Chen and Liu *Boundary Value Problems* (2016) 2016:161
DOI 10.1186/s13661-016-0669-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Lower bound for the blow-up time for some nonlinear parabolic equations

Wenhui Chen and Yan Liu*

*Correspondence:
liuyan99021324@tom.com
Department of Applied Mathematics, Guangdong University of Finance, Guangzhou, 510521, P.R. China

Abstract
In this paper, we study the blow-up phenomenon for some nonlinear parabolic problems. Using the technique of differential inequalities, the lower bound for the blow-up time is determined if a blow-up does really occur. Our result is obtained in a bounded domain $\Omega \in \mathbb{R}^N$ for any $N \geq 3$.

Keywords: lower bound; blow-up time; nonlinear parabolic problems

1 Introduction

Payne et al. [1] studied the blow-up phenomenon for solutions to the following family of mixed problems:

$$\frac{\partial u}{\partial t} = \left(\rho\left(|\nabla u|^2\right)u_{,i}\right)_{,i} + f(u) \quad \text{in } \Omega \times \left(0, t^*\right), \tag{1.1}$$

$$u(x,0) = g(x) \geq 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = 0 \quad \text{in } \partial\Omega \times \left(0, t^*\right). \tag{1.3}$$

They obtained a lower bound for the blow-up time t^* if the blow-up does really occur together with a criterion for getting a blow-up. Moreover, they proposed conditions that ensure that a blow-up cannot occur. In this paper, we continue the work of Payne, Philippin, and Schaefer. In [1], they obtained the lower bound for the blow-up time of solutions in a bounded domain $\Omega \in \mathbb{R}^N$ for $N = 3$. If one is interested in generalizations to the case $N > 3$, then one important tool, which is important for proving the results obtained in [1], namely, the Sobolev inequality is no longer applicable. There are only a few papers dealing with a lower bound for the blow-up time when $N > 3$ (see [2, 3]). Our goal is to get a lower bound for the blow-up time of the solutions to (1.1)-(1.3) in $\Omega \in \mathbb{R}^N$ for any $N \geq 3$.

The study of finite-time blow-up of solutions to parabolic problems under a homogeneous Dirichlet boundary condition and Neumann condition has earned great attention (see [4–10]). Recently, some papers began to consider the blow-up phenomena of these problems under the Robin boundary conditions (see [11–14]). Many methods have been used to study equations (1.1)-(1.3) (see [15–17]).

In this paper, Ω is a bounded star-shaped domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$. The operator ∇ is the gradient operator, and t^* is the possible blow-up time. Furthermore, i stands for the partial differentiation with respect to x_i, $i = 1, 2, 3, \ldots, N$. The repeated index indicates Einstein's summation convention over the indices. We assume that ρ is a positive C^1 function that satisfies

$$\rho(s) + 2s\rho'(s) > 0, \quad s > 0, \tag{1.4}$$

so that $(\rho u_{,i})_{,i}$ is an elliptic operator. We also assume that ρ and f satisfy the conditions

$$0 < f(s) \leq a_1 + a_2 s^p, \quad s > 0, \tag{1.5}$$

and

$$\rho(s) \geq b_1 + b_2 s^q, \quad s > 0, \tag{1.6}$$

where $p > 1$ and $0 < 2q < p-1$, and a_1, a_2, b_1, b_2 are positive constants. Using the maximum principle, we can get that u is nonnegative in x and $t \in [0, t^*)$.

In the further discussions, we will use the following Hölder inequality:

$$\int_\Omega w^{x_1+x_2}\,dx \leq \left(\int_\Omega w^{\frac{x_1}{\alpha}}\,dx\right)^{\alpha} \left(\int_\Omega w^{\frac{x_2}{1-\alpha}}\,dx\right)^{1-\alpha}, \tag{1.7}$$

where $0 < \alpha < 1$, and x_1, x_2 are positive constants.

2 Lower bound for the blow-up time

In this section, we define the auxiliary function $\varphi = \varphi(t)$ as follows (see [1]):

$$\varphi(t) = \int_\Omega u^{2(n-1)(q+1)+2}\,dx = \int_\Omega u^\sigma\,dx \quad \text{with } \sigma = 2(n-1)(q+1)+2. \tag{2.1}$$

We establish the following theorem.

Theorem 1 *Assume that $u = u(x,t)$ is the classical nonnegative solution of the mixed problem* (1.1)-(1.3) *in a bounded domain $\Omega \in \mathbb{R}^N$ $(N \geq 3)$. Then the quantity $\varphi(t)$ defined in* (2.1) *satisfies the differential inequality*

$$\varphi'(t) \leq \sigma a_1 |\Omega|^{\frac{1}{\sigma}} \big[\phi(t)\big]^{\frac{\sigma-1}{\sigma}} + k_1\big[\phi(t)\big]^{\frac{(N-2)\alpha}{N\alpha-2}} + k_2\big[\phi(t)\big]^{\frac{(N-2)\alpha'}{N\alpha'-2}}, \tag{2.2}$$

which yields that the blow-up time t^* is bounded from below. We have

$$t^* \geq \int_{\phi(0)}^{+\infty} \frac{d\xi}{\sigma a_1 |\Omega|^{\frac{1}{\sigma}} [\xi]^{\frac{\sigma-1}{\sigma}} + k_1[\xi]^{\frac{(N-2)\alpha}{N\alpha-2}} + k_2[\xi]^{\frac{(N-2)\alpha'}{N\alpha'-2}}}, \tag{2.3}$$

where $|\Omega|$ is the volume of the domain Ω, and k_1, k_2 are positive constants that will be defined later.

Proof First, we compute

$$\begin{aligned}\varphi'(t) &= \sigma \int_\Omega u^{\sigma-1}\big[\big(\rho\big(|\nabla u|^2\big)u_{,i}\big)_{,i} + f(u)\big]\,dx \\ &= -\sigma(\sigma-1)\int_\Omega u^{\sigma-2}\rho\big(|\nabla u|^2\big)|\nabla u|^2\,dx + \sigma\int_\Omega u^{\sigma-1}f(u)\,dx\end{aligned}$$

$$\begin{aligned}&\le -\sigma(\sigma-1)\int_\Omega u^{2(n-1)(q+1)}|\nabla u|^2\big(b_1+b_2|\nabla u|^{2q}\big)\,dx\\&\quad+\sigma\int_\Omega u^{\sigma-1}\big(a_1+a_2u^p\big)\,dx.\end{aligned}\tag{2.4}$$

Using the equality

$$\big|\nabla u^n\big|^{2(q+1)}=\big|nu^{n-1}\nabla u\big|^{2(q+1)}=n^{2(q+1)}u^{2(n-1)(q+1)}|\nabla u|^{2(q+1)}$$

and the Hölder inequality, we get

$$\varphi'(t)\le-\frac{\sigma(\sigma-1)b_2}{n^{2(q+1)}}\int_\Omega\big|\nabla u^n\big|^{2(q+1)}\,dx+\sigma a_1|\Omega|^{\frac{1}{\sigma}}\big[\phi(t)\big]^{\frac{\sigma-1}{\sigma}}+\sigma a_2\int_\Omega u^{\sigma+p-1}\,dx.\tag{2.5}$$

If we set $v=u^n$, then we obtain

$$\varphi'(t)\le-\frac{\sigma(\sigma-1)b_2}{n^{2(q+1)}}\int_\Omega|\nabla v|^{2(q+1)}\,dx+\sigma a_1|\Omega|^{\frac{1}{\sigma}}\big[\phi(t)\big]^{\frac{\sigma-1}{\sigma}}+\sigma a_2\int_\Omega v^{2(q+1)+\frac{\gamma}{n}}\,dx,\tag{2.6}$$

where $\gamma=p-1-2q>0$. After application of the Hölder and Schwarz inequalities, it follows

$$\begin{aligned}\int_\Omega\big|\nabla v^{q+1}\big|^2\,dx&\le(q+1)^2\Big(\int_\Omega|\nabla v|^{2(q+1)}\,dx\Big)^{\frac{1}{q+1}}\Big(\int_\Omega|v|^{2(q+1)}\,dx\Big)^{\frac{q}{q+1}}\\&\le(q+1)\int_\Omega|\nabla v|^{2(q+1)}\,dx+(q+1)q\int_\Omega|v|^{2(q+1)}\,dx.\end{aligned}\tag{2.7}$$

Combining (2.6) and (2.7), we easily obtain

$$\begin{aligned}\varphi'(t)&\le-\frac{\sigma(\sigma-1)b_2}{n^{2(q+1)}(q+1)}\int_\Omega\big|\nabla v^{q+1}\big|^2\,dx+\frac{q\sigma(\sigma-1)b_2}{n^{2(q+1)}}\int_\Omega v^{2(q+1)}\,dx+\sigma a_1|\Omega|^{\frac{1}{\sigma}}\big[\phi(t)\big]^{\frac{\sigma-1}{\sigma}}\\&\quad+\sigma a_2\int_\Omega v^{2(q+1)+\frac{\gamma}{n}}\,dx.\end{aligned}\tag{2.8}$$

We choose x_1, x_2, and α such that

$$x_1+x_2=2(q+1),\qquad x_1\cdot\frac{1}{\alpha}=\frac{\sigma}{n},\qquad x_2\cdot\frac{1}{1-\alpha}=(q+1)\frac{2N}{N-2},$$

so that

$$\begin{aligned}&x_1=\frac{\sigma}{n}\frac{2(q+1)\frac{2}{N-2}}{2(q+1)\frac{N}{N-2}-\frac{\sigma}{n}},\qquad x_2=2(q+1)-\frac{\sigma}{n}\frac{2(q+1)\frac{2}{N-2}}{2(q+1)\frac{N}{N-2}-\frac{\sigma}{n}},\\&\alpha=\frac{2(q+1)\frac{2}{N-2}}{2(q+1)\frac{N}{N-2}-\frac{\sigma}{n}}.\end{aligned}$$

Then the Hölder inequality (1.7) yields

$$\int_\Omega v^{2(q+1)}\,dx\le\Big(\int_\Omega v^{\frac{\sigma}{n}}\,dx\Big)^{\alpha}\Big(\int_\Omega v^{(q+1)\frac{2N}{N-2}}\,dx\Big)^{1-\alpha}.\tag{2.9}$$

We follow the same procedure for x_1', x_2', and α', that is, we choose them such that

$$x_1' + x_2' = 2(q+1) + \frac{\gamma}{n}, \qquad x_1 \cdot \frac{1}{\alpha'} = \frac{\sigma}{n}, \qquad x_2' \cdot \frac{1}{1-\alpha'} = (q+1)\frac{2N}{N-2},$$

so that

$$\begin{aligned} x_1' &= \frac{\sigma}{n} \frac{2(q+1)\frac{2}{N-2} - \frac{\gamma}{n}}{2(q+1)\frac{N}{N-2} - \frac{\sigma}{n}}, \\ x_2' &= 2(q+1) + \frac{\gamma}{n} - \frac{\sigma}{n} \frac{2(q+1)\frac{2}{N-2} - \frac{\gamma}{n}}{2(q+1)\frac{N}{N-2} - \frac{\sigma}{n}}, \\ \alpha' &= \frac{2(q+1)\frac{2}{N-2} - \frac{\gamma}{n}}{2(q+1)\frac{N}{N-2} - \frac{\sigma}{n}}, \end{aligned}$$

and obtain

$$\int_\Omega v^{2(q+1)+\frac{\gamma}{n}}\,dx \le \left(\int_\Omega v^{\frac{\sigma}{n}}\,dx\right)^{\alpha'} \left(\int_\Omega v^{(q+1)\frac{2N}{N-2}}\,dx\right)^{1-\alpha'}. \tag{2.10}$$

Stressing the Sobolev inequality gives $W_0^{1,2} \hookrightarrow L^{\frac{2N}{N-2}}$ for $N \ge 3$. Consequently, we get

$$\|v^{q+1}\|_{L^{\frac{2N}{N-2}}}^{\frac{2N}{N-2}(1-\alpha)} \le c_1^{\frac{2N}{N-2}(1-\alpha)} \|\nabla v^{q+1}\|_{L^2}^{\frac{2N}{N-2}(1-\alpha)} \tag{2.11}$$

and

$$\|v^{q+1}\|_{L^{\frac{2N}{N-2}}}^{\frac{2N}{N-2}(1-\alpha')} \le c_1^{\frac{2N}{N-2}(1-\alpha')} \|\nabla v^{q+1}\|_{L^2}^{\frac{2N}{N-2}(1-\alpha')}, \tag{2.12}$$

where c_1 is the best embedding constant (see [18]).

A combination of (2.9) and (2.11) leads to

$$\int_\Omega v^{2(q+1)}\,dx \le c_1^{\frac{2N(1-\alpha)}{N-2}} \left(\int_\Omega v^{\frac{\sigma}{n}}\,dx\right)^{\alpha} \left(\int_\Omega |\nabla v^{q+1}|^2\,dx\right)^{\frac{N(1-\alpha)}{N-2}}. \tag{2.13}$$

An application of the Young inequality yields

$$\begin{aligned} \int_\Omega v^{2(q+1)}\,dx \le{}& \frac{N\alpha-2}{N-2} c_1^{\frac{2N(1-\alpha)}{N\alpha-2}} \varepsilon_1^{-\frac{N(1-\alpha)}{N\alpha-2}} \left(\int_\Omega v^{\frac{\sigma}{n}}\,dx\right)^{\frac{(N-2)\alpha}{N\alpha-2}} \\ &+ \frac{N(1-\alpha)}{N-2}\varepsilon_1 \int_\Omega |\nabla v^{q+1}|^2\,dx, \end{aligned} \tag{2.14}$$

where ε_1 is a positive constant to be determined later.

A combination of (2.9) and (2.11) also leads to

$$\begin{aligned} \int_\Omega v^{2(q+1)+\frac{\gamma}{n}}\,dx \le{}& \frac{N\alpha'-2}{N-2} c_1^{\frac{2N(1-\alpha')}{N\alpha'-2}} \varepsilon_2^{-\frac{N(1-\alpha')}{N\alpha'-2}} \left(\int_\Omega v^{\frac{\sigma}{n}}\,dx\right)^{\frac{(N-2)\alpha'}{N\alpha'-2}} \\ &+ \frac{N(1-\alpha')}{N-2}\varepsilon_2 \int_\Omega |\nabla v^{q+1}|^2\,dx, \end{aligned} \tag{2.15}$$

where ε_2 is a positive constant to be determined later.

Combining (2.8), (2.14), and (2.15), we obtain

$$\begin{aligned}\varphi'(t) \le &-\left[\frac{\sigma(\sigma-1)b_2}{n^{2(q+1)}(q+1)} - \frac{q\sigma(\sigma-1)b_2}{n^{2(q+1)}}\frac{N(1-\alpha)}{N-2}\varepsilon_1 - \sigma a_2\frac{N(1-\alpha')}{N-2}\varepsilon_2\right]\int_\Omega \left|\nabla v^{q+1}\right|^2 dx \\ &+ \sigma a_1|\Omega|^{\frac{1}{\sigma}}\left[\phi(t)\right]^{\frac{\sigma-1}{\sigma}} + \frac{N\alpha-2}{N-2}c_1^{\frac{2N(1-\alpha)}{N\alpha-2}}\varepsilon_1^{-\frac{N(1-\alpha)}{N\alpha-2}}\frac{q\sigma(\sigma-1)b_2}{n^{2(q+1)}}\left[\phi(t)\right]^{\frac{(N-2)\alpha}{N\alpha-2}} \\ &+ \frac{N\alpha'-2}{N-2}c_1^{\frac{2N(1-\alpha')}{N\alpha'-2}}\varepsilon_2^{-\frac{N(1-\alpha')}{N\alpha'-2}}\left[\phi(t)\right]^{\frac{(N-2)\alpha'}{N\alpha'-2}}. \end{aligned} \tag{2.16}$$

By choosing ε_1 and ε_2 small enough such that

$$\frac{\sigma(\sigma-1)b_2}{n^{2(q+1)}(q+1)} - \frac{q\sigma(\sigma-1)b_2}{n^{2(q+1)}}\frac{N(1-\alpha)}{N-2}\varepsilon_1 - \sigma a_2\frac{N(1-\alpha')}{N-2}\varepsilon_2 \ge 0 \tag{2.17}$$

we get the differential inequality

$$\varphi'(t) \le \sigma a_1|\Omega|^{\frac{1}{\sigma}}\left[\phi(t)\right]^{\frac{\sigma-1}{\sigma}} + k_1\left[\phi(t)\right]^{\frac{(N-2)\alpha}{N\alpha-2}} + k_2\left[\phi(t)\right]^{\frac{(N-2)\alpha'}{N\alpha'-2}} \tag{2.18}$$

with $k_1 = \frac{N\alpha-2}{N-2}c_1^{\frac{2N(1-\alpha)}{N\alpha-2}}\varepsilon_1^{-\frac{N(1-\alpha)}{N\alpha-2}}$ and $k_2 = \frac{N\alpha'-2}{N-2}c_1^{\frac{2N(1-\alpha')}{N\alpha'-2}}\varepsilon_2^{-\frac{N(1-\alpha')}{N\alpha'-2}}$.

Inequality (2.18) can be rewritten as

$$\frac{d\phi}{\sigma a_1|\Omega|^{\frac{1}{\sigma}}[\phi(t)]^{\frac{\sigma-1}{\sigma}} + k_1[\phi(t)]^{\frac{(N-2)\alpha}{N\alpha-2}} + k_2[\phi(t)]^{\frac{(N-2)\alpha'}{N\alpha'-2}}} \le dt. \tag{2.19}$$

An integration of (2.19) from 0 to t leads to

$$\int_{\phi(0)}^{\phi(t)} \frac{d\xi}{\sigma a_1|\Omega|^{\frac{1}{\sigma}}[\xi]^{\frac{\sigma-1}{\sigma}} + k_1[\xi]^{\frac{(N-2)\alpha}{N\alpha-2}} + k_2[\xi]^{\frac{(N-2)\alpha'}{N\alpha'-2}}} \le t. \tag{2.20}$$

Taking the limit as $t \longrightarrow t^*$, we obtain

$$\int_{\phi(0)}^{+\infty} \frac{d\xi}{\sigma a_1|\Omega|^{\frac{1}{\sigma}}[\xi]^{\frac{\sigma-1}{\sigma}} + k_1[\xi]^{\frac{(N-2)\alpha}{N\alpha-2}} + k_2[\xi]^{\frac{(N-2)\alpha'}{N\alpha'-2}}} \le t^*, \tag{2.21}$$

and the proof is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for helpful and very careful reading this paper. The work was supported by the National Natural Science Foundation of China (Grant ♯ 11471126), Foundation for the Training of the Excellent Young Teachers in Higher Education of Guangdong, China (Grant ♯ *Yq*2013121), the Project of innovation and strengthen of the University of Guangdong University of Finance (Grant ♯ 3-7), and Special Funds for the Cultivation of Guangdong College Student's Scientific and Technological Innovation ('Climbing Program' Special Funds).

Received: 15 June 2016 Accepted: 23 August 2016 Published online: 31 August 2016

References

1. Payne, LE, Philippin, GA, Schaefer, PW: Blow-up phenomena for some nonlinear parabolic problems. Nonlinear Anal. **69**, 3495-3502 (2008)
2. Bao, AG, Song, XF: Bounds for the blow up time of the solutions to quasi-linear parabolic problems. Z. Angew. Math. Phys. **65**, 115-123 (2014)
3. Li, HX, Gao, WJ, Han, YZ: Lower bounds for the blow up time of solutions to a nonlinear parabolic problems. Electron. J. Differ. Equ. **2014**, 20 (2014)
4. Liu, DM, Mu, CL, Qiao, X: Lower bounds estimate for the blow up time of a nonlinear nonlocal porous medium equation. Acta Math. Sci. **32**, 1206-1212 (2012)
5. Song, JC: Lower bounds for blow-up time in a non-local reaction-diffusion problem. Appl. Math. Lett. **5**, 793-796 (2011)
6. Payne, LE, Song, JC: Lower bounds for blow-up time in a nonlinear parabolic problem. J. Math. Anal. Appl. **354**, 394-396 (2009)
7. Payne, LE, Schaefer, PW: Lower bounds for blow-up time in parabolic problems under Dirichlet conditions. J. Math. Anal. Appl. **328**, 1196-1205 (2007)
8. Payne, LE, Schaefer, PW: Lower bounds for blow-up time in parabolic problems under Neumann conditions. Appl. Anal. **85**, 1301-1311 (2006)
9. Payne, LE, Philippin, GA, Schaefer, PW: Bounds for blow-up time in nonlinear parabolic problems. J. Math. Anal. Appl. **338**, 438-447 (2008)
10. Payne, LE, Song, JC: Lower bounds for the blow-up time in a temperature dependent Navier-Stokes flow. J. Math. Anal. Appl. **335**, 371-376 (2007)
11. Li, YF, Liu, Y, Lin, CH: Blow-up phenomena for some nonlinear parabolic problems under mixed boundary conditions. Nonlinear Anal., Real World Appl. **11**, 3815-3823 (2010)
12. Payne, LE, Schaefer, PW: Blow-up in parabolic problems under Robin boundary conditions. Appl. Anal. **87**, 699-707 (2008)
13. Ding, J: Global and blow-up solutions for nonlinear parabolic equations with Robin boundary conditions. Comput. Math. Appl. **65**(11), 1808-1822 (2013)
14. Enache, C: Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition. Appl. Math. Lett. **24**(3), 288-292 (2011)
15. Schaefer, PW: Blow up phenomena in some porous medium problems. Dyn. Syst. Appl. **18**, 103-110 (2009)
16. Payne, LE, Schaefer, PW: Bounds for the blow-up time for the heat equation under nonlinear boundary conditions. Proc. R. Soc. Edinb. A **139**, 1289-1296 (2009)
17. Payne, LE, Philippin, GA, Vernier Piro, S: Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition, I. Z. Angew. Math. Phys. **61**, 999-1007 (2010)
18. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)