Chen and Bao *Boundary Value Problems* (2016) 2016:153
DOI 10.1186/s13661-016-0661-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence, nonexistence, and multiplicity of solutions for the fractional $p\&q$-Laplacian equation in $\mathbb{R}^N$

Caisheng Chen[1*] and Jinfeng Bao[1,2]

*Correspondence: cshengchen@hhu.edu.cn
[1] College of Science, Hohai University, Nanjing, 210098, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the existence, nonexistence, and multiplicity of solutions to the following fractional $p\&q$-Laplacian equation:

$$\begin{aligned}(-\Delta)_p^s u + a(x)|u|^{p-2}u + (-\Delta)_q^s u + b(x)|u|^{q-2}u + \mu(x)|u|^{r-2}u \\ = \lambda h(x)|u|^{m-2}u, \quad x \in \mathbb{R}^N, \end{aligned} \tag{0.1}$$

where λ is a real parameter, $(-\Delta)_p^s$ and $(-\Delta)_q^s$ are the fractional $p\&q$-Laplacian operators with $0 < s < 1 < q < p, r > 1$ and $sp < N$, and the functions $a(x), b(x), \mu(x)$, and $h(x)$ are nonnegative in $\mathbb{R}^N$. Three cases on p, q, r, m are considered: $p < m < r < p_s^*$, $\max\{p, r\} < m < p_s^*$, and $1 < m < q < r < p_s^*$. Using variational methods, we prove the existence, nonexistence, and multiplicity of solutions to Eq. (0.1) depending on λ, p, q, r, m and the integrability properties of the ratio h^{r-p}/μ^{m-p}. Our results extend the previous work in Bartolo et al. (J. Math. Anal. Appl. 438:29-41, 2016) and Chaves et al. (Nonlinear Anal. 114:133-141, 2015) to the fractional $p\&q$-Laplacian equation (0.1).

MSC: 35R11; 35A15; 35J60; 35J92

Keywords: fractional $p\&q$-Laplacian equation; $(PS)_c$ condition; variational methods

1 Introduction and the main result

In this paper, we study the existence, nonexistence, and multiplicity of solutions to the following fractional $p\&q$-Laplacian equation:

$$(-\Delta)_p^s u + a(x)|u|^{p-2}u + (-\Delta)_q^s u + b(x)|u|^{q-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $(-\Delta)_p^s$ and $(-\Delta)_q^s$ are the fractional $p\&q$-Laplacian operators with $0 < s < 1 < q < p, r > 1$ and $sp < N$. The nonlinearity $f(x,u) = \lambda h(x)|u|^{m-2}u - \mu(x)|u|^{r-2}u$ can be seen as a competitive interplay of two nonlinearities. The coefficients $a(x), b(x), \mu(x), h(x)$ are assumed to be positive in $\mathbb{R}^N$, and other exact assumptions will be given further.

The fractional t-Laplacian operator $(-\Delta)_t^s$ with $0 < s < 1 < t$ and $st < N$ is defined along a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_t^s\varphi(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|\varphi(x)-\varphi(y)|^{t-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ts}}\,dy,\quad \forall x\in\mathbb{R}^N, \tag{1.2}$$

where $B_\varepsilon(x) := \{y\in\mathbb{R}^N : |x-y|<\varepsilon\}$; see [3–6] and the references therein.

When $p = q$, Eq. (1.1) is reduced to the fractional p-Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u),\quad x\in\mathbb{R}^N, \tag{1.3}$$

and when $s = 1$, Eq. (1.1) is the $p\&q$-Laplacian equation

$$-\Delta_p u + a(x)|u|^{p-2}u - \Delta_q u + b(x)|u|^{q-2}u = f(x,u),\quad x\in\mathbb{R}^N. \tag{1.4}$$

Equation (1.4) comes from a general reaction-diffusion system

$$u_t = \operatorname{div}\big(D(u)\nabla u\big) + f(x,u),\quad x\in\mathbb{R}^N, t>0, \tag{1.5}$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration, and the first term on the right-hand side of (1.5) corresponds to the diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to sources and loss processes. Typically, in chemical and biological applications, the reaction term $f(x,u)$ is a polynomial of u with variable coefficients [7, 8].

The solution of (1.4) has been studied by many authors; for example, see [1, 2, 7, 9–13] and the references therein. In the literature cited, the authors always assume that the potentials $a(x), b(x)$ satisfy one of the following conditions:

(A_1) $a(x), b(x)\in C(\mathbb{R}^N)$ and $a(x), b(x)\ge c_0$ in $\mathbb{R}^N$ for some constant $c_0 > 0$. Furthermore, for each $d > 0$, $\operatorname{meas}(\{x\in\mathbb{R}^N : a(x), b(x)\le d\}) < \infty$.
(A_2) $\lim_{|x|\to\infty} a(x) = +\infty, \lim_{|x|\to\infty} b(x) = +\infty$.
(A_3) $a(x), b(x)\ge c_0 > 0$ in $\mathbb{R}^N$, and $a(x)^{-1}, b(x)^{-1}\in L^1(\mathbb{R}^N)$.

It is well known that one of assumptions (A_1), (A_2), and (A_3) guarantees that the embedding $W^{1,t}(\mathbb{R}^N)\hookrightarrow L^r(\mathbb{R}^N)$ is compact for each $t\le r<t^* = \frac{tN}{N-t}$ with $1<t<N$. As far as we know, there are few papers that deal with a general bounded potential case for problem (1.4).

Now let us recall some advances of our problem. Pucci and Rădulescu [14] first studied the nonnegative solutions of the equation

$$-\Delta_p u + |u|^{p-2}u + h(x)|u|^{r-2}u = \lambda|u|^{m-2}u,\quad x\in\mathbb{R}^N, \tag{1.6}$$

where $h(x) > 0$ satisfies

$$\int_{\mathbb{R}^N}\big[h(x)\big]^{\frac{m}{m-r}}\,dx = H_1\in\mathbb{R}^+ = (0,\infty), \tag{1.7}$$

and $\lambda > 0$, and $2\le p<m<\min\{r,p^*\}$ with $p^* = pN/(N-p)$ if $N>p$ and $p* = \infty$ if $N\le p$. They showed the nonexistence of nontrivial solutions to (1.6) if λ is small enough and the existence of at least two nontrivial solutions for (1.6) if λ is large enough.

Autuori and Pucci [15] generalized (1.6) to the quasilinear elliptic equation

$$-\operatorname{div} A(x,\nabla u) + a(x)|u|^{p-2}u + h(x)|u|^{r-2}u = \lambda\omega(x)|u|^{m-2}u, \quad x\in\mathbb{R}^N, \tag{1.8}$$

where $A(x,\nabla u)$ acts like the p-Laplacian, $\max\{2,p\} < m < \min\{r,p^*\}$, and the coefficients ω and h are related by the integrability condition

$$\int_{\mathbb{R}^N}\left[\frac{\omega(x)^r}{h(x)^m}\right]^{\frac{1}{r-m}} dx = H_2 \in \mathbb{R}^+. \tag{1.9}$$

By imposing a strong convexity condition of the p-Laplacian type on the potential of A, the authors extend completely the result of [14]. Moreover, Autuori and Pucci [15] proposed two open questions: the deletion of the restriction $\max\{2,p\} < m$ and the replacement of (1.9) by the assumption that $\omega(\omega/h)^{(m-p)/(r-m)}$ is in $L^{N/p}(\mathbb{R}^N)$.

Later, Autuori and Pucci [16] studied the existence and multiplicity of solution to the following elliptic equation involving the fractional Laplacian:

$$(-\Delta)^s u + a(x)u + h(x)|u|^{r-2}u = \lambda\omega(x)|u|^{m-2}u, \quad x\in\mathbb{R}^N, \tag{1.10}$$

where $\lambda > 0$, $0 < s < 1$, $2s < N$, $2 < m < \min\{r, 2_s^*\}$, $2_s^* = 2N/(N-2s)$, and $(-\Delta)^s$ is the fractional Laplacian operator. The coefficients ω and h are related by condition (1.9). The authors proved the existence of entire solutions of (1.10) by using a direct variational method and the mountain pass theorem.

More recently, Xiang et al. [6] investigated the fractional p-Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u + b(x)|u|^{r-2}u = \lambda a(x)|u|^{m-2}u, \quad x\in\mathbb{R}^N, \tag{1.11}$$

where $\lambda > 0$, $p < m < \min\{r, p_s^*\}$, $p_s^* = pN/(N-ps)$, and $a(x)$ and $b(x)$ are related by the condition $a(a/b)^{(r-p)/(m-r)} \in L^{N/ps}(\mathbb{R}^N)$.

Up to now, it is worth noting that there is much attention on equations like (1.6), (1.8), and (1.11) with $1 < m < r$. From the papers mentioned, it is natural to ask whether the existence, nonexistence, and multiplicity of solutions to Eq. (1.1) is admitted if $1 < r < m < p_s^*$ and $1 < m < r < p_s^*$? Clearly, equations like (1.6), (1.8), and (1.11) are contained in (1.1).

In this paper, motivated by [5, 6], we will answer this interesting question, extend the $p\&q$-Laplacian (1.4), which has been studied deeply in [1, 2], to the fractional $p\&q$-Laplacian equation (1.1), and investigate the existence, nonexistence, and multiplicity of solutions depending on λ and according to the integrability properties of the ratio h^{r-p}/μ^{m-p}.

For this purpose, we apply a version of symmetric mountain pass lemma in [17]. Also, we adapt some ideas developed by Pucci et al. [18] and Xiang et al. [6]. Note that although the idea was earlier used for other problems, the adaptation to the procedure to our problem is not trivial at all since the parameters r, m satisfy $1 < r < m$ and we must consider our problem for a suitable space, and so we need more delicate estimates and new technique. Our results, which are new even in the canonical case $p = q = 2$, generalize the main results of [1, 2] in several directions. Furthermore, we weaken the conditions in those papers, and assumptions (A_1)-(A_3) are not necessary for our results.

In order to state our main theorems, we recall some fractional Sobolev spaces and norms. The fractional Sobolev space $W^{s,t}(\mathbb{R}^N)$ $(0 < s < 1 < t)$ with $st < N$ is defined by

$$W^{s,t}(\mathbb{R}^N) = \left\{ u \in L^t(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{t}+s}} \in L^t(\mathbb{R}^{2N}) \right\}. \tag{1.12}$$

This space is endowed with the natural norm

$$\|u\|_{W^{s,t}} = \left([u]_{s,t}^t + \|u\|_t^t\right)^{1/t}, \tag{1.13}$$

whereas $[u]_{s,t}$ denotes the Gagliardo seminorm given by

$$[u]_{s,t} = \left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^t}{|x-y|^{N+ts}} \, dx \, dy \right)^{1/t}. \tag{1.14}$$

The spaces X_p and X_q denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norms

$$\|u\|_{X_p} = \left([u]_{s,p}^p + \|u\|_{p,a}^p\right)^{1/p}, \qquad \|u\|_{X_q} = \left([u]_{s,q}^q + \|u\|_{q,b}^q\right)^{1/q}, \tag{1.15}$$

respectively, in which the functions $a(x), b(x)$ satisfy

(H_0) $a(x), b(x) \in C(\mathbb{R}^N)$ and $a(x), b(x) \geq c_0 > 0$ in $\mathbb{R}^N$ for some constant c_0.

In general, let $\|u\|_{t,\rho} = (\int_{\mathbb{R}^N} \rho|u|^t \, dx)^{1/t}$ with $t \geq 1$ and $\rho = \rho(x) \geq 0, \not\equiv 0$ a.e. in $\mathbb{R}^N$. In particular, denote $\|u\|_t = (\int_{\mathbb{R}^N} |u|^t \, dx)^{1/t}$ or $\|u\|_{L^t(\Omega)} = (\int_\Omega |u|^t \, dx)^{1/t}$ with the domain $\Omega \subset \mathbb{R}^N$. Let $E = X_p \cap X_q$ with $1 < q < p < N$. The norm of $u \in E$ is defined by

$$\|u\|_E = \|u\|_{X_p} + \|u\|_{X_q}. \tag{1.16}$$

Lemma 1.1 [3, 19] *Let $0 < s < 1 < t$ with $st < N$. In addition, assume (H_0). Then, $Y_t \equiv W^{s,t}(\mathbb{R}^N)$ is a uniformly convex Banach space, and there exists a positive constant $S_0 = S_0(N, t, s)$ such that*

$$\|u\|_{t_s^*} \leq S_0 [u]_{s,t}, \quad \forall u \in Y_t, \tag{1.17}$$

and

$$\|u\|_r \leq S_r \|u\|_{Y_t}, \quad \forall u \in Y_t, \tag{1.18}$$

where $t = p$ or q, $t_s^* = \frac{tN}{N-ts}$ is the fractional critical exponent, and S_r is a constant depending on s, r, t, N. For convenience, we denote $S_{t_s^*}$ by S_0. Consequently, the space Y_t is continuously embedded in $L^r(\mathbb{R}^N)$ for any $r \in [t, t_s^*]$. Moreover, the embedding $Y_t \hookrightarrow L^r(\mathbb{R}^N)$ is locally compact whenever $1 < r < t_s^*$.

Clearly, from definitions (1.13) and (1.15) and assumption (H_0), we see that

$$\min\{1, c_0\} \|u\|_{Y_t} \leq \|u\|_{X_t}, \quad \forall u \in Y_t, \text{where } t = p, q. \tag{1.19}$$

Let $J(u) : E \to \mathbb{R}$ be the energy functional associated to Eq. (1.1) defined by

$$J(u) = \frac{1}{p}\|u\|_{X_p}^p + \frac{1}{q}\|u\|_{X_q}^q + \frac{1}{r}\|u\|_{r,\mu}^r - \frac{\lambda}{m}\|u\|_{m,h}^m, \quad \forall u \in E, \tag{1.20}$$

where the norms $\|\cdot\|_{X_p}$ and $\|\cdot\|_{X_p}$ are defined by (1.15).

From the embedding inequalities (1.18) and assumptions (H_0)-(H_4) below, we see that the functional J is well defined and $J \in C^1(E, \mathbb{R})$ with

$$\begin{aligned} J'(u)\varphi = {} & \int\int_{\mathbb{R}^{2N}} \left[\frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}} + \frac{|u(x)-u(y)|^{q-2}}{|x-y|^{N+qs}}\right]\bigl(u(x)-u(y)\bigr)\bigl(\varphi(x)-\varphi(y)\bigr)\,dx\,dy \\ & + \int_{\mathbb{R}^N} \bigl[a(x)\bigl|u(x)\bigr|^{p-2} + b(x)|u|^{q-2} + \mu(x)|u|^{r-2} - \lambda h(x)|u|^{m-2}\bigr] \\ & \times u(x)\varphi(x)\,dx, \quad \forall \varphi \in E. \end{aligned} \tag{1.21}$$

A function $u \in E$ is said to be a (weak) solution of Eq. (1.1) if $J'(u)\varphi = 0$ for any $\varphi \in E$.

Throughout this paper, we let $0 < s < 1 < q < p$ with $sp < N$. Our main results are as follows.

Theorem 1.2 *Assume* (H_0) *and*

(H_1) $p < m < p_s^*$; *$h(x)$ is a positive weight satisfying* $h(x) \in L^\gamma(\mathbb{R}^N)$ *with* $\gamma = \frac{p_s^*}{p_s^*-m}$.

(H_2) $p < m < r < p_s^*$; *the functions $\mu(x)$ and $h(x)$ are positive and* $\mu(x), h(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. *Furthermore, $h(x)$ and $\mu(x)$ are related by the condition*

$$\int_{\mathbb{R}^N} \left[\frac{h(x)^{(r-p)/(r-m)}}{\mu(x)^{(m-p)/(r-m)}}\right]^{N/ps} dx = H_1 \in \mathbb{R}^+. \tag{1.22}$$

Then there exist constants $\lambda_2 \geq \lambda_1 > 0$ such that Eq. (1.1) *has*

(i) *only the trivial weak solution if $\lambda < \lambda_1$;*

(ii) *at least two nontrivial weak solutions if $\lambda \geq \lambda_2$.*

Theorem 1.3 *Let* $\max\{p, r\} < m < p_s^*$. *Assume that* (H_0) *and* (H_1) *hold. In addition, suppose that $\mu(x)$ are nonnegative and $\mu(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then Eq.* (1.1) *admits*

(i) *only the trivial solution if $\lambda \leq 0$;*

(ii) *infinitely many weak solutions $u_n \in E$ such that $J(u_n) \to \infty$ as $n \to \infty$ if $\lambda > 0$.*

Theorem 1.4 *Let* $0 < s < 1 < m < q \leq p < p_s^*$ *and* $q \leq r < p_s^*$. *Assume* (H_0) *and*

(H_3) $\mu(x) \geq 0$ *in* $\mathbb{R}^N$ *and* $\mu(x) \in L^\sigma_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $\sigma = \frac{p_s^*}{p_s^*-r}$;

(H_4) $h(x) \in L^\delta(\mathbb{R}^N)$ *with* $\delta = \frac{q}{q-m}$, *and there exist $d_0 > 0$ and $x_0 = (x_1^0, x_2^0, \ldots, x_N^0) \in \mathbb{R}^N$ such that $h(x) > 0$ in $B_{d_0}(x_0)$, where* $B_{d_0}(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < d_0\}$.

Then Eq. (1.1) *with $\lambda > 0$ admits infinitely many solutions $u_n \in E$ with $u_n \to 0$ in E.*

Remark 1.5 From Theorem 1.2, we know that it still remains an open problem to verify whether $\lambda_1 = \lambda_2$. In addition, the nonlinear function $f(x,u) = \lambda h(x)|u|^{m-2}u - \mu(x)|u|^{r-2}u$ with $p < m < r$ fails to satisfy the Ambrosetti-Rabinowitz condition. Furthermore, for $s = 1$ in (1.1), our results and context are more general than those in [1, 2].

The paper is organized as follows: In Section 2, we give some preliminaries, will set up the variational framework for problem (1.1), and prove that the functional associated to (1.1) satisfies the $(PS)_c$ condition. The proofs of Theorems 1.2 and 1.3 are given in Section 2. Finally, Section 3 is devoted to the proof of Theorem 1.4.

2 Preliminaries

To prove our main results, we need to establish some lemmas.

Lemma 2.1 *Let* (H_0) *and one of assumptions* (H_1) *and* (H_4) *hold. Then, if* $\{u_n\}$ *is a bounded sequence in E, then there exists* $u \in E \cap L^m(\mathbb{R}^N, h)$ *such that, up to a subsequence,* $u_n \to u$ *strongly in* $L^m(\mathbb{R}^N, h)$ *as* $n \to \infty$.

Proof We first choose a constant $\beta > 0$ such that $\|u_n\|_E \le \beta$ for all $n \ge 1$. If (H_1) is satisfied, then for any $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\left(\int_{B_R^c} \left|h(x)\right|^\gamma dx\right)^{1/\gamma} < 2^{-m}\beta^{-m}\varepsilon \quad \text{for all } R \ge R_0. \tag{2.1}$$

Then, it follows from the Hölder inequality and Lemma 1.1 that, for $R \ge R_0$,

$$\int_{B_R^c} h(x)\left|u_n(x) - u(x)\right|^m dx \le \|h\|_{L^\gamma(B_R^c)}\|u_n - u\|^m_{L^{p_s^*}(B_R^c)} \le 2^m\beta^m\|h\|_{L^\gamma(B_R^c)} < \varepsilon. \tag{2.2}$$

By Lemma 1.1, up to a subsequence, we obtain $u_n \to u$ strongly in $L^m(B_{R_0})$ and $u_n(x) \to u(x)$ a.e. in B_{R_0} as $n \to \infty$. Thus $h(x)|u_n(x) - u(x)|^m \to 0$ a.e. in B_{R_0} as $n \to \infty$. Similarly, for each measurable subset $\Omega \subset B_{R_0}$, we have

$$\int_{\Omega} h(x)\left|u_n(x) - u(x)\right|^m dx \le \|h\|_{L^\gamma(\Omega)}\|u_n - u\|^m_{L^{p_s^*}(\Omega)} \le 2^m\beta^m\|h\|_{L^\gamma(\Omega)}. \tag{2.3}$$

Since $h(x) \in L^\gamma(\mathbb{R}^N)$, we obtain that the sequence $\{h(x)|u_n(x) - u(x)|^m\}$ is uniformly integrable and bounded in $L^1(B_{R_0})$. Furthermore, an application of the Vitali convergence theorem gives

$$\lim_{n\to\infty}\int_{B_{R_0}} h(x)\left|u_n(x) - u(x)\right|^m dx = 0. \tag{2.4}$$

Then the conclusion that $u_n \to u$ strongly in $L^m(\mathbb{R}^N, h)$ follows from (2.2) and (2.4).

If (H_4) is satisfied, then for any $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\|h\|_{L^\delta(B_R^c)} = \left(\int_{B_R^c} \left|h(x)\right|^\delta dx\right)^{1/\delta} < 2^{-m}\beta^{-m}\varepsilon \quad \text{for all } R \ge R_0 \tag{2.5}$$

and

$$\int_{B_R^c} h(x)\left|u_n(x) - u(x)\right|^m dx \le \|h\|_{L^\delta(B_R^c)}\|u_n - u\|^m_{L^q(B_R^c)} \le 2^m\beta^m\|h\|_{L^\delta(B_R^c)} < \varepsilon. \tag{2.6}$$

Similarly, we can derive (2.4). Then combining (2.4) with (2.6), we have $u_n \to u$ in $L^m(\mathbb{R}^N, h)$. □

Lemma 2.2 *Let* (H_0) *and one of assumptions* (H_1) *and* (H_4) *hold. If* $\{u_n\}$ *is a bounded* $(PS)_c$ *sequence of the functional* J *defined by* (1.20), *then the functional* J *satisfies* $(PS)_c$ *condition.*

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence, that is,

$$J(u_n) \to c \quad \text{and} \quad \|J'(u_n)\|_{E'} \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Since the sequence $\{u_n\}$ is bounded in E, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } E, \qquad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N, \\ &u_n \to u \quad \text{strongly in } L^t_{\text{loc}}(\mathbb{R}^N), \end{aligned} \tag{2.8}$$

where $t = p$ or q. We now prove that $u_n \to u$ in E. Let $\varphi \in E$ be fixed and denote by T_φ the linear functional on E defined by

$$T_\varphi(v) = A_\varphi(v) + B_\varphi(v), \quad \forall \varphi \in E, \tag{2.9}$$

where $A_\varphi(v)$ and $B_\varphi(v)$ are the linear functionals defined by

$$\begin{aligned} A_\varphi(v) &= \int\int_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\big(v(x)-v(y)\big)\,dx\,dy, \quad \forall \varphi \in E, \\ B_\varphi(v) &= \int\int_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^{q-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+qs}}\big(v(x)-v(y)\big)\,dx\,dy, \quad \forall \varphi \in E, \end{aligned} \tag{2.10}$$

respectively. Clearly, by the Hölder inequality, T_φ is also continuous, and

$$\begin{aligned} \big|T_\varphi(v)\big| &\le \big|A_\varphi(v)\big| + \big|B_\varphi(v)\big| \le \|\varphi\|^{p-1}_{X_p}\|v\|_{X_p} + \|\varphi\|^{q-1}_{X_q}\|v\|_{X_q} \\ &\le \big(\|\varphi\|^{p-1}_E + \|\varphi\|^{q-1}_E\big)\|v\|_E, \quad \forall v \in E. \end{aligned} \tag{2.11}$$

Furthermore, the fact that $u_n \rightharpoonup u$ weakly in E implies that $\lim_{n\to\infty} A_u(u_n - u) = \lim_{n\to\infty} B_u(u_n - u) = 0$, and so

$$\lim_{n\to\infty} T_u(u_n - u) = 0. \tag{2.12}$$

On the other hand, as $n \to \infty$, we have

$$\begin{aligned} o_n(1) &= \big(J'(u_n) - J'(u)\big)(u_n - u) \\ &= T_{u_n}(u_n - u) - T_u(u_n - u) + \Phi_n + \Psi_n - \lambda P_n + Z_n, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \Phi_n &= \int_{\mathbb{R}^N} a(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)\,dx, \\ \Psi_n &= \int_{\mathbb{R}^N} b(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n - u)\,dx, \end{aligned} \tag{2.14}$$

$$Z_n = \int_{\mathbb{R}^N} \mu(x)\big(|u_n|^{r-2}u_n - |u|^{r-2}u\big)(u_n - u)\,dx,$$
$$P_n = \int_{\mathbb{R}^N} h(x)\big(|u_n|^{m-2}u_n - |u|^{m-2}u\big)(u_n - u)\,dx.$$

From (2.13) and $Z_n \geq 0$, we obtain, for large n,

$$T_{u_n}(u_n - u) - T_u(u_n - u) + \Phi_n + \Psi_n \leq \lambda P_n + o_n(1). \tag{2.15}$$

Note that, by Lemma 2.1, $P_n \to 0$ as $n \to \infty$.

Let us now recall the well-known vector inequalities: for all $\xi, \eta \in \mathbb{R}^N$,

$$\begin{aligned} &|\xi - \eta|^p \leq c_p\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)(\xi - \eta) \quad \text{for } p \geq 2, \quad \text{and} \\ &|\xi - \eta|^p \leq C_p\big[\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)(\xi - \eta)\big]^{p/2}\big(|\xi|^p + |\eta|^p\big)^{(2-p)/2} \quad \text{for } 1 < p < 2, \end{aligned} \tag{2.16}$$

where c_p and C_p are positive constants depending only on p. Assume first that $p > q \geq 2$. Then by (2.16) we have $\|u_n - u\|_{p,a}^p \leq c_p\Phi_n$ and

$$\begin{aligned} [u_n - u]_{s,p}^p &= \int\int_{\mathbb{R}^{2N}} \big|u_n(x) - u_n(y) - u(x) + u(y)\big|^p |x - y|^{-(N+sp)}\,dx\,dy \\ &\leq c_p \int\int_{\mathbb{R}^{2N}} \big[\big|u_n(x) - u_n(y)\big|^{p-2}\big(u_n(x) - u_n(y)\big) - \big|u(x) - u(y)\big|^{p-2} \\ &\quad \times \big(u(x) - u(y)\big)\big]\big(u_n(x) - u(x) - u_n(y) + u(y)\big)|x - y|^{-(N+sp)}\,dx\,dy \\ &= c_p\big[A_{u_n}(u_n - u) - A_u(u_n - u)\big]. \end{aligned} \tag{2.17}$$

Similarly, we have $\|u_n - u\|_{q,b}^q \leq c_q\Psi_n$ and

$$\begin{aligned} [u_n - u]_{s,q}^q &= \int\int_{\mathbb{R}^{2N}} \big|u_n(x) - u_n(y) - u(x) + u(y)\big|^q |x - y|^{-(N+sq)}\,dx\,dy \\ &\leq c_q \int\int_{\mathbb{R}^{2N}} \big[\big|u_n(x) - u_n(y)\big|^{q-2}\big(u_n(x) - u_n(y)\big) - \big|u(x) - u(y)\big|^{q-2} \\ &\quad \times \big(u(x) - u(y)\big)\big]\big(u_n(x) - u(x) - u_n(y) + u(y)\big)|x - y|^{-(N+sq)}\,dx\,dy \\ &= c_q\big[B_{u_n}(u_n - u) - B_u(u_n - u)\big]. \end{aligned} \tag{2.18}$$

Let $C_0 = \min\{c_p^{-1}, c_q^{-1}\}$. By (2.17) and (2.18) we see that

$$\begin{aligned} T_{u_n}(u_n - u) - T_u(u_n - u) &= A_{u_n}(u_n - u) - A_u(u_n - u) + B_{u_n}(u_n - u) - B_u(u_n - u) \\ &\geq C_0\big([u_n - u]_{s,p}^p + [u_n - u]_{s,q}^q\big). \end{aligned} \tag{2.19}$$

Then the application of (2.15) yields

$$C_0\big(\|u_n - u\|_{X_p}^p + \|u_n - u\|_{X_q}^q\big) \leq \lambda P_n + o_n(1) \to 0 \quad \text{as } n \to \infty. \tag{2.20}$$

In conclusion, $u_n \to u$ in E as $n \to \infty$.

Finally, it remains to consider the case $1 < p < 2$. By (2.8) there exists $\beta > 0$ such that $\|u_n\|_E \leq \beta$ for all $n \geq 1$. Now from (2.16) and the Hölder inequality it follows that

$$\begin{aligned}[u_n - u]_{s,p}^p &\leq C_p\big[A_{u_n}(u_n - u) - A_u(u_n - u)\big]^{p/2}\big([u_n]_{s,p}^p + [u]_{s,p}^p\big)^{(2-p)/2} \\ &\leq C_p\big[A_{u_n}(u_n - u) - A_u(u_n - u)\big]^{p/2}\big([u_n]_{s,p}^{p(2-p)/2} + [u]_{s,p}^{p(2-p)/2}\big) \\ &\leq D_p\big[A_{u_n}(u_n - u) - A_u(u_n - u)\big]^{p/2} \end{aligned} \tag{2.21}$$

and

$$\|u_n - u\|_{p,a}^p \leq D_p \Phi_n^{p/2}, \tag{2.22}$$

where we have applied the inequality

$$(x + y)^{(2-p)/2} \leq x^{(2-p)/2} + y^{(2-p)/2} \quad \text{for all } x, y \geq 0 \text{ and } 1 < p < 2, \tag{2.23}$$

and $D_p = 2C_p\beta^{p(2-p)/2}$. Similarly, for $1 < q < 2$, we have

$$[u_n - u]_{s,q}^q \leq D_q\big[B_{u_n}(u_n - u) - B_u(u_n - u)\big]^{q/2}, \qquad \|u_n - u\|_{q,b}^q \leq D_q \Psi_n^{q/2} \tag{2.24}$$

with $D_q = 2C_q\beta^{q(2-q)/2}$. Then, by (2.21), (2.22), and (2.24) we get

$$\begin{aligned} &T_{u_n}(u_n - u) - T_u(u_n - u) + \Phi_n + \Psi_n \\ &\quad \geq C_1\big([u_n - u]_{s,q}^2 + [u_n - u]_{s,p}^2 + \|u_n - u\|_{p,a}^2 + \|u_n - u\|_{q,b}^2\big) \end{aligned} \tag{2.25}$$

with some $C_1 > 0$. Then (2.15) and (2.25) imply that $u_n \to u$ in E as $n \to \infty$. Therefore, J satisfies the $(PS)_c$ condition, and we complete the proof of Lemma 2.2. □

Lemma 2.3 *Under the assumptions of Theorem* 1.2, *suppose that $u \in E$ is a nontrivial weak solution of* (1.1). *Then there exists $\lambda_1 > 0$ such that $\lambda \geq \lambda_1$.*

Proof Since $u \in E$ is a nontrivial weak solution of (1.1), we have $J'(u)\varphi = 0$ for all $\varphi \in E$. In particular, choosing $\varphi = u$, we have

$$\|u\|_{X_p}^p + \|u\|_{X_q}^q + \|u\|_{r,\mu}^r = \lambda\|u\|_{m,h}^m. \tag{2.26}$$

By the Young inequality with $\epsilon > 0$ we see that

$$cd \leq \epsilon\theta^{-1}c^p + \tau^{-1}\epsilon^{1/(1-\theta)}d^\tau, \quad \tau^{-1} + \theta^{-1} = 1, \theta > 1. \tag{2.27}$$

Taking $0 < \alpha < \beta, c = k_1 > 0, d = t^\alpha, \tau = \frac{\beta}{\alpha}, \epsilon = (k_2\beta/\alpha)^{-\alpha/(\beta-\alpha)}$, $k_2 > 0$, it follows from (2.27) that

$$k_1 t^\alpha - k_2 t^\beta \leq k_0 k_1 (k_1/k_2)^{\alpha/(\beta-\alpha)}, \quad \forall t \geq 0, \tag{2.28}$$

with $k_0 = (1-\alpha/\beta)(\beta/\alpha)^{-\alpha/(\beta-\alpha)} < 1$. Furthermore, let $k_1 = \lambda h(x), k_2 = \frac{1}{2}\mu(x)$, $\alpha = m-p$, and $\beta = r-p$. Then from (2.28) we obtain

$$\lambda h(x)|u|^{m-p} - \frac{1}{2}\mu(x)|u|^{r-p} \le c_1\lambda^{\frac{r-p}{r-m}} g(x), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.29}$$

where $c_1 = 2^{(m-p)/(r-m)}$ and $g(x) = [h(x)^{r-p}/\mu(x)^{m-p}]^{\frac{1}{r-m}}$. By (H_2) we know $g(x) \in L^{\frac{N}{sp}}(\mathbb{R}^N)$. So, the application of (1.17) and (2.29) yields

$$\begin{aligned}\lambda \int_{\mathbb{R}^N} h(x)|u|^m\,dx - \frac{1}{2}\int_{\mathbb{R}^N} \mu(x)|u|^r\,dx &\le c_1\lambda^{\frac{r-p}{r-m}} \int_{\mathbb{R}^N} g(x)|u|^p\,dx \\ &\le c_1 G S_0^p \lambda^{\frac{r-p}{r-m}} [u]_{s,p}^p\end{aligned} \tag{2.30}$$

with $G = \|g\|_{L^{\frac{N}{sp}}(\mathbb{R}^N)}$. Then, from (2.26) and (2.30) we see that

$$[u]_{s,p}^p \le c_1 G S_0^p \lambda^{\frac{r-p}{r-m}} [u]_{s,p}^p. \tag{2.31}$$

This implies that $\lambda \ge \lambda_1 \equiv (c_1^{-1} S_0^{-p} G^{-1})^{(r-m)/(r-p)}$ and completes the proof of Lemma 2.3. □

Lemma 2.4 *Under the assumptions of Theorem* 1.2, *the functional J is coercive in E.*

Proof Letting $k_1 = \frac{\lambda}{m}h(x), k_2 = \frac{1}{2r}\mu(x)$, $\alpha = m-p, \beta = r-p$, and $t = |u(x)|$ in (2.28), we conclude that

$$f(x,u) := \frac{\lambda}{m}h(x)|u|^m - \frac{1}{2r}\mu(x)|u|^r \le c_2 g(x)|u|^p, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.32}$$

where $c_2 = (2r)^{\frac{m-p}{r-m}} m^{\frac{p-r}{r-m}} \lambda^{\frac{r-p}{r-m}}$ and $g(x) = [h(x)^{r-p}/\mu(x)^{m-p}]^{\frac{1}{r-m}}$. Since $g(x) \in L^{\frac{N}{ps}}(\mathbb{R}^N)$, for any small $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$c_2\left(\int_{B_{R_1}^c} \left|g(x)\right|^{N/ps} dx\right)^{ps/N} \le \varepsilon \tag{2.33}$$

and

$$c_2 \int_{B_{R_1}^c} \left|g(x)\right||u|^p\,dx \le c_2\left(\int_{B_{R_1}^c} \left|g(x)\right|^{N/ps} dx\right)^{ps/N} \|u\|_{L^{p_s^*}(\mathbb{R}^N)}^p \le \varepsilon S_0^p [u]_{s,p}^p, \tag{2.34}$$

where S_0 is the embedding constant in (1.17). So, it follows from (2.32)-(2.34) that

$$\begin{aligned}J(u) &= \frac{1}{p}\|u\|_{X_p}^p + \frac{1}{q}\|u\|_{X_q}^q + \frac{1}{r}\|u\|_{r,\mu}^r - \frac{\lambda}{m}\|u\|_{m,h}^m \\ &\ge \frac{1}{p}\|u\|_{X_p}^p + \frac{1}{q}\|u\|_{X_q}^q - \int_{B_R} f(x,u)\,dx - c_2\int_{B_R^c} g|u|^p\,dx.\end{aligned} \tag{2.35}$$

For fixed $R_1 > 0$ and for any $\tau > 0$ and $\omega > 0$, we decompose $B_{R_1} = X \cup Y \cup Z$ as follows:

$$\begin{aligned}&X = \left\{x \in B_{R_1} : 0 \le h(x) < \omega \text{ and } \mu(x) > \tau\right\}, \qquad Z = \left\{x \in B_{R_1} : h(x) \ge \omega\right\}, \\ &Y = \left\{x \in B_{R_1} : 0 \le h(x) < \omega \text{ and } 0 \le \mu(x) < \tau\right\}.\end{aligned} \tag{2.36}$$

Obviously, the sets X, Y, and Z are Lebesgue measurable. Note that the assumption $h(x), \mu(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ implies that $\mathrm{meas}(Y) \to 0$ as $\tau \to 0$ and $\mathrm{meas}(Z) \to 0$ as $\omega \to \infty$.

On the other hand, letting $k_1 = \frac{\lambda}{m}h(x), k_2 = \frac{1}{2r}\mu(x), t = |u(x)|, \alpha = m$, and $\beta = r$ in (2.28), we derive

$$f(x,u) := \frac{\lambda}{m}h(x)|u|^m - \frac{1}{2r}\mu(x)|u|^r \le c_3 g_1(x) \tag{2.37}$$

with $c_3 = (2r)^{m/r}(\lambda/m)^{1+m/r}, g_1(x) = [h(x)/\mu(x)]^{m/r}$. Then,

$$\int_X f(x,u)\,dx \le c_3 \int_X g_1(x)\,dx \le C_1, \tag{2.38}$$

where $C_1 = C_1(\omega, \tau, R) > 0$ is a constant. Furthermore, it follows from (2.32) and (2.34) that

$$\int_{Y\cup Z} f(x,u)\,dx \le c_2 \int_{Y\cup Z} g(x)|u|^p\,dx \le c_2 \left(\int_{Y\cup Z} |g|^{N/ps}\,dx\right)^{ps/N} \|u\|^p_{L^{p_s^*}(B_{R_1})}. \tag{2.39}$$

For any $\varepsilon > 0$, we can choose large $\omega > 0$ and small $\tau > 0$ such that $\mathrm{meas}(Y \cup Z)$ is so small that

$$c_2 \left(\int_{Y\cup Z} |g|^{N/ps}\,dx\right)^{ps/N} \le \varepsilon. \tag{2.40}$$

From (1.13) and (2.38)-(2.40) we obtain

$$\int_{B_R} f(x,u)\,dx \le C_1 + \varepsilon\|u\|^p_{L^{p_s^*}(B_{R_1})} \le C_1 + \varepsilon S_0^p [u]^p_{s,p}. \tag{2.41}$$

Thus, combining (2.34) and (2.35) with (2.41) yields

$$J(u) \ge \frac{1}{p}\|u\|^p_{X_p} + \frac{1}{q}\|u\|^q_{X_q} - 2\varepsilon S_0^p [u]^p_{s,p} - C_1 \ge \frac{1}{2p}\|u\|^p_{X_p} + \frac{1}{q}\|u\|^q_{X_q} - C_1, \tag{2.42}$$

where $0 < 2\varepsilon S_0^p \le 1/2p$. Hence, J is coercive in E. □

Lemma 2.5 *Under the assumptions of Theorem* 1.2, *there exists* $u \in E$ *such that* $d = J(u) = \inf_{v\in E} J(v)$ *and* u *is a weak solution of* (1.1).

Proof By Lemma 2.4 we see that $d > -\infty$. Let $\{u_n\}$ be a minimizing sequence for d in E, which is bounded in E by Lemma 2.4. Without loss of generality, we may assume that $\{u_n\}$ is nonnegative, converges to weakly to some u in E, and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Moreover, by the weak lower semicontinuity of the norms we have

$$\frac{1}{p}\|u\|^p_{X_p} + \frac{1}{q}\|u\|^q_{X_q} + \frac{1}{r}\|u\|^r_{r,\mu} \le \liminf_{n\to\infty}\left[\frac{1}{p}\|u_n\|^p_{X_p} + \frac{1}{q}\|u_n\|^q_{X_q} + \frac{1}{r}\|u_n\|^r_{r,\mu}\right]. \tag{2.43}$$

Then from Lemma 2.1 and (2.43) it follows

$$J(u) \le \liminf_{n\to\infty} J(u_n) = d. \tag{2.44}$$

On the other hand, since $u \in E$, we have that $J(u) \geq d$, which shows that $J(u) = d$. Therefore, u is a global minimum for J, and hence it is a critical point, namely a weak solution of (1.1). □

Lemma 2.6 *Under the assumptions of Theorem* 1.2, *there exists* $\lambda_2 > 0$ *such that for all* $\lambda > \lambda_2$, *Eq.* (1.1) *admits a global nontrivial minimum* $u_0 \in E$ *of* J *with* $J(u_0) < 0$.

Proof Clearly, $J(0) = 0$. Consider the constrained minimization problem

$$\lambda_2 = \inf\left\{\frac{1}{p}\|u\|_{X_p}^p + \frac{1}{q}\|u\|_{X_q}^q + \frac{1}{r}\|u\|_{r,\mu}^r : u \in E \text{ and } \|u\|_{m,h}^m = m\right\}. \tag{2.45}$$

Let u_n be a minimizing sequence of (2.45), which is clearly bounded in E, so that we can assume, without loss of generality, that it converges weakly to some $u_0 \in E$ with $\|u_0\|_{m,h}^m = m$ and

$$\lambda_2 = \frac{1}{p}\|u_0\|_{X_p}^p + \frac{1}{q}\|u_0\|_{X_q}^q + \frac{1}{r}\|u_0\|_{r,\mu}^r > 0. \tag{2.46}$$

Thus, $J(u_0) = \lambda_2 - \lambda < 0$ for any $\lambda > \lambda_2$, and

$$d = J(u_0) = \inf_{u \in E} J(u) < 0 \quad \text{for all } \lambda > \lambda_2. \tag{2.47}$$

This completes the proof. □

Next, we show that if $\lambda > \lambda_2$, then problem (1.1) admits a second nontrivial weak solution $e \neq u_0$ by the mountain pass theorem.

Lemma 2.7 *Suppose that assumptions* (H_0)-(H_1) *are satisfied. Then, for all* $e \in E$ *and* $\lambda > 0$, *there exist* $\alpha > 0$ *and* $\rho \in (0, \|e\|_E)$ *such that* $J(u) \geq \alpha$ *for all* $u \in E$ *with* $\|u\|_E = \rho$.

Proof Let $u \in E$. From (H_1), (1.18), and (1.19) with $t = p$ we obtain

$$\int_{\mathbb{R}^N} h(x)|u|^m \, dx \leq \|h\|_\gamma \|u\|_{p_s^*}^m \leq S_0^m \|h\|_\gamma \|u\|_E^m. \tag{2.48}$$

Then,

$$J(u) \geq p^{-1}\|u\|_{X_p}^p + q^{-1}\|u\|_{X_q}^q - \lambda S_0^m H \|u\|_E^m \geq p^{-1}\|u\|_E^p - \lambda S_0^m H \|u\|_E^m, \tag{2.49}$$

where $H = \|h\|_\gamma$, $\|u\|_E = \rho$, and

$$0 < \rho < \min\left\{1, \|e\|_E, \left(\lambda p S_0^m H\right)^{\frac{1}{p-m}}\right\}, \tag{2.50}$$

so that

$$J(u) \geq \rho^p\left(p^{-1} - \lambda S_0^m H \rho^{m-p}\right) \equiv \alpha > 0. \tag{2.51}$$

Thus, we finish the proof of Lemma 2.7. □

Lemma 2.8 *Under the assumptions of Theorem* 1.2 *and* $\lambda > \lambda_2$, *Eq.* (1.1) *admits a nontrivial weak solution* $u \in E$ *such that* $J(u) > 0$.

Proof By Lemma 2.6, for all $\lambda > \lambda_2$, there exists a nontrivial weak solution $u_0 \in E$ with $J(u_0) < 0$. Taking $e = u_0$ in Lemma 2.7, we get that J satisfies the geometrical structure of Theorem A.3 of [15]. Thus, for all $\lambda > \lambda_2$ there exists a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c > 0 \quad \text{and} \quad \|J'(u_n)\|_{E'} \to 0 \quad \text{as } n \to \infty, \tag{2.52}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} J(\gamma(t)) \quad \text{with } \Gamma = \{\gamma \in C([0,1];E) : \gamma(0) = 0, \gamma(1) = u_0\}. \tag{2.53}$$

Since J is coercive in E, the sequence $\{u_n\}$ is bounded in E. By Lemma 2.2 there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \to u$ in E as $n \to \infty$. Therefore, $J(u) = \lim_{n\to\infty} J(u_n) = c > 0$, and $J'(u)\varphi = \lim_{n\to\infty} J'(u_n)\varphi = 0$ for all $\varphi \in E$. So, u is a weak solution of (1.1) with $J(u) > 0$. □

Proof of Theorem 1.2 The application of Lemma 2.2 shows that problem (1.1) has only a trivial solution if $\lambda < \lambda_1$. By Lemmas 2.6 and 2.8 it follows that, for all $\lambda > \lambda_2$, problem (1.1) admits at least two nontrivial weak solutions in E, one with negative energy and the other with positive energy. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 We first prove, under the assumptions in Theorem 1.3, that any $(PS)_c$ sequence $\{u_n\}$ is bounded in E. Let the sequence $\{u_n\}$ satisfy (2.7). Then, for large n, we have

$$\begin{aligned} c + 1 + \|u_n\|_E &\ge J(u_n) - \frac{1}{m} J'(u_n)u_n \\ &= \left(\frac{1}{p} - \frac{1}{m}\right)\|u_n\|_{X_p}^p + \left(\frac{1}{q} - \frac{1}{m}\right)\|u_n\|_{X_q}^q + \left(\frac{1}{r} - \frac{1}{m}\right)\|u_n\|_{r,\mu}^r. \end{aligned} \tag{2.54}$$

Since $m > \max\{p, r\}$, it follows from (1.19) that $\{\|u_n\|_E\}$ is bounded. Furthermore, by Lemma 2.2 there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in E$ such that $u_n \to u$ in E and J satisfies the $(PS)_c$ condition.

From (2.26) it follows that if $u \in E$ is a nontrivial solution, then $\lambda > 0$. This proves part (i). In the following, we prove part (ii). We now verify the conditions in Theorem 6.5 in [17]. Clearly, the functional J defined by (1.20) is even, and $J(0) = 0$. By Lemma 2.7 there exist $\alpha, \rho > 0$ such that $J(u) \ge \alpha$ for all $u \in E$ with $\|u\|_E = \rho$.

On the other hand, for any finite-dimensional subspace $E_0 \subset E$, it is well known that any norms in E_0 are equivalent. So, there exist $d_1, d_2 > 0$ such that

$$d_1\|u\|_E \le \|u\|_{r,\mu} \le d_2\|u\|_E, \qquad d_1\|u\|_E \le \|u\|_{m,h} \le d_2\|u\|_E, \quad \forall u \in E_0. \tag{2.55}$$

Then, from (1.20) we have

$$J(u) \le \frac{1}{q}\left(\|u\|_E^p + \|u\|_E^q\right) + \frac{1}{r} d_2^r \|u\|_E^r - \frac{\lambda}{m} d_1^m \|u\|_E^m, \quad \forall u \in E_0. \tag{2.56}$$

Since $\lambda > 0$ and $m > \max\{p, r\}$, there exists $R = R(E_0) > \rho$ such that $J(u) < 0$ for $u \in E_0$ and $\|u\|_E \geq R$. Therefore, all conditions are verified. Then an application of Theorem 6.5 in [17] shows that Eq. (1.1) admits infinitely many solutions $u_n \in E$ with $J(u_n) \to \infty$ as $n \to \infty$. This completes the proof of Theorem 1.3. □

3 Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. The main tool for this purpose is the following symmetric mountain pass lemma. First, we introduce the concept of genus.

Definition 3.1 [17] Let E be a Banach space, and A a subset of E. The set A is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set A that does not contain the origin, we define the genus $\gamma(A)$ of A as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If such k does not exist, then we define $\gamma(A) = \infty$. We set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$.

Lemma 3.1 [20] (Symmetric mountain pass lemma) *Let E be an infinite-dimensional Banach space and $J \in C^1(E, \mathbb{R})$ such that*:

(I) *J is even and bounded from below, $J(0) = 0$, and J verifies the $(PS)_c$ condition.*

(II) *for each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} J(u) < 0$.*

Then one of the following two results holds:

(1) *there exists a sequence $\{u_k\}$ such that $J'(u_k) = 0, J(u_k) < 0$, and $\{u_k\}$ converges to zero.*

(2) *there exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $J'(u_k) = 0, J(u_k) = 0, u_k \neq 0, \lim_{k\to\infty} u_k = 0, J'(v_k) = 0, J(v_k) < 0, \lim_{k\to\infty} J(v_k) = 0$, and $\{v_k\}$ converges to a nonzero limit.*

We now establish the following:

Lemma 3.2 *Let the assumptions in Theorem* 1.4 *be satisfied. Then, for each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that*

$$\sup_{u \in A_k} J(u) < 0. \tag{3.1}$$

Proof We use the following geometric construction introduced by Kajikiya [20]. Let d_0 and $x_0 = (x_1^0, x_2^0, \ldots, x_N^0)$ be fixed by assumption (H_4) and consider the cube

$$D(d) = \left\{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \left|x_i - x_i^0\right| < d, 1 \leq i \leq N\right\}. \tag{3.2}$$

We choose small $d > 0$ such that the cube $D(d) \subset \Omega_0 := B_{d_0}(x_0)$. Note that $h(x) > 0$ in $D(d)$. Fix $k \in \mathbb{N}$ arbitrarily. Let $n \in \mathbb{N}$ be the smallest integer such that $n^N \geq k$. We divide $D(d)$ equally into n^N small cubes, denoted D_i, $1 \leq i \leq n^N$, by planes parallel to each face of $D(d)$. The edge of D_i has the length of $z = \frac{2d}{n}$. We construct new cubes E_i in D_i such that E_i has the same center as that of D_i. The faces of E_i and D_i are parallel, and the edge of E_i has the length $\frac{d}{n}$. Then, let the functions $\psi_i(x) \in C^1(\mathbb{R}^N), 1 \leq i \leq k$, be such that

$$\begin{aligned} &\operatorname{supp}(\psi_i) \subset D_i,\ \operatorname{supp}(\psi_i) \cap \operatorname{supp}(\psi_j) = \emptyset\ (i \neq j), \\ &\psi_i(x) = 1,\ x \in E_i,\ 0 \leq \psi_i(x) \leq 1,\ x \in \mathbb{R}^N. \end{aligned} \tag{3.3}$$

Denote

$$V_k = \left\{(t_1, t_2, \dots, t_k) \in \mathbb{R}^k : \max_{1\le i\le k} |t_i| = 1\right\} \tag{3.4}$$

and

$$W_k = \left\{\sum_{i=1}^{k} t_i\psi_i(x) : (t_1, t_2, \dots, t_k) \in V_k\right\} \subset E. \tag{3.5}$$

Clearly, V_k is the surface of k-dimensional, cube and W_k is a closed symmetric set in E such that $0 \notin W_k$. It is easy to see that V_k is homeomorphic to the sphere S^{k-1} by an odd mapping (take, e.g., the radial projection $V_k \to S^{k-1}$). Hence, $\gamma(V_k) = k$. Moreover, $\gamma(W_k) = \gamma(V_k) = k$ because the mapping $(t_1, t_2, \dots, t_k) \longmapsto \sum_{i=1}^{k} t_i\psi_i(x)$ is homeomorphic and odd. On the other hand, since W_k is bounded in E, there is a constant $\alpha_k > 0$ such that

$$\|u\|_E \le \alpha_k, \quad \forall u \in W_k. \tag{3.6}$$

Let $z > 0$ and $u = \sum_{i=1}^{k} t_i\psi_i(x) \in W_k$. Then,

$$\begin{aligned} J(zu) &= \frac{z^p}{p}\|u\|_{X_p}^p + \frac{z^q}{q}\|u\|_{X_q}^q + \frac{z^r}{r}\|u\|_{r,\mu}^r - \frac{1}{m}\int_{\mathbb{R}^N} h|zu|^m\,dx \\ &\le \frac{z^p}{p}\alpha_k^p + \frac{z^q}{q}\alpha_k^q + \frac{z^r}{r}\alpha_k^r S_r^r\|\mu\|_{L^\sigma(\Omega_0)} - \frac{1}{m}\sum_{i=1}^{k}\int_{D_i} h|zt_i\psi_i|^m\,dx, \end{aligned} \tag{3.7}$$

where S_r is the embedding constant in (1.18), and $\sigma = p_s^*/(p_s^* - r)$. By (3.7) there exists an integer $j \in [1, k]$ such that $|t_j| = 1$ and $|t_i| \le 1$ for $i \ne j$. Hence,

$$\sum_{i=1}^{k}\int_{D_i} h|zt_i\psi_i|^m\,dx = \sum_{i\ne j}\int_{D_i} h|zt_i\psi_i|^m\,dx + \int_{D_j\setminus E_j} h|zt_j\psi_j|^m\,dx + \int_{E_j} h|zt_j\psi_j|^m\,dx. \tag{3.8}$$

Since $\psi_j(x) = 1$ for $x \in E_j$ and $|t_j| = 1$, we have

$$\int_{E_j} h|zt_j\psi_j|^m\,dx = |z|^m\int_{E_j} h\,dx. \tag{3.9}$$

On the over hand, since $D(d) \subset \Omega_0$, by (H_4) we obtain

$$\sum_{i\ne j}\int_{D_i} h|zt_i\psi_i|^m\,dx + \int_{D_j\setminus E_j} h|zt_j\psi_j|^m\,dx \ge 0. \tag{3.10}$$

Then, it follows from (3.7)-(3.10) that

$$\frac{J(zu)}{z^q} \le \frac{z^{p-q}}{p}\alpha_k^p + \frac{1}{q}\alpha_k^q + \frac{z^{r-q}}{r}\alpha_k^r S_r^r\|\mu\|_{L^\sigma(\Omega_0)} - \frac{1}{m}z^{m-q}\inf_{1\le i\le k}\left(\int_{E_i} h\,dx\right). \tag{3.11}$$

Since $h(x) > 0$ in E_i and $m \in (1, q)$, we have

$$\lim_{z\to 0^+}\sup_{u\in W_k}\frac{J(zu)}{z^q} = -\infty. \tag{3.12}$$

We fix $z > 0$ small such that

$$\sup\{J(u) : u \in A_k\} < 0, \quad \text{where } A_k = zW_k \in \Gamma_k, \tag{3.13}$$

which completes the proof of (3.1) and thus of Lemma 3.2. □

Proof of Theorem 1.4 Evidently, $J(0) = 0$, and J is an even functional. Then, by Lemma 2.2, J satisfies the $(PS)_c$ condition. Furthermore, by Lemma 3.2 conditions (I) and (II) in Lemma 3.1 are satisfied. Thus, by Lemma 3.1 problem (1.1) admits infinitely many solutions $u_n \in E$ with $u_n \to 0$ in E. Thus, the proof of Theorem 1.4 is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by CSC. JFB prepared the manuscript in part. All steps of the proofs in this research are performed by CSC. All authors read and approved the final manuscript.

Author details
[1]College of Science, Hohai University, Nanjing, 210098, P.R. China. [2]College of Science, Nanjing Forestry University, Nanjing, 210037, P.R. China.

Acknowledgements
This work is supported by the Fundamental Research Funds for the Central Universities of China (2015B31014).

Received: 24 May 2016 Accepted: 8 August 2016 Published online: 17 August 2016

References

1. Bartolo, R, Candela, AM, Salvatore, A: On a class of superlinear (p,q)-Laplacian type equations on $\mathbb{R}^N$. J. Math. Anal. Appl. **438**, 29-41 (2016)
2. Chaves, MF, Ercole, G, Miyagaki, OH: Existence of a nontrivial solution for the (p,q)-Laplacian in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition. Nonlinear Anal. **114**, 133-141 (2015)
3. Di Nezza, E, Palatucci, G, Valdinoci, E: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521-573 (2012)
4. Franzina, G, Palatucci, G: Fractional p-eigenvalues. Riv. Mat. Univ. Parma **5**, 315-328 (2014)
5. Pucci, P, Xiang, MQ, Zhang, BL: Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. Calc. Var. **54**, 2785-2806 (2015)
6. Xiang, MQ, Zhang, BL, Rădulescu, V: Existence of solutions for perturbed fractional p-Laplacian equations. J. Differ. Equ. **260**, 1392-1413 (2016)
7. Cherfils, L, Il'yszov, Y: On the stationary solutions of generalized reaction diffusion equation with p&q-Laplacian. Commun. Pure Appl. Anal. **4**, 9-22 (2005)
8. He, CJ, Li, GB: The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacian. Ann. Acad. Sci. Fenn., Math. **33**, 337-371 (2008)
9. Barile, S, Figueiredo, GM: Existence of least energy positive, negative and nodal solutions for a class of p&q-problems with potentials vanishing at infinity. J. Math. Anal. Appl. **427**, 1205-1233 (2015)
10. He, CJ, Li, GB: The existence of a nontrivial solution to the p&q-Laplacian problem with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$. Nonlinear Anal. **68**, 1100-1119 (2008)
11. Li, GB, Liang, XY: The existence of nontrivial solutions to nonlinear elliptic equation of p-q-Laplacian type on $\mathbb{R}^N$. Nonlinear Anal. **71**, 2316-2334 (2009)
12. Wu, MZ, Yang, ZD: A class of p-q-Laplacian type equation with potentials eigenvalue problem in R^N. Bound. Value Probl. **2009**, Article ID 185319 (2009)
13. Yin, HH, Yang, ZD: Multiplicity of positive solutions to a p-q-Laplacian equation involving critical nonlinearity. Nonlinear Anal. **75**, 3021-3035 (2012)
14. Pucci, P, Rădulescu, V: Combined effects in quasilinear elliptic problems with lack of compactness. Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. **22**, 189-205 (2011)
15. Autuori, G, Pucci, P: Existence of entire solutions for a class of quasilinear elliptic equations. Nonlinear Differ. Equ. Appl. **20**, 977-1009 (2013)
16. Autuori, G, Pucci, P: Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$. J. Differ. Equ. **255**, 2340-2362 (2013)
17. Struwe, M: Variational Methods, 3rd edn. Springer, New York (2000)
18. Pucci, P, Xiang, MQ, Zhang, BL: Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations. Adv. Nonlinear Anal. **5**(1), 27-55 (2016)
19. Molica Bisci, G, Rădulescu, V, Servadei, R: Variational Methods for Nonlocal Fractional Problems. Cambridge University Press, Cambridge (2016)
20. Kajikiya, R: A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations. J. Funct. Anal. **225**, 352-370 (2005)