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Existence of a positive solution for problems with (p,q)-Laplacian and convection term in $\mathbb{R}^N$

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Abstract

This paper provides a positive solution for the (p,q)-Laplace equation in $\mathbb{R}^N$ with a nonlinear term depending on the gradient. The solution is constructed as the limit of positive solutions in bounded domains. Strengthening the growth condition, it is shown that the solution is also bounded. The positivity of the solution is obtained through a new comparison principle. Finally, under a stronger growth condition, we show the existence of a vanishing at infinity solution.

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1 Introduction

In this paper, we study the existence of a (positive) solution for the following quasi-linear elliptic equation:

$$\text{(P)}\quad \begin{cases} -\Delta_p u + \lambda_1 |u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2 |u|^{q-2}u = f(x,u,\nabla u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

In the left-hand side of the equation in (P), we have the p-Laplacian Δ_p and the q-Laplacian Δ_q with $1 < q < p < +\infty$ and the constants $\mu \geq 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$. The problem covers the corresponding statement with p-Laplacian in the principal part, for which it is sufficient to take $\mu = 0$. Here $-\Delta_p$ is regarded as the operator $-\Delta_p : W^{1,p}(\mathbb{R}^N) \to W^{-1,p'}(\mathbb{R}^N)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, defined by

$$\langle -\Delta_p u, v\rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v\,dx \quad \text{for all } u, v \in W^{1,p}(\mathbb{R}^N).$$

The right-hand side of the equation in (P) is in the form of convection term, meaning a nonlinearity that depends on the point x in $\mathbb{R}^N$, on the solution u, and on its gradient ∇u.

The existence of positive solutions for problems with p-Laplacian and convection term on a bounded domain has been studied in [1–3]. In the case where the principal part of the equation is driven by the (p,q)-Laplacian operator with $1 < q < p$ and by a nonhomogeneous operator, the existence of a positive solution of elliptic problems with convex term on a bounded domain has been investigated in [4] and [5], respectively. Results of this type when the principal part of the equation is expressed through a general Leray-Lions operator can be found in [6]. Essential features of the present work are the dependence on the gradient ∇u, which prevents the use of variational methods, and the unboundedness of the domain, which produces lack of compactness.

We assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the growth condition:

(F0) $f(x,0,\xi) \equiv 0$ for all $x \in \mathbb{R}^N, \xi \in \mathbb{R}^N$;

(F1) there exist constants $r_1, r_2 \in (0, p-1)$ and continuous nonnegative functions $a_0 \in L^{p'}(\mathbb{R}^N)$ and $a_i \in L^{\tilde{r}_i}(\mathbb{R}^N)$ $(i = 1,2)$, where $1/p + 1/p' = 1$ and $\tilde{r}_i = p/(p - r_i - 1) = (p/(r_i + 1))'$ $(i = 1,2)$, such that

$$\bigl|f(x,t,\xi)\bigr| \le a_0(x) + a_1(x)|t|^{r_1} + a_2(x)|\xi|^{r_2} \tag{1}$$

for all $(x,t,\xi) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$.

In this setting, by a solution of problem (P) we mean any function $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ when $\mu > 0$ or $u \in W^{1,p}(\mathbb{R}^N)$ when $\mu = 0$ such that $u(x) > 0$ for a.e. $x \in \mathbb{R}^N$ and

$$\begin{aligned} &\int_{\mathbb{R}^N} \bigl(|\nabla u|^{p-2} + \mu|\nabla u|^{q-2}\bigr)\nabla u \nabla\varphi\,dx + \int_{\mathbb{R}^N} \bigl(\lambda_1|u|^{p-2} + \mu\lambda_2|u|^{q-2}\bigr)u\varphi\,dx \\ &\quad= \int_{\mathbb{R}^N} f(x,u,\nabla u)\varphi\,dx \quad \text{for all } \varphi \in C_0^\infty\bigl(\mathbb{R}^N\bigr). \end{aligned}$$

In order to show the positivity of a solution, we will need an additional growth condition when t is small:

(F2) there exist constants $\delta_0 > 0$ and $r_0 \in (0, p-1)$ if $\mu = 0$ or $r_0 \in (0, q-1)$ if $\mu > 0$ and a continuous positive function b_0 such that

$$b_0(x)t^{r_0} \le f(x,t,\xi) \quad \text{for all } 0 < t \le \delta_0, x \in \mathbb{R}^N, \xi \in \mathbb{R}^N. \tag{2}$$

We mention that the fact that condition (F2) is supposed only for $t > 0$ small is a significant improvement with respect to all the previous works. A direct consequence is that f is allowed to change sign.

For example, the following nonlinearity satisfies our assumptions (F0) $\sim$ (F2):

$$f(x,t,\xi) = a_1(x)|t|^{r_0} + a_2(x)|\xi|^{r_2}\sin t,$$

with $a_1 \in L^{p'}(\mathbb{R}^N) \cap L^{\tilde{r}_1}(\mathbb{R}^N)$ and $a_2 \in L^{\tilde{r}_2}(\mathbb{R}^N)$, where $\tilde{r}_i = p/(p - r_i - 1)$ $(i = 1,2)$.

Our main result provides the existence of a (positive) solution for problem (P).

Theorem 1 *Under assumptions* (F0)-(F2), *problem* (P) *admits a* (*positive*) *solution* $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \cap C^1_{\mathrm{loc}}(\mathbb{R}^N)$ *if* $\mu > 0$ *and* $u \in W^{1,p}(\mathbb{R}^N) \cap C^1_{\mathrm{loc}}(\mathbb{R}^N)$ *if* $\mu = 0$.

The proof is based on a priori estimates obtained through hypotheses (F0)-(F2) for approximate solutions on bounded domains and the use of comparison arguments. In this respect, we establish several comparison principles that ultimately determine the positivity of solutions.

Under a stronger version of the growth condition (F1), we show that any positive solution disappear at infinity.

Theorem 2 *Assume* (F0)-(F2). *If one of the following conditions holds, then any* (*positive*) *solution u of problem* (P) *satisfies that $u(x) \to 0$ as $|x| \to \infty$*:

(i) $N \le p$;

(ii) $N > p$ *and* $p^2 < p^*$ (*if and only if* $p < N < p^2/(p-1)$);

(iii) $N \ge p^2/(p-1)$, *and for the functions* a_i $(i = 0,1,2)$ *in* (F1), *there exist* $R_* > 0$ *and* γ_i *such that*

$$\gamma_0, \gamma_1 > \frac{p^*}{p^* - p}, \qquad \gamma_2 > \frac{pp^*}{(p - r_2)(p^* - p)}, \tag{3}$$

$$\sup_{x_0 \in \mathbb{R}^N} \int_{B(x_0, 2R_*)} \left|a_i(x)\right|^{\gamma_i} dx < \infty \tag{4}$$

for $i = 0,1,2$, *where* $p^* := pN/(N-p)$ *if* $N > p$.

Since we are looking for positive solutions of problem (P), without any loss of generality, we will suppose in the sequel that $f(x,t,s) \equiv 0$ for all $t \le 0$ and $(x,s) \in \mathbb{R}^N \times \mathbb{R}^N$.

The rest of the paper is organized as follows. In Section 2, we present comparison principles related to problem (P). Section 3 deals with approximate solutions on bounded domains. Section 4 is devoted to the proof of Theorem 1. In Section 5, we give a proof of Theorem 2 after we show the boundedness of a solution.

2 Comparison principles

In this section, we assume that D is a bounded domain in $\mathbb{R}^N$. We consider the operator denoted $-\Delta_p$ from $W^{1,p}(D)$ to $W^{1,p}(D)^*$ defined by

$$\langle -\Delta_p u, v\rangle = \int_D |\nabla u|^{p-2} \nabla u \nabla v \, dx \quad \text{for all } u, v \in W^{1,p}(D).$$

First we recall the following result.

Lemma 1 ([7], Lemma 2.1) *Let $w_1, w_2 \in L^\infty(D)$ satisfy $w_i \ge 0$ a.e. on D, $w_i^{1/q} \in W^{1,p}(D)$ for $i = 1,2$, and $w_1 = w_2$ on ∂D. If $w_1/w_2, w_2/w_1 \in L^\infty(D)$, then*

$$0 \le \left\langle -\Delta_p w_1^{1/q} - \mu \Delta_q w_1^{1/q}, \frac{w_1 - w_2}{w_1^{(q-1)/q}} \right\rangle - \left\langle -\Delta_p w_2^{1/q} - \mu \Delta_q w_2^{1/q}, \frac{w_1 - w_2}{w_2^{(q-1)/q}} \right\rangle. \tag{5}$$

Lemma 1 leads to a comparison principle for a subsolution and a supersolution of the problem

$$\begin{cases} -\Delta_p u + \lambda_1 |u|^{p-2} u - \mu \Delta_q u + \mu \lambda_2 |u|^{q-2} u = g(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{6}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function.

We say that $u_1 \in W^{1,p}(D)$ is a subsolution of problem (6) if $u_1 \leq 0$ on ∂D and

$$\begin{aligned}&\int_D (|\nabla u_1|^{p-2}\nabla u_1 \nabla\varphi + \lambda_1|u_1|^{p-2}u_1\varphi + \mu|\nabla u_1|^{q-2}\nabla u_1\nabla\varphi + \mu\lambda_2|u_1|^{q-2}u_1\varphi\,dx \\ &\quad\leq \int_D g(u_1)\varphi\,dx\end{aligned}$$

for all $\varphi \in W_0^{1,p}(D)$ with $\varphi \geq 0$ in D, provided that the integral $\int_D g(u_1)\varphi\,dx$ exists. We say that $u_2 \in W^{1,p}(D)$ is a supersolution of (6) if the reversed inequalities are satisfied with u_2 in place of u_1 for all $\varphi \in W_0^{1,p}(D)$ with $\varphi \geq 0$ in D.

Theorem 3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $t^{1-q}g(t)$ is nonincreasing for $t > 0$ if $\mu > 0$ and $t^{1-p}g(t)$ is nonincreasing for $t > 0$ if $\mu = 0$. Assume that u_1 and u_2 are a positive subsolution and a positive supersolution of problem* (6), *respectively. If $u_2(x) > u_1(x) = 0$ for all $x \in \partial D$ and $u_i \in C^1(\overline{D})$ for $i = 1, 2$, then $u_2 \geq u_1$ in D.*

Proof We prove the result only for $\mu > 0$ because the case $\mu = 0$ is easier. Suppose by contradiction that the set $D_0 = \{x \in D : u_1(x) > u_2(x)\}$ is nonempty. Let U be a connected component of D_0. Noting that $\inf_{\overline{D}} u_2 > 0$, we see that $\overline{U} \subset D$, $u_1 = u_2$ on ∂U, and $u_i/u_j \in L^\infty(U)$ for $i, j = 1, 2$. So, $(u_1^q - u_2^q)/u_1^{q-1}, (u_1^q - u_2^q)/u_2^{q-1} \in W_0^{1,p}(U)$, and extending by 0 on $D \setminus U$, we can take them as test functions in the above definitions of subsolution and supersolution for problem (6). It follows that

$$\begin{aligned}&\left\langle -\Delta_p u_1 - \mu\Delta_q u_1, \frac{u_1^q - u_2^q}{u_1^{q-1}}\right\rangle - \left\langle -\Delta_p u_2 - \mu\Delta_q u_2, \frac{u_1^q - u_2^q}{u_2^{q-1}}\right\rangle \\ &\quad\leq \int_U \left(-\lambda_1 u_1^{p-1} - \mu\lambda_2 u_1^{q-1} + g(u_1)\right)\frac{u_1^q - u_2^q}{u_1^{q-1}}\,dx \\ &\qquad - \int_U \left(-\lambda_1 u_2^{p-1} - \mu\lambda_2 u_2^{q-1} + g(u_2)\right)\frac{u_1^q - u_2^q}{u_2^{q-1}}\,dx \\ &\quad= -\lambda_1 \int_U \left(u_1^{p-q} - u_2^{p-q}\right)\left(u_1^q - u_2^q\right)dx + \int_U \left(\frac{g(u_1)}{u_1^{q-1}} - \frac{g(u_2)}{u_2^{q-1}}\right)\left(u_1^q - u_2^q\right) \\ &\quad< 0.\end{aligned}$$

The last inequality is obtained through our assumption that $g(t)/t^{q-1}$ is nonincreasing for $t > 0$ and $\lambda_1 > 0$.

On the other hand, note that we can apply Lemma 1 with $w_i = u_i^q$ $(i = 1, 2)$ and U in place of D. The conclusion provided by (5) in Lemma 1 contradicts the above inequality in the case where U is nonempty. Therefore, $D_0 = \emptyset$, which completes the proof. □

The next theorem points out that the condition of supersolution in Theorem 3 can be relaxed to a weaker notion of supersolution directly related to the given subsolution.

Theorem 4 *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous function such that $t^{1-q}g(t)$ is nonincreasing for $t > 0$ if $\mu > 0$ and $t^{1-p}g(t)$ is nonincreasing for $t > 0$ if $\mu = 0$. Assume that $u_1 \in C^1(\overline{D})$ is a positive subsolution of problem* (6) *and that $h : \overline{D} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous function* *such that* $h(x,t,\xi) \geq g(t)$ *for all* $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$, *and* $t \in (0, \|u_1\|_{L^\infty(D)}]$. *If* $u_2 \in C^1(\overline{D})$ *is a* (*positive*) *solution of*

$$\begin{cases} -\Delta_p u + \lambda_1 |u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2 |u|^{q-2}u = h(x,u,\nabla u) & \textit{in } D, \\ u > 0 & \textit{in } D \end{cases} \tag{7}$$

such that $u_2(x) > u_1(x) = 0$ *for all* $x \in \partial D$, *then* $u_2 \geq u_1$ *in* D.

Proof The conclusion can be achieved following the same argument as in Theorem 3. Indeed, arguing by contradiction, let us assume that the set $D_0 := \{x \in D : u_1(x) > u_2(x)\}$ is not empty. Let U be a connected component of D_0. Then $u_1 = u_2$ on ∂U and $g(u_2) \leq h(x, u_2, \nabla u_2)$ in U. Proceeding as in the proof of Theorem 3, we have

$$\left\langle -\Delta_p u_1 - \mu\Delta_q u_1, \frac{u_1^q - u_2^q}{u_1^{q-1}} \right\rangle - \left\langle -\Delta_p u_2 - \mu\Delta_q u_2, \frac{u_1^q - u_2^q}{u_2^{q-1}} \right\rangle < 0.$$

This leads to a contradiction by applying Lemma 1. □

In the case where u_1 and u_2 satisfy the homogeneous Dirichlet boundary condition we can state the following:

Theorem 5 *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $t^{1-q}g(t)$ *is nonincreasing for* $t > 0$ *if* $\mu > 0$ *and* $t^{1-p}g(t)$ *is nonincreasing for* $t > 0$ *if* $\mu = 0$. *Assume that* $u_1 \in C_0^1(\overline{D})$ *is a positive subsolution of problem* (6) *and that* $h : \overline{D} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is continuous and such that* $h(x,t,\xi) \geq g(t)$ *for all* $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$, *and* $t \in (0, \|u_1\|_{L^\infty(D)}]$. *If* $u_2 \in C_0^1(\overline{D})$ *is a* (*positive*) *solution of problem* (7) *such that* $u_1/u_2 \in L^\infty(D)$ *and* $u_2/u_1 \in L^\infty(D)$, *then* $u_2 \geq u_1$ *in* D.

Proof Due to the assumptions $u_1/u_2 \in L^\infty(D)$ and $u_2/u_1 \in L^\infty(D)$, it turns out that $(u_1^q - u_2^q)/u_1^{q-1}, (u_1^q - u_2^q)/u_2^{q-1} \in W_0^{1,p}(U)$ with U introduced in the proof of Theorem 3. Then we can conclude as in the proof of Theorem 4. □

3 Solution on a bounded domain

In this section, we assume that D is a bounded domain in $\mathbb{R}^N$ with C^2 boundary ∂D. For $r \geq 1$, we denote by $\|u\|_{L^r(D)}$ the usual norm on the space $L^r(D)$. We endow $W_0^{1,p}(D)$ with the norm $\|u\|_D^p = \|\nabla u\|_{L^p(D)}^p + \lambda_1 \|u\|_{L^p(D)}^p$, which is equivalent to the usual one.

We focus on the existence of a (positive) solution for the problem

$$\text{(PD)} \quad \begin{cases} -\Delta_p u + \lambda_1 |u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2 |u|^{q-2}u = f(x,u,\nabla u) & \text{in } D, \\ u > 0 & \text{in } D, \\ u(x) = 0 & \text{on } \partial D. \end{cases}$$

Here we impose the following hypotheses: $f : \overline{D} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

($\widetilde{\text{F0}}$) $f(x,0,\xi) \equiv 0$ for all $x \in D, \xi \in \mathbb{R}^N$;

($\widetilde{\text{F1}}$) there exist constants $r_1, r_2 \in (0, p-1)$ and continuous nonnegative functions a_i ($i = 0, 1, 2$) on $\overline{D}$ such that

$$\left|f(x,t,\xi)\right| \leq a_0(x) + a_1(x)|t|^{r_1} + a_2(x)|\xi|^{r_2} \tag{8}$$

for all $(x,t,\xi) \in D \times \mathbb{R} \times \mathbb{R}^N$;

($\widetilde{\text{F2}}$) there exist constants $\delta_0 > 0$ and $r_0 \in (0, p-1)$ if $\mu = 0$ or $r_0 \in (0, q-1)$ if $\mu > 0$ and a continuous function b_0 such that $\inf_{x\in D} b_0(x) > 0$ and

$$b_0(x)t^{r_0} \leq f(x,t,\xi) \quad \text{for all } 0 < t \leq \delta_0, x \in D, \xi \in \mathbb{R}^N. \tag{9}$$

We say that $u \in W_0^{1,p}(D)$ is a solution of (PD) if $u(x) > 0$ for a.e. $x \in D$ and

$$\begin{aligned}&\int_D \left(|\nabla u|^{p-2} + \mu|\nabla u|^{q-2}\right)\nabla u \nabla \varphi \, dx + \int_D \left(\lambda_1|u|^{p-2} + \mu\lambda_2|u|^{q-2}\right)u\varphi \, dx \\ &\quad = \int_D f(x,u,\nabla u)\varphi \, dx \quad \text{for all } \varphi \in W_0^{1,p}(D).\end{aligned}$$

The existence of a solution for problem (PD) is stated as follows.

Theorem 6 *Under assumptions* ($\widetilde{\widetilde{\text{F0}}}$)-($\widetilde{\text{F2}}$), *problem* (PD) *admits a* (*positive*) *solution* $u \in C_0^1(\overline{D})$ *such that* $\partial u/\partial \nu < 0$ *on* ∂D, *where* ν *stands for the outer normal to* ∂D.

In the proof of Theorem 6, we utilize the following approximate equation:

$$(\mathrm{PD}_\varepsilon) \qquad \begin{cases} -\Delta_p u + \lambda_1|u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2|u|^{q-2}u = f(x,u,\nabla u) + \varepsilon\psi & \text{in } D, \\ u > 0 & \text{in } D, \\ u(x) = 0 & \text{on } \partial D, \end{cases}$$

with $\varepsilon > 0$ and a nonnegative function $0 \not\equiv \psi \in C(\overline{D})$.

Lemma 2 *Under* ($\widetilde{\widetilde{\text{F0}}}$)-($\widetilde{\text{F2}}$), *for any* $\varepsilon > 0$ *and a nonnegative function* $0 \not\equiv \psi \in C(D)$, *problem* ($\mathrm{PD}_\varepsilon$) *admits a* (*positive*) *solution* $u_\varepsilon \in C_0^1(\overline{D})$ *such that* $\partial u_\varepsilon/\partial \nu < 0$ *on* ∂D.

Proof We argue as in [5], Proposition 8, and [7]. Fix $\varepsilon > 0$ and consider a Schauder basis $\{e_1, \dots, e_m, \dots\}$ of $W_0^{1,p}(D)$ (refer to [4, 8] for its existence). For each $m \in \mathbb{N}$, we define the m-dimensional subspace $V_m := \operatorname{span}\{e_1, \dots, e_m\}$ of $W_0^{1,p}(D)$. The map $T_m \colon \mathbb{R}^m \to V_m$ defined by $T_m(\xi_1, \dots, \xi_m) = \sum_{i=1}^m \xi_i e_i$ is a linear isomorphism. Let $T_m^* : V_m^* \to (\mathbb{R}^m)^*$ be the dual map of T_m. Identifying $\mathbb{R}^m$ and $(\mathbb{R}^m)^*$, we may regard T_m^* as a map from V_m^* to $\mathbb{R}^m$. Define the maps A_m and B_m from V_m to V_m^* as follows:

$$\left\langle A_m(u), v\right\rangle := \int_D \left(|\nabla u|^{p-2} + \mu|\nabla u|^{q-2}\right)\nabla u \nabla v \, dx$$

and

$$\left\langle B_m(u), v\right\rangle := -\int_D \left(\lambda_1|u|^{p-2} + \mu\lambda_2|u|^{q-2}\right)uv \, dx + \int_D \left(f(x,u,\nabla u) + \varepsilon\psi\right)v \, dx$$

for all $u, v \in V_m$.

By ($\widetilde{\mathrm{F}}1$) and Hölder's inequality, we have

$$\begin{aligned}&\langle A_m(u) - B_m(u), u\rangle\\&\quad \geq \|u\|_D^p - d\bigl(\|u\|_{L^1(D)} + \|u\|_{L^{r_1+1}(D)}^{r_1+1} + \|u\|_D^{r_2+1}\bigr) - \varepsilon\|\psi\|_{L^\infty(D)}\|u\|_{L^1(D)}\end{aligned}\tag{10}$$

for all $u \in V_m$, where d is a positive constant independent of m and u. Because of $r_1 + 1 < p$ and $r_2 + 1 < p$, we easily see that $A_m - B_m$ is coercive on V_m, whence $T_m^* \circ (A_m - B_m) \circ T_m$ is coercive on $\mathbb{R}^m$. By a well-known consequence of Brouwer's fixed point theorem it follows that there exists $y_m \in \mathbb{R}^m$ such that $(T_m^* \circ (A_m - B_m) \circ T_m)(y_m) = 0$, and hence $A_m(u_m) - B_m(u_m) = 0$ with $u_m = T_m(y_m) \in V_m$.

Writing (10) with $u = u_m \in W_0^{1,p}(D)$ shows the boundedness of the sequence $\|u_m\|_D$. Thus, along a subsequence, u_m converges to some u_0 weakly in $W_0^{1,p}(D)$ and strongly in $L^p(D)$.

We claim that

$$u_m \to u_0 \quad \text{in } W_0^{1,p}(D) \text{ as } m \to \infty.\tag{11}$$

Let P_m denote the projection onto V_m, that is, $P_m u = \sum_{i=1}^m \xi_i e_i$ for $u = \sum_{i=1}^\infty \xi_i e_i$. Since u_m, $P_m u_0 \in V_m$ and $A_m(u_m) - B_m(u_m) = 0$ in V_m^*, we obtain

$$\begin{aligned}&\langle A_m(u_m), u_m - P_m u_0\rangle\\&\quad = \langle B_m(u_m), u_m - P_m u_0\rangle\\&\quad = \langle B_m(u_m), u_m - u_0\rangle + \langle B_m(u_m), u_0 - P_m u_0\rangle\\&\quad = \int_D \bigl(-\lambda_1|u_m|^{p-2}u_m - \mu\lambda_2|u_m|^{q-2}u_m + f(x, u_m, \nabla u_m)\bigr)(u_m - P_m u_0)\,dx\\&\qquad + \int_D \varepsilon\psi(u_m - P_m u_0)\,dx\\&\quad \to 0 \quad \text{as } m \to \infty,\end{aligned}$$

where we use ($\widetilde{\mathrm{F}}1$), the boundedness of $\|u_m\|_D$, $u_m \to u_0$ in $L^p(\Omega)$, and $P_m u_0 \to u_0$ in $W_0^{1,p}(D)$. This leads to

$$\lim_{m\to\infty} \int_D |\nabla u_m|^{p-2}\nabla u_m \nabla(u_m - u_0)\,dx = 0.$$

In view of the (S_+)-property of $-\Delta_p$ (see, e.g., [9], Proposition 3.5, or refer to (22) in the proof of Theorem 1), we obtain (11).

Now let us prove that u_0 is a solution of $(\mathrm{PD}_\varepsilon)$. Fix $l \in \mathbb{N}$ and $\varphi \in V_l$. For each $m \geq l$, letting $m \to \infty$ in $\langle A_m(u_m), \varphi\rangle = \langle B_m(u_m), \varphi\rangle$ and making use of (11), we have

$$\begin{aligned}&\int_D \bigl(|\nabla u_0|^{p-2} + \mu|\nabla u_0|^{q-2}\bigr)\nabla u_0 \nabla\varphi\,dx + \int_D \bigl(\lambda_1|u_0|^{p-2} + \mu\lambda_2|u_0|^{q-2}\bigr)u_0\varphi\,dx\\&\quad = \int_D \bigl(f(x, u_0, \nabla u_0) + \varepsilon\psi\bigr)\varphi\,dx.\end{aligned}\tag{12}$$

Since l is arbitrary, equality (12) holds for every $\varphi \in \bigcup_{l\geq 1} V_l$. In fact, the density of $\bigcup_{l\geq 1} V_l$ in $W_0^{1,p}(D)$ guarantees that (12) holds for every $\varphi \in W_0^{1,p}(D)$. This means that u_0 is a solution of $(\mathrm{PD}_\varepsilon)$. Acting with $-u_0^-$ (where $u_0^- := \max\{0, -u_0\}$) and taking into account that $\psi \geq 0$ and $f(x,t,\xi) = 0$ for $t \leq 0$, we see that

$$\begin{aligned}
&\|u_0^-\|_D^p + \mu \|\nabla u_0^-\|_{L^q(D)}^q + \mu\lambda_2 \|u_0^-\|_{L^q(D)}^q \\
&\quad = \int_{u_0<0} \big(f(x,u_0,\nabla u_0) + \varepsilon\psi\big) u_0 \, dx = \varepsilon \int_{u_0<0} \psi u_0 \, dx \leq 0,
\end{aligned}$$

whence $u_0 \geq 0$ a.e. in D. Moreover, $u_0 \not\equiv 0$ because we assumed that $\psi \not\equiv 0$ and $\varepsilon > 0$. Next, we observe that hypothesis $(\widetilde{\mathrm{F}1})$ allows us to refer to [10], Theorem 7.1 (see also [11] and [5]), from which we infer that $u \in L^\infty(D)$. Furthermore, the regularity result up to the boundary in [12], Theorem 1, and [13], p.320, ensures that $u \in C_0^{1,\beta}(\overline{D})$ with some $\beta \in (0,1)$. Applying the strong maximum principle in [14], Theorem 5.4.1, and the boundary point lemma in [14], Theorem 5.5.1 (note that $f(x,t,\xi) \geq 0$ for $0 \leq t \leq \delta_0$) entails that $u > 0$ in D and $\partial u/\partial\nu < 0$ on ∂D. Altogether, we have established that the conclusion of lemma is fulfilled for $u_\varepsilon = u_0$. □

We will also need the following result.

Lemma 3 *Let* $1 < q < p < +\infty$, $\lambda_1 > 0$, $\lambda_2 \geq 0$, *and* $\mu \geq 0$. *For any constants* $b > 0$ *and* $0 < r < p-1$ *with* $0 < r < q-1$ *if* $\mu > 0$, *the problem*

$$\begin{cases} -\Delta_p u + \lambda_1 |u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2 |u|^{q-2}u = bu^r & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases} \tag{13}$$

admits a solution $u_b \in C_0^1(\overline{D})$ *satisfying* $\lambda_1 \|u_b\|_{L^\infty(D)}^{p-r-1} \leq b$ *and* $\partial u_b/\partial\nu < 0$ *on* ∂D.

Proof We can proceed for the existence of a solution of (13) along the lines of the proof of [7], Lemma 3. For readers' convenience, we outline the proof in the case where $\mu > 0$. Given the constants $b > 0$ and $0 < r < q-1$, we define the functional $I : W_0^{1,p}(D) \to \mathbb{R}$ by

$$I(u) = \frac{1}{p}\int_D \big(|\nabla u|^p + \lambda_1 |u|^p\big)\, dx + \frac{\mu}{q}\int_D \big(|\nabla u|^q + \lambda_2 |u|^q\big)\, dx - \frac{b}{r+1}\int_D \big(u^+\big)^{r+1}\, dx$$

for all $u \in W_0^{1,p}(D)$, where $u^+ = \max\{0, u\}$. Notice that I is of class C^1. By using the Sobolev embedding theorem we have the estimate

$$I(u) \geq \frac{1}{p}\|u\|_D^p - c\|u\|_D^{r+1} \quad \text{for all } u \in W_0^{1,p}(D)$$

with a constant $c > 0$ independent of u. Since $p > r+1$, I is bounded from below and coercive. Having that I is sequentially weakly lower semicontinuous too, there exists $u_b \in W_0^{1,p}(D)$ such that

$$I(u_b) = \inf_{u \in W_0^{1,p}(D)} I(u)$$

(see, e.g., [15], Theorems 1.1, 1.2). In addition, because of $r+1<q<p$, taking any positive smooth function v and a sufficiently small $t>0$, we have

$$I(tv)=t^{r+1}\left(\frac{t^{p-r-1}}{p}\|v\|_D+\frac{\mu t^{q-r-1}}{q}\left(\|\nabla v\|^q_{L^q(D)}+\lambda_2\|v\|^q_{L^q(D)}\right)-\frac{\|v\|^{r+1}_{L^{r+1}(D)}}{r+1}\right)<0.$$

This ensures that $\inf_{u\in W_0^{1,p}(D)} I(u)<0$, and hence u_b is a nontrivial critical point of I. By the regularity theory we infer that $u_b\in C_0^1(\overline{D})$. Taking $-u_b^-$ as a test function in the equation $I'(u_b)=0$, we see that $u_b\geq 0$. Then the strong maximum principle enables us to derive that $u_b>0$ in D, so u_b is a solution of problem (13), and $\partial u_b/\partial\nu<0$ on ∂D.

Taking $u_b^{\alpha+1}$ with $\alpha>0$ as a test function in (13), by using Hölder's inequality and that $r+1<p$ we get

$$\lambda_1\|u_b\|^{p+\alpha}_{L^{p+\alpha}(D)}\leq b\int_D u_b^{r+\alpha+1}\,dx\leq b\|u_b\|^{r+\alpha+1}_{L^{p+\alpha}(D)}|D|^{(p-r-1)/(p+\alpha)},$$

where $|D|$ denotes the Lebesgue measure of D, and hence

$$\lambda_1\|u_b\|^{p-r-1}_{L^{p+\alpha}(D)}\leq b|D|^{(p-r-1)/(p+\alpha)}.$$

Letting $\alpha\to\infty$, we conclude that $\lambda_1\|u_b\|^{p-r-1}_{L^\infty(D)}\leq b$. □

Proof of Theorem 6 Using the data δ_0, r_0, and b_0 in ($\widetilde{\mathrm{F}}2$), we fix a positive constant b such that

$$b\leq\min\left\{\inf_{x\in D}b_0(x),\lambda_1\delta^{p-r_0-1}\right\}. \tag{14}$$

Then, according to Lemma 3, there exists a (positive) solution u_b of

$$\begin{cases}-\Delta_p u+\lambda_1|u|^{p-2}u-\mu\Delta_q u+\mu\lambda_2|u|^{q-2}u=bu^{r_0} & \text{in } D,\\ u>0 & \text{in } D,\\ u=0 & \text{on } \partial D,\end{cases}$$

satisfying

$$\|u_b\|_{L^\infty(D)}\leq\left(\frac{b}{\lambda_1}\right)^{\frac{1}{p-r_0-1}}\leq\delta_0 \tag{15}$$

(note (14)). Let u_ε ($\varepsilon>0$) be a positive solution of problem (PD_ε) obtained by Lemma 2. Let us observe that $u_b/u_\varepsilon, u_\varepsilon/u_b\in L^\infty(D)$ because u_b and u_ε are positive functions belonging to $C_0^1(\overline{D})$ and satisfying $\partial u_i/\partial\nu<0$ on ∂D ($i=b$ and $i=\varepsilon$). On the basis of (15), we are able to apply Theorem 5 with $u_1=u_b$, $u_2=u_\varepsilon$, $g(t)=bt^{r_0}$, and $h(x,t,\xi)=f(x,t,\xi)+\varepsilon\psi$ because for any $0<t\leq\|u_b\|_{L^\infty(D)}$, we have that

$$h(x,t,\xi)=f(x,t,\xi)+\varepsilon\psi\geq f(x,t,\xi)\geq bt^{r_0}=g(t)$$

by (14) and ($\widetilde{\mathrm{F}}2$). In this way, we see that $u_\varepsilon\geq u_b$ in D for every $\varepsilon>0$.

Using the growth condition (8) of f, taking u_ε as a test function in (PD_ε), we obtain the inequality

$$\begin{aligned}
\|u_\varepsilon\|_D^p &\le \int_D \bigl(a_0 u_\varepsilon + a_1 u_\varepsilon^{r_1+1} + a_2|\nabla u_\varepsilon|^{r_2} u_\varepsilon\bigr)\,dx + \varepsilon\|\psi\|_{L^{p'}(D)}\|u_\varepsilon\|_{L^p(D)} \\
&\le \|a_0\|_{L^{p'}(D)}\|u_\varepsilon\|_{L^p(D)} + \|a_1\|_{L^{\tilde r_1}(D)}\|u_\varepsilon\|_{L^p(D)}^{r_1+1} \\
&\quad + \|a_2\|_{L^{\tilde r_2}(D)}\|\nabla u_\varepsilon\|_{L^p(D)}^{r_2}\|u_\varepsilon\|_{L^p(D)} + \varepsilon\|\psi\|_{L^{p'}(D)}\|u_\varepsilon\|_{L^p(D)} \\
&\le \lambda_1^{-1/p}\|a_0\|_{L^{p'}(D)}\|u_\varepsilon\|_D + \lambda_1^{-(r_1+1)/p}\|a_1\|_{L^{\tilde r_1}(D)}\|u_\varepsilon\|_D^{r_1+1} \\
&\quad + \lambda_1^{-1/p}\|a_2\|_{L^{\tilde r_2}(D)}\|u_\varepsilon\|_D^{r_2+1} + \lambda_1^{-1/p}\varepsilon\|\psi\|_{L^{p'}(D)}\|u_\varepsilon\|_D
\end{aligned}$$

for every $\varepsilon > 0$. This shows the boundedness of $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ in $W_0^{1,p}(D)$ because $p > r_1 + 1$, $r_2 + 1$ (note $0 < \varepsilon \le 1$). Thus, we can find a sequence $\varepsilon_n \to 0^+$ such that $u_n := u_{\varepsilon_n}$ is weakly convergent to some u in $W_0^{1,p}(D)$ and strongly in $L^r(D)$ (for all $r \in [1, p^*)$). On the other hand, taking $u_n - u$ as a test function, we easily see that

$$\begin{aligned}
U_n &:= \int_D \bigl(|\nabla u_n|^{p-2} + \mu|\nabla u_n|^{q-2}\bigr)\nabla u_n \nabla(u_n - u)\,dx \\
&\le -\lambda_1 \int_D u_n^{p-1}(u_n - u)\,dx - \mu\lambda_2 \int_D u_n^{q-1}(u_n - u)\,dx \\
&\quad + \int_D \bigl(a_0 + a_1 u_n^{r_1} + a_2|\nabla u_n|^{r_2}\bigr)(u_n - u)\,dx \\
&\quad + \varepsilon_n\|\psi\|_{L^{p'}(D)}\|u_n - u\|_{L^p(D)} \\
&\le \lambda_1\|u_n\|_{L^p(D)}^{p-1}\|u_n - u\|_{L^p(D)} + \mu\lambda_2\|u_n\|_{L^q(D)}^{q-1}\|u_n - u\|_{L^q(D)} \\
&\quad + \|a_0\|_{L^{p'}(D)}\|u_n - u\|_{L^p(D)} + \|a_1\|_{L^\infty(D)}\|u_n\|_{L^{p' r_1}(D)}^{r_1}\|u_n - u\|_{L^p(D)} \\
&\quad + \|a_2\|_{L^{\tilde r_2}(D)}\|\nabla u_n\|_{L^p(D)}^{r_2}\|u_n - u\|_{L^p(D)} + \varepsilon\|\psi\|_{L^{p'}(D)}\|u_n - u\|_{L^p(D)} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, $\limsup_{n\to\infty} U_n \le 0$. According to the (S_+) property of $-\Delta_p$, this ensures that u_n is strongly convergent to u in $W_0^{1,p}(D)$ (refer to (22)). Hence, u is a solution of (PD). Since we already know that $u_n \ge u_b$ in D for every n, in the limit, we obtain that $u \ge u_b$ in D. This completes the proof. □

4 Proof of Theorem 1

In this section, we denote $B_n := B_n(0)$ the open ball with center at the origin and radius n. The spaces $W^{1,p}(B_n)$ and $W^{1,q}(B_n)$ are equipped with the norms

$$\|u\|_{p,n}^p := \int_{B_n} \bigl(|\nabla u|^p + \lambda_1|u|^p\bigr)\,dx$$

and

$$\|u\|_{q,n}^q := \int_{B_n} \bigl(|\nabla u|^q + \lambda_2|u|^q\bigr)\,dx,$$

respectively.

Proof of Theorem 1 By applying Theorem 6 with $D = B_n$ ($n \in \mathbb{N}$) we obtain a (positive) solution $v_n \in C_0^1(\overline{B_n})$ of the problem

$$(\mathrm{P}_n) \qquad \begin{cases} -\Delta_p u + \lambda_1 |u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2 |u|^{q-2}u = f(x,u,\nabla u) & \text{in } B_n, \\ u > 0 & \text{in } B_n, \\ u(x) = 0 & \text{on } \partial B_n. \end{cases}$$

We claim that there exists a positive constant C such that

$$\|v_n\|_{p,n} \leq C \quad \text{for all } n \in \mathbb{N} \tag{16}$$

and

$$\|v_n\|_{q,n} \leq C \quad \text{for all } n \in \mathbb{N}, \text{provided that } \mu > 0. \tag{17}$$

Indeed, acting with v_n in (P_n) as a test function, through assumption (F1) and Hölder's and Young's inequalities we obtain

$$\begin{aligned} \|v_n\|_{p,n}^p + \mu\|v_n\|_{q,n}^q &= \int_{B_n} f(x,v_n,\nabla v_n)v_n\,dx \\ &\leq \|a_0\|_{L^{p'}(B_n)}\|v_n\|_{L^p(B_n)} + \|a_1\|_{L^{\tilde{r}_1}(B_n)}\|v_n\|_{L^p(B_n)}^{r_1+1} \\ &\quad + \|a_2\|_{L^{\tilde{r}_2}(B_n)}\|v_n\|_{L^p(B_n)}\|\nabla v_n\|_{L^p(B_n)}^{r_2} \\ &\leq \frac{\lambda_1}{2}\|v_n\|_{L^p(B_n)}^p + \frac{1}{2}\|\nabla v_n\|_{L^p(B_n)}^p \\ &\quad + C\big(\|a_0\|_{L^{p'}(B_n)}^{p'} + \|a_1\|_{L^{\tilde{r}_1}(B_n)}^{\tilde{r}_1} + \|a_2\|_{L^{\tilde{r}_2}(B_n)}^{\tilde{r}_2}\big), \end{aligned}$$

where C is a positive constant independent of n. It turns out that

$$\begin{aligned} \frac{1}{2}\|v_n\|_{p,n}^p + \mu\|v_n\|_{q,n}^q &\leq C\big(\|a_0\|_{L^{p'}(B_n)}^{p'} + \|a_1\|_{L^{\tilde{r}_1}(B_n)}^{\tilde{r}_1} + \|a_2\|_{L^{\tilde{r}_2}(B_n)}^{\tilde{r}_2}\big) \\ &\leq C\big(\|a_0\|_{L^{p'}(\mathbb{R}^N)}^{p'} + \|a_1\|_{L^{\tilde{r}_1}(\mathbb{R}^N)}^{\tilde{r}_1} + \|a_2\|_{L^{\tilde{r}_2}(\mathbb{R}^N)}^{\tilde{r}_2}\big), \end{aligned}$$

whence (16) and (17) follow.

Fix $m \in \mathbb{N}$. If $n \geq m+1$, then by (16) we have

$$\|v_n\|_{p,m+1} \leq \|v_n\|_{p,n} \leq C. \tag{18}$$

Therefore, there exists $v \in W^{1,p}(B_{m+1})$ such that

$$v_n \rightharpoonup v \quad \text{in } W^{1,p}(B_{m+1}), W^{1,q}(B_{m+1}), \tag{19}$$

$$v_n \to v \quad \text{in } L^p(B_{m+1}), L^q(B_{m+1}), \tag{20}$$

$$v_n(x) \to v(x) \quad \text{for a.e. } x \in B_{m+1} \tag{21}$$

as $n \to \infty$.

Let us show that v_n converges to v strongly in $W^{1,p}(B_m)$ and $W^{1,q}(B_m)$. To this end, fix $l \in \mathbb{N}$ and choose a smooth function ψ_l satisfying $0 \leq \psi_l \leq 1$, $\psi_l(r) = 1$ if $r \leq m$, and $\psi_l(r) = 0$ if $r \geq m + 1/l$. Setting $\eta_l(x) := \psi_l(|x|)$, we note that $(v_n - v)\eta_l \in W_0^{1,p}(B_{m+1}) \subset W_0^{1,p}(B_n)$ for any $n \geq m + 1$. Denote

$$V_n = \int_{B_m} \left(|\nabla v_n|^{p-2} + \mu|\nabla v_n|^{q-2}\right)\nabla v_n(\nabla v_n - \nabla v)\,dx.$$

Using $(v_n - v)\eta_l$ as a test function in (P_n) and invoking the growth condition (F1), we obtain

$$\begin{aligned}
V_n &= \int_{|x|<m+1/l} \left(f(x, v_n, \nabla v_n) - \lambda_1 v_n^{p-1} - \mu\lambda_2 v_n^{q-1}\right)(v_n - v)\eta_l\,dx \\
&\quad - \int_{m\leq|x|<m+1/l} \left(|\nabla v_n|^{p-2} + \mu|\nabla v_n|^{q-2}\right)\nabla v_n \nabla(v_n - v))\eta_l\,dx \\
&\quad - \int_{m\leq|x|<m+1/l} \left(|\nabla v_n|^{p-2} + \mu|\nabla v_n|^{q-2}\right)\nabla v_n \nabla\eta_l(v_n - v)\,dx \\
&\leq \int_{B_{m+1}} \left(a_0(x) + a_1(x)v_n^{r_1} + a_2(x)|\nabla v_n|^{r_2} + \lambda_1 v_n^{p-1} + \mu\lambda_2 v_n^{q-1}\right)|v_n - v|\,dx \\
&\quad + \int_{m\leq|x|<m+1/l} \left(|\nabla v_n|^{p-1} + \mu|\nabla v_n|^{q-1}\right)|\nabla v|\,dx \\
&\quad + d_l \int_{m\leq|x|<m+1/l} \left(|\nabla v_n|^{p-1} + \mu|\nabla v_n|^{q-1}\right)|v_n - v|\,dx \\
&\equiv I_n^1 + I_n^2 + I_n^3,
\end{aligned}$$

where $d_l := \sup_{|x|<m+1/l} |\nabla\eta_l(x)|$.

By Hölder's inequality, (16) and (17), we have

$$\begin{aligned}
I_n^1 &\leq \|v_n - v\|_{L^p(B_{m+1})}\Big\{\|a_0\|_{L^{p'}(B_{m+1})} + \|a_1\|_{L^{\tilde{r}_1}(B_{m+1})}\|v_n\|_{L^p(B_{m+1})}^{r_1} \\
&\quad + \|a_2\|_{L^{\tilde{r}_2}(B_{m+1})}\|\nabla v_n\|_{L^p(B_{m+1})}^{r_2} + \lambda_1\|v_n\|_{L^p(B_{m+1})}^{p-1} + \mu\lambda_2\|v_n\|_{L^{p'(q-1)}(B_{m+1})}^{q-1}\Big\} \\
&\leq C_1\|v_n - v\|_{L^p(B_{m+1})},
\end{aligned}$$

where C_1 is a positive constant independent of v_n, n, m, and l. Again by Hölder's inequality the following estimates follow:

$$\begin{aligned}
I_n^2 &\leq \|\nabla v_n\|_{L^p(B_{m+1})}^{p-1}\left(\int_{m\leq|x|<m+1/l} |\nabla v|^p\,dx\right)^{1/p} \\
&\quad + \mu\|\nabla v_n\|_{L^q(B_{m+1})}^{q-1}\left(\int_{m\leq|x|<m+1/l} |\nabla v|^q\,dx\right)^{1/q}, \\
I_n^3 &\leq d_l\|v_n - v\|_{L^p(B_{m+1})}\|\nabla v_n\|_{L^p(B_{m+1})}^{p-1} + d_l\mu\|v_n - v\|_{L^q(B_{m+1})}\|\nabla v_n\|_{L^q(B_{m+1})}^{q-1}.
\end{aligned}$$

Thereby, from (16), (17), and (20) we derive

$$\begin{aligned}
&\limsup_{n\to\infty} V_n \\
&\quad \leq C^{p-1}\left(\int_{m\leq|x|<m+1/l} |\nabla v|^p\,dx\right)^{1/p} + \mu C^{q-1}\left(\int_{m\leq|x|<m+1/l} |\nabla v|^q\,dx\right)^{1/q}
\end{aligned}$$

for all $l \in \mathbb{N}$. Thus, letting $l \to \infty$, we obtain that $\limsup_{n\to\infty} V_n \leq 0$. As known from (19), v_n weakly converges to v in $W^{1,p}(B_m)$ and $W^{1,q}(B_m)$, so we may write

$$\begin{aligned} V_n + o(1) &= \int_{B_m} \bigl(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\bigr)(\nabla v_n - \nabla v)\,dx \\ &\quad + \mu \int_{B_m} \bigl(|\nabla v_n|^{q-2}\nabla v_n - |\nabla v|^{q-2}\nabla v\bigr)(\nabla v_n - \nabla v)\,dx \\ &\geq \bigl(\|\nabla v_n\|^{p-1}_{L^p(B_m)} - \|\nabla v\|^{p-1}_{L^p(B_m)}\bigr)\bigl(\|\nabla v_n\|_{L^p(B_m)} - \|\nabla v\|_{L^p(B_m)}\bigr) \\ &\quad + \mu\bigl(\|\nabla v_n\|^{p-1}_{L^q(B_m)} - \|\nabla v\|^{q-1}_{L^q(B_m)}\bigr)\bigl(\|\nabla v_n\|_{L^q(B_m)} - \|\nabla v\|_{L^q(B_m)}\bigr) \\ &\geq 0. \end{aligned} \tag{22}$$

What we have shown entails $\lim_{n\to\infty} V_n = 0$, $\lim_{n\to\infty} \|\nabla v_n\|_{L^p(B_m)} = \|\nabla v\|_{L^p(B_m)}$ and $\lim_{n\to\infty} \|\nabla v_n\|_{L^q(B_m)} = \|\nabla v\|_{L^q(B_m)}$ if $\mu > 0$. This implies that v_n converges to v strongly in $W^{1,p}(B_m)$ and $W^{1,q}(B_m)$ because the spaces $W^{1,p}(B_m)$ and $W^{1,q}(B_m)$ are uniformly convex.

Recalling that $v_n > 0$ in B_m, we infer that v is a nonnegative solution of the problem

$$-\Delta_p v + \lambda_1|v|^{p-2}v - \mu\Delta_q v + \mu\lambda_2|v|^{q-2}v = f(x, v, \nabla v) \quad \text{in } B_m, v \geq 0 \text{ on } \partial B_m.$$

Now, by a diagonal argument and (21) there exist a relabeled subsequence of $\{v_n\}$ and a function $v \in W^{1,p}(\mathbb{R}^N)$ such that

$$\begin{aligned} &v_n \to v \quad \text{in } W^{1,p}_{\text{loc}}\bigl(\mathbb{R}^N\bigr), \\ &v_n(x) \to v(x) \quad \text{for a.e } x \in \mathbb{R}^N. \end{aligned}$$

These convergence properties ensure that v is a solution of problem (P).

The next step in the proof is to show that v does not vanish in Ω. To do this, we fix $m \in \mathbb{N}$ and a positive constant b_m satisfying $b_m \leq \inf_{x\in B_m} b_0(x)$, where the function b_0 appears in assumption (F2). Moreover, choosing b_m even smaller, Lemma 3 provides a solution u_m of the problem

$$\begin{cases} -\Delta_p u + \lambda_1|u|^{p-2}u - \mu\Delta_q u + \mu\lambda_2|u|^{q-2}u = b_m u^{r_0} & \text{in } B_m, \\ u > 0 & \text{in } B_m, \\ u(x) = 0 & \text{on } \partial B_m \end{cases}$$

such that $\|u_m\|_{L^\infty(B_m)} \leq \delta_0$, where δ_0 is given in assumption (F2). It follows from hypothesis (F2) that if $t \leq \|u_m\|_{L^\infty(B_m)}$, then $f(x,t,\xi) \geq b_0(x)t^{r_0}$ for all $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$. We are thus in a position to apply Theorem 4 to the functions u_m and v_n with $n > m$ in place of $u_1 = u_m$ and $u_2 = v_n$, respectively, which renders $v_n \geq u_m$ in B_m for every $n > m$. This enables us to deduce that $v \geq u_m$ in B_m, so $v(x) > 0$ for almost every $x \in \mathbb{R}^N$ because m was arbitrary.

Furthermore, since $\lambda_1 > 0$ and v_n weakly converges to v in $W^{1,p}(\mathbb{R}^N)$ (we can extend $v_n(x) = 0$ if $|x| \geq n$ (so $\|v_n\|_{p,n} = \|\nabla v_n\|_{L^p(\mathbb{R}^N)} + \lambda_1\|v_n\|_{L^p(\mathbb{R}^N)}$)), by means of (16) and (17), we can check that $v \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ if $\mu > 0$ and $v \in W^{1,p}(\mathbb{R}^N)$ if $\mu = 0$. According to the iteration process, it is proved that v is bounded on any bounded sets (see Section 5.1). Hence, the regularity theory as in [7] leads to $v \in C^1_{\text{loc}}(\mathbb{R}^N)$. The proof of Theorem 1 is complete. □

5 Proof of Theorem 2

Throughout this section, we fix any (positive) solution v of (P) (belonging to $W^{1,p}(\mathbb{R}^N)$). Define $v_M := \max\{v, M\}$ for $M > 0$. Here, we choose $\overline{p}^*$ satisfying $p^2 < \overline{p}^*$ if $N \leq p$ and set $\overline{p}^* = p^* = Np/(N-p)$ if $N > p$. For $R' > R > 0$, we take a smooth function $\eta_{R,R'}$ such that $0 \leq \eta_{R,R'} \leq 1$, $\|\eta'_{R,R'}\|_\infty \leq 2/(R'-R)$, $\eta_{R,R'}(t) = 1$ if $t \leq R$, and $\eta_{R,R'} = 0$ if $t \geq R'$.

5.1 Boundedness of solutions

Lemma 4 *Let $x_0 \in \mathbb{R}^N$, $M > 0$, $R' > R > 0$, $\tilde{p} > 1$, $\gamma_i > 1$, and $1/\gamma_i + 1/\gamma_i' = 1$ $(i = 0,1)$. Denote $\eta(x) := \eta_{R,R'}(|x - x_0|)$. Assume that $\gamma_i' \leq \tilde{p}$ $(i = 0,1)$ and $v \in L^{\tilde{p}(p+\alpha)}(B(x_0,R'))$ with $\alpha \geq 0$. Then*:

$$\int_{B(x_0,R')} a_0 v v_M^\alpha \eta^p \, dx \leq \|a_0\|_{L^{\gamma_0}(B(x_0,R'))} \|v\|^{1+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'}, \tag{23}$$

$$\int_{B(x_0,R')} a_1 v^{r_1+1} v_M^\alpha \eta^p \, dx \leq \|a_1\|_{L^{\gamma_1}(B(x_0,R'))} \|v\|^{r_1+1+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'}, \tag{24}$$

where $B_{R'} := (1 + |B(0,R')|)$, and $|B(0,R')|$ denotes the Lebesgue measure of the ball $B(0,R')$. Moreover, if $\gamma_2 > p/(p - r_2)$ and $\gamma_3 := \frac{(p-r_2)\gamma_2}{(p-r_2)\gamma_2 - p} \leq \tilde{p}$, then

$$\begin{aligned}\int_{B(x_0,R')} a_2 |\nabla v|^{r_2} v v_M^\alpha \eta^p \, dx &\leq \frac{1}{4} \int_{B(x_0,R')} |\nabla v|^p v_M^\alpha \eta^p \, dx \\ &\quad + 4^{\frac{r_2}{p-r_2}} \|a_2\|^{\frac{p}{p-r_2}}_{L^{\gamma_2}(B(x_0,R'))} \|v\|^{\frac{p}{p-r_2}+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'}.\end{aligned} \tag{25}$$

Proof According to Hölder's inequality, we easily show our assertions (23) and (24). So, we prove (25) only. By Young's inequality and recalling that $r_2 < p - 1$ and $\eta^p \geq \eta^{p^2/r_2}$, we have

$$\begin{aligned}\int_{B(x_0,R')} a_2 |\nabla v|^{r_2} v v_M^\alpha \eta^p \, dx &\leq \frac{1}{4} \int_{B(x_0,R')} |\nabla v|^p v_M^\alpha \eta^p \, dx \\ &\quad + 4^{\frac{r_2}{p-r_2}} \int_{B(x_0,R')} a_2^{\frac{p}{p-r_2}} v^{\frac{p}{p-r_2}} v_M^\alpha \, dx.\end{aligned}$$

Moreover, because of $\gamma_2 > p/(p-r_2)$, $p > p/(p-r_2)$, and $\tilde{p} \geq \gamma_3$, applying Hölder's inequality, we obtain

$$\begin{aligned}\int_{B(x_0,R')} a_2^{\frac{p}{p-r_2}} v^{\frac{p}{p-r_2}} v_M^\alpha \, dx &\leq \|a_2\|^{\frac{p}{p-r_2}}_{L^{\gamma_2}(B(x_0,R'))} \|v\|^{\frac{p}{p-r_2}+\alpha}_{L^{\gamma_3(\frac{p}{p-r_2}+\alpha)}(B(x_0,R'))} \\ &\leq \|a_2\|^{\frac{p}{p-r_2}}_{L^{\gamma_2}(B(x_0,R'))} \|v\|^{\frac{p}{p-r_2}+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} \big(1 + \big|B\big(0,R'\big)\big|\big).\end{aligned}$$

Hence, (25) follows. □

Lemma 5 *Let $x_0 \in \mathbb{R}^N$, $R' > R > 0$, $\tilde{p} > 1$, $\gamma_i > 1$, and $1/\gamma_i + 1/\gamma_i' = 1$ $(i = 0,1)$. Assume that $\gamma_i' \leq \tilde{p}$ $(i = 0,1)$, $\gamma_2 > p/(p - r_2)$, and $\gamma_3 := \frac{(p-r_2)\gamma_2}{(p-r_2)\gamma_2 - p} \leq \tilde{p}$. If $v \in L^{\tilde{p}(p+\alpha)}(B(x_0,R'))$ with $\alpha \geq 0$, then*

$$\|v\|^{p+\alpha}_{L^{\frac{\overline{p}^*}{p}(p+\alpha)}(B(x_0,R))} \leq 2^p (p+\alpha)^p C_*^p B_{R'} (C_{R'} + D_{R,R'}) \max\big\{1, \|v\|_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}\big\}^{p+\alpha} \tag{26}$$

with

$$B_{R'} := \left(1 + \left|B\left(0, R'\right)\right|\right),$$

$$C_{R'} := \left\{ \|a_0\|_{L^{\gamma_0}(B(x_0,R'))} + \|a_1\|_{L^{\gamma_1}(B(x_0,R'))} + 4^{\frac{r_2}{p-r_2}} \|a_2\|^{\frac{p}{p-r_2}}_{L^{\gamma_2}(B(x_0,R'))} \right\},$$

$$D_{R,R'} := \left\{ \frac{2^{3p-2}p^p + 2^{p-1}}{(R'-R)^p} + \frac{\mu 2^{3q-2}p^q}{(R'-R)^q} \right\},$$

where C_* is the positive constant from embedding from $W^{1,p}(\mathbb{R}^N)$ to $L^{\overline{p}^*}(\mathbb{R}^N)$.

Proof Taking $v v_M^{\alpha} \eta^p \in W_0^{1,p}(B(x_0, R'))$ (for $M > 0$) as a test function, where $\eta(x) = \eta_{R,R'}(|x - x_0|)$, by Lemma 4 and (F1) we obtain

$$\begin{aligned}
&\|a_0\|_{L^{\gamma_0}(B(x_0,R'))} \|v\|^{1+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'} \\
&\quad + \|a_1\|_{L^{\gamma_1}(B(x_0,R'))} \|v\|^{r_1+1+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'} \\
&\quad + 4^{\frac{r_2}{p-r_2}} \|a_2\|_{L^{\gamma_2}(B(x_0,R'))} \|v\|^{\frac{p}{p-r_2}+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'} + \frac{1}{4} \int_{B(x_0,R')} |\nabla v|^p v_M^{\alpha} \eta^p \, dx \\
&\geq \int_{B(x_0,R')} |\nabla v|^p v_M^{\alpha} \eta^p \, dx + \lambda_1 \int_{B(x_0,R')} v_M^{p+\alpha} \eta^p \, dx - \frac{2p}{R'-R} \int_{B(x_0,R')} |\nabla v|^{p-1} v_M^{\alpha} v \eta^{p-1} \, dx \\
&\quad + \mu \left\{ \int_{B(x_0,R')} |\nabla v|^q v_M^{\alpha} \eta^p \, dx - \frac{2p}{R'-R} \int_{B(x_0,R')} |\nabla v|^{q-1} v_M^{\alpha} v \eta^{p-1} \, dx \right\},
\end{aligned} \tag{27}$$

where we use $|\nabla \eta| \leq 2/(R' - R)$. According to Young's and Hölder's inequalities, for $j = p, q$, we see that

$$\begin{aligned}
&\frac{2p}{R'-R} \int_{B(x_0,R')} |\nabla v|^{j-1} v_M^{\alpha} v \eta^{p-1} \, dx \\
&\leq \frac{1}{4} \int_{B(x_0,R')} |\nabla v|^j v_M^{\alpha} \eta^p \, dx + \frac{2^j p^j 4^{j-1}}{(R'-R)^j} \int_{B(x_0,R')} v^{j+\alpha} \eta^{p-j} \, dx \\
&\leq \frac{1}{4} \int_{B(x_0,R')} |\nabla v|^j v_M^{\alpha} \eta^p \, dx + \frac{2^{3j-2} p^j}{(R'-R)^j} \|v\|^{j+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} B_{R'}.
\end{aligned} \tag{28}$$

Consequently, because of $\mu \geq 0$ and $p + \alpha > r_1 + 1 + \alpha, p/(p - r_2) + \alpha$, it follows from (27) and (28) that

$$\begin{aligned}
&B_{R'} \left(C_{R'} + \frac{2^{3p-2}p^p}{(R'-R)^p} + \frac{\mu 2^{3q-2}p^q}{(R'-R)^q} \right) \max\left\{1, \|v\|_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}\right\}^{p+\alpha} \\
&\geq \frac{1}{2} \int_{B(x_0,R')} |\nabla v|^p v_M^{\alpha} \eta^p \, dx + \lambda_1 \int_{B(x_0,R')} v_M^{p+\alpha} \eta^p \, dx.
\end{aligned} \tag{29}$$

Moreover, by using

$$\begin{aligned}
&\left\| \nabla \left(v_M^{1+\alpha/p} \eta \right) \right\|^p_{L^p(\mathbb{R}^N)} \\
&\leq 2^{p-1} \left\{ \left\| \eta \nabla \left(v_M^{1+\alpha/p} \right) \right\|^p_{L^p(\mathbb{R}^N)} + \left\| v_M^{1+\alpha/p} \nabla \eta \right\|^p_{L^p(\mathbb{R}^N)} \right\} \\
&\leq 2^{p-1} \left(1 + \frac{\alpha}{p} \right)^p \int_{B(x_0,R')} |\nabla v|^p v_M^{\alpha} \eta^p \, dx + \frac{2^{2p-1}}{(R'-R)^p} \int_{B(x_0,R')} v_M^{p+\alpha} \, dx
\end{aligned}$$

and Hölder's inequality, due to the embedding from $W^{1,p}(\mathbb{R}^N)$ to $L^{\overline{p}^*}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_0,R')} |\nabla v|^p v_M^{\alpha}\eta^p\,dx + \lambda_1 \int_{B(x_0,R')} v_M^{p+\alpha}\eta^p\,dx \\
&\quad\geq 2^{-p}p^p(p+\alpha)^{-p}\bigl\{ \bigl\| \nabla\bigl(v_M^{1+\alpha/p}\eta\bigr)\bigr\|_{L^p(\mathbb{R}^N)}^p + \lambda_1 \bigl\| v_M^{1+\alpha/p}\eta\bigr\|_{L^p(\mathbb{R}^N)}^p \bigr\} \\
&\qquad - \frac{2^{p-1}p^p}{(p+\alpha)^p(R'-R)^p}\int_{B(x_0,R')} v_M^{p+\alpha}\,dx \\
&\quad\geq 2^{-p}p^p(p+\alpha)^{-p}\bigl\| v_M^{1+\alpha/p}\eta\bigr\|_{W^{1,p}(\mathbb{R}^N)}^p \\
&\qquad - \frac{2^{p-1}}{(R'-R)^p}\|v\|_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}^{p+\alpha}\bigl(1+\bigl|B\bigl(0,R'\bigr)\bigr|\bigr) \\
&\quad\geq 2^{-p}p^p(p+\alpha)^{-p}C_*^{-p}\bigl\| v_M^{1+\alpha/p}\eta\bigr\|_{L^{\overline{p}^*}(\mathbb{R}^N)}^p - \frac{2^{p-1}}{(R'-R)^p}\|v\|_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}^{p+\alpha}B_{R'} \\
&\quad\geq 2^{-p}p^p(p+\alpha)^{-p}C_*^{-p}\|v_M\|_{L^{\overline{p}^*(p+\alpha)/p}(B(x_0,R))}^{p+\alpha} \\
&\qquad - \frac{2^{p-1}}{(R'-R)^p}\|v\|_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}^{p+\alpha}B_{R'}.
\end{aligned} \tag{30}$$

Therefore, (29) and (30) lead to

$$\begin{aligned}
&2^{-p}p^p(p+\alpha)^{-p}C_*^{-p}\|v_M\|_{L^{\overline{p}^*(p+\alpha)/p}(B(x_0,R))}^{p+\alpha} \\
&\quad\leq B_{R'}(C_{R'}+D_{R,R'})\max\bigl\{1, \|v\|_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}\bigr\}^{p+\alpha}.
\end{aligned} \tag{31}$$

Applying Fatou's lemma and letting $M \to \infty$ in (31), our conclusion follows. □

Proposition 1 *Under the assumptions in Theorem* 2, *we have that* $v \in L^\infty(\mathbb{R}^N)$.

Proof First, in the case of $N > p$, we note that

$$\begin{aligned}
&\gamma_j' < \frac{p^*}{p} \quad\Longleftrightarrow\quad \gamma_j > \frac{p^*}{p^*-p} \quad (j=0,1), \\
&\gamma_2 > \frac{p}{p-r_2} \quad\text{and}\quad \gamma_3 := \frac{(p-r_2)\gamma_2}{(p-r_2)\gamma_2 - p} < \frac{p^*}{p} \quad\Longleftrightarrow\quad \gamma_2 > \frac{pp^*}{(p-r_2)(p^*-p)}.
\end{aligned}$$

In the cases of (i) and (ii) (case $p < \overline{p}^*/p$), we take $\gamma_0 = p'$ and $\gamma_j = \tilde{r}_j$ $(j = 1,2)$. Then, we have $\gamma_0' = p$, $\gamma_1' = \tilde{r}_1' = p/(r_1+1) \leq p$, $\gamma_2 = \tilde{r}_2 = p/(p-r_2-1) > p/(p-r_2)$, and $\gamma_3 = (p-r_2)\tilde{r}_2/((p-r_2)\tilde{r}_2 - p) = p - r_2 \leq p$. Choose $\tilde{p}$ such that

$$\begin{aligned}
&\bigl(\max\bigl\{\gamma_0',\gamma_1',\gamma_3\bigr\} =\bigr)\tilde{p} = p\left(< \frac{\overline{p}^*}{p}\right) \text{ in the cases of (i) and (ii)}, \\
&\max\bigl\{\gamma_0',\gamma_1',\gamma_3\bigr\} \leq \tilde{p} < \frac{p^*}{p} \text{ in the case of (iii)}.
\end{aligned}$$

Let R_* be the positive constant satisfying (4) in the case of (iii) and any positive constant in the cases of (i) and (ii). Put

$$A_i := \begin{cases} \|a_i\|_{L^{\gamma_i}(\mathbb{R}^N)} & \text{if (i) and (ii)}, \\ \sup_{x\in\mathbb{R}^N}\|a_i\|_{L^{\gamma_i}(B(x,2R_*))} & \text{if (iii)} \end{cases}$$

for $i = 0, 1, 2$. Define the sequences $\{\alpha_n\}$, $\{R'_n\}$, and $\{R_n\}$ by

$$\alpha_0 := \frac{\overline{p}^*}{\tilde{p}} - p > 0, \qquad \tilde{p}(p + \alpha_{n+1}) = \frac{\overline{p}^*}{p}(p + \alpha_n),$$
$$R'_n := \left(1 + 2^{-n}\right)R_*, \qquad R_n := R'_{n+1}.$$

Recall that $v \in W^{1,p}(\mathbb{R}^N)$, and using the embedding of $W^{1,p}(\mathbb{R}^N)$ to $L^{\overline{p}^*}(\mathbb{R}^N)$, we see that $v \in L^{\overline{p}^*}(\mathbb{R}^N) = L^{\tilde{p}(p+\alpha_0)}(\mathbb{R}^N)$.

Fix any $x_0 \in \mathbb{R}^N$. Then Lemma 5 guarantees that if $v \in L^{\tilde{p}(p+\alpha_n)}(B(x_0, R'_n))$, then $v \in L^{\frac{\overline{p}^*}{p}(p+\alpha_n)}(B(x_0, R_n)) = L^{\tilde{p}(p+\alpha_{n+1})}(B(x_0, R'_{n+1}))$. Noting that

$$B_{R'_n} \leq \left(1 + \left|B(0, 2R_*)\right|\right) =: B_0,$$
$$C_{R'_n} \leq A_0 + A_1 + 4^{\frac{r_2}{p-r_2}} A_2^{\frac{p}{p-r_2}} + 1 =: C_0,$$
$$D_{R_n, R'_n} \leq \frac{(1 + p^p)2^{p(n+3)}}{R_*^p} + \frac{\mu q^q 2^{q(n+3)}}{R_*^q} =: D_n \leq C' 2^{p(n+3)}$$

for any $n \geq 0$ with sufficiently large C' independent of n and setting

$$b_n := \max\left\{1, \|v\|_{L^{\tilde{p}(p+\alpha_n)}(B(x_0, R'_n))}\right\},$$

by Lemma 5 we obtain

$$b_{n+1} \leq C^{\frac{1}{p+\alpha_n}} (p + \alpha_n)^{\frac{p}{p+\alpha_n}} (C_0 + D_n)^{\frac{1}{p+\alpha_n}} b_n \tag{32}$$

for every $n \geq 0$ with $C := 2^p(C_* + 1)^p B_0$. Put $P := \tilde{p}p/\overline{p}^* < 1$. Then, because of $p + \alpha_{n+1} = (p + \alpha_n)/P$, $\alpha_{n+1} > \alpha_n/P > \alpha_0(1/P)^{n+1} \to \infty$ as $n \to \infty$. Moreover, we see that

$$S_1 := \sum_{n=0}^{\infty} \frac{1}{p + \alpha_n} = \frac{1}{p + \alpha_0} \sum_{n=0}^{\infty} P^n = \frac{1}{(p + \alpha_0)(1 - P)} < \infty,$$
$$S_2 := \ln \prod_{n=0}^{\infty} (p + \alpha_n)^{\frac{p}{p+\alpha_n}} = \frac{p}{p + \alpha_0} \sum_{n=0}^{\infty} P^n \left(\ln(p + \alpha_0) + n \ln P^{-1}\right) < \infty,$$

and

$$S_3 := \ln \prod_{n=0}^{\infty} (C_0 + D_n)^{\frac{1}{p+\alpha_n}} = \sum_{n=0}^{\infty} \frac{P^n}{p + \alpha_0} \ln(C_0 + D_n)$$
$$\leq \sum_{n=0}^{\infty} \frac{P^n}{p + \alpha_0} p(n + 3) \ln\left(C_0 + C'\right) 2 < \infty.$$

As a result, by iteration in (32) and the equality $\tilde{p}(p + \alpha_0) = \overline{p}^*$ we obtain

$$\|v\|_{L^{\frac{\overline{p}^*}{p}(p+\alpha_n)}(B(x_0, R_*))} \leq b_n \leq C^{S_1} e^{S_2} e^{S_3} \max\left\{1, \|v\|_{L^{\overline{p}^*}(B(x_0, 2R_*))}\right\}$$

for every $n \geq 1$. Letting $n \to \infty$, this ensures that

$$\|v\|_{L^\infty(B(x_0, R_*))} \leq C^{S_1} e^{S_2} e^{S_3} \max\left\{1, \|v\|_{L^{\overline{p}^*}(B(x_0, 2R_*))}\right\}. \tag{33}$$

Recalling that $v \in W^{1,p}(\mathbb{R}^N)$ and using the embedding of $W^{1,p}(\mathbb{R}^N)$ to $L^{\overline{p}^*}(\mathbb{R}^N)$, (33) yields that

$$\begin{aligned}\|v\|_{L^\infty(B(x_0,R_*))} &\le C^{S_1}e^{S_2}e^{S_3}\max\bigl\{1,\|v\|_{L^{\overline{p}^*}(\mathbb{R}^N)}\bigr\}\\ &\le C^{S_1}e^{S_2}e^{S_3}\max\bigl\{1,C_*\|v\|_{W^{1,p}(\mathbb{R}^N)}\bigr\},\end{aligned}$$

whence v is bounded in $\mathbb{R}^N$ because $x_0 \in \mathbb{R}^N$ is arbitrary and the constant $C^{S_1}e^{pS_2}e^{S_3}$ is independent of x_0. □

5.2 Proof of Theorem 2

Proof of Theorem 2 Since v is bounded in $\mathbb{R}^N$ by Proposition 1, we put $M_0 := \|v\|_{L^\infty(\mathbb{R}^N)}$. Then, as in Lemma 4, we see that

$$\int_{B(x_0,R')} a_0 v v_M^\alpha \eta^p\,dx \le \|a_0\|_{L^{\gamma_0}(B(x_0,R'))}M_0\|v\|^\alpha_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}B_{R'}, \tag{34}$$

$$\int_{B(x_0,R')} a_1 v^{r_1+1} v_M^\alpha \eta^p\,dx \le \|a_1\|_{L^{\gamma_1}(B(x_0,R'))}M_0^{1+r_1}\|v\|^\alpha_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}B_{R'}, \tag{35}$$

$$\begin{aligned}&\int_{B(x_0,R')} a_2|\nabla v|^{r_2} v v_M^\alpha \eta^p\,dx\\ &\quad\le \frac{1}{4}\int_{B(x_0,R')}|\nabla v|^p v_M^\alpha\eta^p\,dx + 4^{\frac{r_2}{p-r_2}}\|a_2\|^{\frac{p}{p-r_2}}_{L^{\gamma_2}(B(x_0,R'))}M_0^{\frac{p}{p-r_2}}\|v\|^\alpha_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}B_{R'},\end{aligned} \tag{36}$$

and

$$\|v\|^{j+\alpha}_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} \le M_0^j\|v\|^\alpha_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))} \quad (j = p,q). \tag{37}$$

Fix any $x_0 \in \mathbb{R}^N$. It follows from the argument as in the proof of Lemma 5 with (34), (35), (36), and (37) that

$$\|v\|^{p+\alpha}_{L^{\frac{\overline{p}^*}{p}(p+\alpha)}(B(x_0,R))} \le 2^p(p+\alpha)^pC_*^pB_{R'}(C_{R'}+D_{R,R'})(M_0+1)^p\|v\|^\alpha_{L^{\tilde{p}(p+\alpha)}(B(x_0,R'))}, \tag{38}$$

provided that $v \in L^{\tilde{p}(p+\alpha)}(B(x_0,R'))$. Choose γ_i ($i = 0,1,2$) and $\tilde{p}$ and define the sequences $\{\alpha_n\}$, $\{R'_n\}$, and $\{R_n\}$ as in the proof of Proposition 1. Set

$$V_n := \|v\|^{\alpha_n}_{L^{\tilde{p}(p+\alpha_n)}(B(x_0,R'_n))}.$$

Then, by the same argument as in the proof of Proposition 1 with (38) we obtain

$$V_n^{\frac{p+\alpha_{n-1}}{\alpha_n}} \le C(p+\alpha_{n-1})^p(C_0+D_{n-1})V_{n-1} \tag{39}$$

with $C := 2^pC_*^pB_0(M_0+1)^p$. Recall that

$$\alpha_n + p = P^{-1}(p+\alpha_{n-1}) \quad\text{and}\quad \frac{p}{p+\alpha_0} = P.$$

Define

$$Q_n := \prod_{k=2}^{n+1}\left(1+\frac{P^k}{1-P^k}\right) = \prod_{k=2}^{n+1}\left(1-P^k\right)^{-1} \quad \text{and} \quad W_n := (C_0 + D_n).$$

Then, inequality (39) leads to

$$\begin{aligned}
\ln V_n &\le \frac{\alpha_n}{p+\alpha_{n-1}}\left(\ln V_{n-1} + \ln C(p+\alpha_{n-1})^p + \ln W_{n-1}\right)\\
&= P^{-1}\left(1-P^{n+1}\right)\left(\ln V_{n-1} + p\ln CP^{-n+1}(p+\alpha_0) + \ln W_{n-1}\right)\\
&\le P^{-1}\left(1-P^{n+1}\right)\ln V_{n-1} + pP^{-1}\ln(C+1)P^{-n+1}(p+\alpha_0) + P^{-1}\ln W_{n-1}\\
&\le P^{-n}\left(\prod_{k=1}^{n}\left(1-P^{k+1}\right)\right)\ln V_0 + p\sum_{k=1}^{n}P^{-k}\ln(C+1)P^{-n+k}(p+\alpha_0)\\
&\quad + \sum_{k=1}^{n}P^{-k}\ln W_{n-k}\\
&= P^{-n}Q_n^{-1}\ln V_0 + p\sum_{k=1}^{n}P^{-k}\ln(C+1)P^{-n+k}(p+\alpha_0) + \sum_{k=1}^{n}P^{-k}\ln W_{n-k}
\end{aligned}$$

for every n because of $\ln(C+1)P^{-n+1}(p+\alpha_0) > 0$ and $\ln W_n > 0$ for all n. Therefore, we have

$$\begin{aligned}
&\ln \|v\|_{L^{\tilde{p}(p+\alpha_n)}(B(x_0,R_n'))}\\
&\quad = \frac{\ln V_n}{\alpha_n} = \frac{P^n \ln V_n}{p+\alpha_0 - pP^n}\\
&\quad \le \frac{Q_n^{-1}\ln V_0}{p+\alpha_0-pP^n} + \frac{\sum_{l=0}^{n-1}P^l\ln(C+1)P^{-l}(p+\alpha_0)}{p+\alpha_0-pP^n} + \frac{\sum_{l=0}^{n-1}P^l\ln W_l}{p+\alpha_0-pP^n}.
\end{aligned} \tag{40}$$

Here, taking a sufficiently large positive constant C' independent of n, we see that

$$\sum_{l=0}^{n-1}P^l\ln(C+1)P^{-l}(p+\alpha_0) \le C'\sum_{l=0}^{\infty}P^l(l+1) =: S_1 < \infty$$

and

$$\sum_{l=0}^{n-1}P^l\ln W_l \le C'\sum_{l=0}^{n-1}P^l(l+3) \le C'\sum_{l=0}^{\infty}P^l(l+3) =: S_2 < \infty.$$

Next, we shall show that $\{Q_n\}$ is a convergent sequence. It is easy see that $\{Q_n\}$ is increasing. Moreover, setting $d_k := \ln(1+\frac{P^k}{1-P^k})$, we see that

$$\lim_{k\to\infty}\frac{d_{k+1}}{d_k} = \lim_{k\to\infty}\frac{\ln(1-P^{k+1})}{\ln(1-P^k)} = \lim_{k\to\infty}\frac{1-P^k}{1-P^{k+1}}P = P < 1$$

by L'Hospital's rule. This implies that

$$\ln Q_n = \sum_{k=2}^{n+1}\ln\left(1+\frac{P^k}{1-P^k}\right) \le \sum_{k=1}^{\infty}\ln\left(1+\frac{P^k}{1-P^k}\right) < \infty.$$

Therefore, $\{Q_n\}$ is bounded from above, whence $\{Q_n\}$ converges, and

$$1 < \frac{1}{1-P^2} = Q_1 \leq Q_\infty := \lim_{n\to\infty} Q_n < \infty.$$

Consequently, letting $n \to \infty$ in (40), we have

$$\|v\|_{L^\infty(B(x_0,R_*))} \leq (pS_1S_2)^{\frac{1}{p+\alpha_0}} \|v\|_{L^{\overline{p}^*}(B(x_0,2R_*))}^{\frac{\alpha_0}{(p+\alpha_0)Q_\infty}}.$$

This yields our conclusion since $\|v\|_{L^{\overline{p}^*}(B(x_0,2R_*))} \to 0$ as $|x_0| \to \infty$, $\alpha_0 > 0$, and the constant pS_1S_2 is independent of x_0. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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