr) \end{aligned}$$

$$
\begin{aligned}
&+ C \sup_{x\in B_R}\left|A_{ij}(x,\bar{u}_R) - A_{ij}(x,u)\right| \|Xu\|_{L^r(B_R)} \\
&\le C\left(\left\|\mathbf{f}(x)\right\|_{L^p(B_R)} + \left\|g(x)\right\|_{L^q(B_R)} + \|u\|_{L^2(B_R)}\right) + C\omega\left(|u-\bar{u}_R|\right)\|Xu\|_{L^r(B_R)} \\
&\le C\left(\left\|\mathbf{f}(x)\right\|_{L^p(\Omega)} + \left\|g(x)\right\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right) + \vartheta\|Xu\|_{L^r(B_R)},
\end{aligned}
$$

where we used the uniform continuity H2 on $A_{ij}(x,u)$ with respect to u in the second last step. If we choose a suitable small $R > 0$ such that the continuity modulus $\omega(\cdot)$ satisfying $C\omega(|u-u_R|) \le \vartheta < 1$, then by employing a standard iteration argument we get

$$
\|Xu\|_{L^r(B_{\frac{R}{2}})} \le C\left(\left\|\mathbf{f}(x)\right\|_{L^p(\Omega)} + \left\|g(x)\right\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right). \tag{3.21}
$$

In fact, let us denote

$$
r_0 = \frac{1}{2}R, \qquad r_k = \left(\frac{1}{2} + \sum_{l=1}^{k}\frac{1}{2^{l+1}}\right)R, \qquad B^{(k)} = B_{r_k}, \quad k = 1,2,\ldots,
$$

and set

$$
A_k = \|Xu\|_{L^r(B^{(k)})}, \qquad B = \left(\left\|\mathbf{f}(x)\right\|_{L^p(\Omega)} + \left\|g(x)\right\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right).
$$

Then

$$
A_k \le \theta A_{k+1} + CB.
$$

Now, multiplying both sides by θ^k with $\theta \in (0,1)$, and summing up with respect to k, it follows that

$$
\sum_{k=0}^{\infty}\theta^k A_k = \sum_{k=1}^{\infty}\theta^k A_k + \sum_{k=0}^{\infty}\theta^k CB.
$$

Since $\theta < 1$, the summation $\sum_{k=0}^{\infty}\theta^k$ is finite. Therefore,

$$
A_0 \le CB,
$$

which implies (3.21). Theorem 1.4 is completely proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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