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Existence and approximation of solution for a nonlinear second-order three-point boundary value problem

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Abstract

A nonlinear second-order ordinary differential equation with four cases of three-point boundary value conditions is studied by investigating the existence and approximation of solutions. First, the integration method is proposed to transform the considered boundary value problems into Hammerstein integral equations. Second, the existence of solutions for the obtained Hammerstein integral equations is analyzed by using the Schauder fixed point theorem. The contraction mapping theorem in Banach spaces is further used to address the uniqueness of solutions. Third, the approximate solution of Hammerstein integral equations is constructed by using a new numerical method, and its convergence and error estimate are analyzed. Finally, some numerical examples are addressed to verify the given theorems and methods.

MSC: 34B10; 45B05

Keywords: three-point boundary value problems; Hammerstein integral equation; existence; approximate solution; convergence and error estimate

1 Introduction

Nonlocal boundary value problems for linear and nonlinear ordinary differential equations are arising in the theory of mathematical physics and some engineering applications [1–3]. They have attracted much attention and lots of interesting observations have been given [4–7]. The existence and approximation of solutions are very important in order to understand various phenomena in physics, engineering, and so on [8]. Here we generally consider the following nonlinear second-order ordinary differential equation:

$$\varphi''(x) + \psi\bigl(x, \varphi(x)\bigr) = g(x), \quad x \in [a, b], \tag{1}$$

where $\psi(x, \varphi(x))$ and $g(x)$ are known, and $\varphi(x)$ is the unknown to be computed. The nonlinear ordinary differential equation (1) can be understood as the case that a linear problem is imposed upon a nonlinear loading $\psi(x, \varphi(x))$. Moreover, the three-point boundary value condition is equipped and it is assumed to be one of the following four cases:

$$\text{I:} \quad \varphi(a) = \alpha, \qquad \varphi(b) + k\varphi(\xi) = \beta, \quad \xi \in (a, b), \tag{2}$$

$$\text{II:}\quad \varphi(a)+k\varphi(\xi)=\alpha,\qquad \varphi(b)=\beta,\quad \xi\in(a,b), \tag{3}$$

$$\text{III:}\quad \varphi(a)=\alpha,\qquad \varphi(b)+k\varphi'(\xi)=\beta,\quad \xi\in(a,b), \tag{4}$$

$$\text{IV:}\quad \varphi(a)+k\varphi'(\xi)=\alpha,\qquad \varphi(b)=\beta,\quad \xi\in(a,b), \tag{5}$$

where α, β, ξ, and k are the known constants. It is found that II and IV can be reformulated by using the variable transformation $u(x)=a+b-x$ from I and III, respectively. That is, under the assumption of $\bar{\varphi}(u)=\varphi(a+b-u)=\varphi(x)$, one has

$$\text{II}':\quad \bar{\varphi}(a)+k\bar{\varphi}(\eta)=\beta,\qquad \bar{\varphi}(b)=\alpha,\quad \eta=a+b-\xi\in(a,b), \tag{6}$$

by using case I, and

$$\text{IV}':\quad \bar{\varphi}(a)-k\bar{\varphi}'(\eta)=\beta,\qquad \bar{\varphi}(b)=\alpha,\quad \eta=a+b-\xi\in(a,b), \tag{7}$$

from case III. Since the constants α, β, ξ, and k are arbitrary, the cases of I and III will be considered mainly in the following sections. When $k=0$, the three-point boundary value problems degenerate to two-point boundary value problems.

It is noted that when $g(x)=0$, the existence of solutions for equation (1) with various boundary value conditions has been studied widely. For example, the method of lower and upper solutions is developed by Ma [9] and the multiplicity solutions for a three-point boundary value problem at resonance were given. Xu [10, 11] considered the singular three-point and m-point boundary value problems, respectively. The multiplicity results and existence of positive solutions were analyzed by using a fixed point index theory. Yao [12] investigated the existence of positive solutions for a second-order three-point boundary value problem and a successive iteration method was given for computing the solutions. Nieto [13] studied the existence of solution for a second-order nonlinear ordinary differential equation with three-point boundary value conditions at resonance. In addition, the problems of nonlinear second-order ordinary differential equations with m-point and integral boundary value conditions were further investigated in [14–19] and so on. On the other hand, when $\psi(x,\varphi(x))=a(x)F(x,\varphi(x))$ and $\psi(x,\varphi(x))$ degenerates to $a(x)\psi(\varphi(x))$, the multipoint boundary value problems of nonlinear second-order ordinary differential equations were dealt with in [20–27]. Recently, the special linear case of $\psi(x,\varphi(x))=q(x)\varphi(x)$ has been studied in [28] and an approximate solution has been given. The case of $g(x)=0$ in equation (1) with impulse three-point boundary value conditions have been studied in [29], where the existence conditions for obtaining a nontrivial solution have been given.

As shown in the above-mentioned work, the existence of the solutions for second-order multipoint boundary value problems is always focused on. Moreover, the approximate solutions of boundary value problems are very important in engineering applications. A monotone iterative technique was developed for the approximate solution of a second-order three-point boundary value problem in [30]. This paper generally focuses on the nonlinear second-order ordinary differential equation with four cases of three-point boundary value conditions in (1)-(5). A general method is proposed to transform the nonlinear three-point boundary value problems into nonlinear Hammerstein integral equations. The existence and uniqueness of solutions for the obtained Hammerstein integral equations are considered by using the Schauder fixed point theorem and the contraction mapping theorem, respectively. A new numerical method is further proposed to construct the approximate solutions of Hammerstein integral equations. Some numerical examples are computed to show the effectiveness of the proposed methods.

2 Hammerstein integral equations

In this section, we will transform the nonlinear second-order ordinary differential equation (1) with three-point boundary value conditions in (2)-(5) into Hammerstein integral equations. Then the existence and uniqueness of solutions for the obtained Hammerstein integral equations will be investigated.

2.1 Transformations

In the following, we apply the integration method to get the following four theorems.

Theorem 1 *When* $(b-a)+k(\xi-a)\neq 0$, *the three-point boundary value problem*

$$\begin{cases} \varphi''(x)+\psi(x,\varphi(x))=g(x), & x\in[a,b],\\ \varphi(a)=\alpha, \quad \varphi(b)+k\varphi(\xi)=\beta, & \xi\in(a,b), \end{cases} \tag{8}$$

can be transformed into the Hammerstein integral equation as follows:

$$\varphi(x)+\int_a^b K_1(x,t)\psi\bigl(t,\varphi(t)\bigr)\,dt=f_1(x), \tag{9}$$

where

$$K_1(x,t)=\begin{cases} \frac{(a-t)[(b-x)+k(\xi-x)]}{(b-a)+k(\xi-a)}, & a\le t\le \min\{x,\xi\}\le b,\\ \frac{(a-x)(b-t)}{(b-a)+k(\xi-a)}, & a\le \max\{x,\xi\}\le t\le b,\\ \frac{(a-t)(b-x)+k(\xi-a)(x-t)}{(b-a)+k(\xi-a)}, & a\le \xi\le t\le x\le b,\\ \frac{(a-x)[(b-t)+k(\xi-t)]}{(b-a)+k(\xi-a)}, & a\le x\le t\le \xi\le b, \end{cases}$$

$$f_1(x)=\int_a^b K_1(x,t)g(t)\,dt+\frac{(b-x)+k(\xi-x)}{(b-a)+k(\xi-a)}\alpha+\frac{(x-a)}{(b-a)+k(\xi-a)}\beta.$$

Proof By integrating twice, from a to x, both sides of the differential equation in (8) with respect to x, one gets

$$\varphi(x)+\int_a^x (x-t)\psi\bigl(t,\varphi(t)\bigr)\,dt=\int_a^x (x-t)g(t)\,dt+\varphi'(a)(x-a)+\varphi(a). \tag{10}$$

Then the unknowns $\varphi'(a)$ and $\varphi(a)$ in (10) will be determined by using the boundary value conditions in (8), respectively. By setting $x=b$ in (10) one obtains

$$\varphi'(a)=\frac{1}{b-a}\biggl[\varphi(b)-\varphi(a)+\int_a^b (b-t)\psi\bigl(t,\varphi(t)\bigr)\,dt-\int_a^b (b-t)g(t)\,dt\biggr]. \tag{11}$$

Applying (11) to (10), it follows that

$$\begin{aligned} &\varphi(x)+\int_a^x (x-t)\psi\bigl(t,\varphi(t)\bigr)\,dt-\int_a^b \frac{(x-a)(b-t)}{(b-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt\\ &\quad=\int_a^x (x-t)g(t)\,dt-\int_a^b \frac{(x-a)(b-t)}{(b-a)}g(t)\,dt+\frac{b-x}{b-a}\varphi(a)+\frac{x-a}{b-a}\varphi(b). \end{aligned} \tag{12}$$

Now we let $x = \xi$ in (10) and obtain

$$\begin{aligned}
&\varphi(x) + \int_a^x (x-t)\psi\bigl(t,\varphi(t)\bigr)\,dt - \int_a^\xi \frac{(x-a)(\xi-t)}{(\xi-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt \\
&\quad = \int_a^x (x-t)g(t)\,dt - \int_a^\xi \frac{(x-a)(\xi-t)}{(\xi-a)}g(t)\,dt + \frac{\xi-x}{\xi-a}\varphi(a) + \frac{x-a}{\xi-a}\varphi(\xi).
\end{aligned} \tag{13}$$

In virtue of (8), (12), and (13), it gives

$$\begin{aligned}
&\varphi(x) + \int_a^x (x-t)\psi\bigl(t,\varphi(t)\bigr)\,dt - \int_a^b \frac{(x-a)(b-t)}{(b-a)+k(\xi-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt \\
&\quad - \int_a^\xi \frac{k(x-a)(\xi-t)}{(b-a)+k(\xi-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt \\
&\quad = \frac{(b-x)+k(\xi-x)}{(b-a)+k(\xi-a)}\alpha + \frac{(x-a)}{(b-a)+k(\xi-a)}\beta + \int_a^x (x-t)g(t)\,dt \\
&\quad - \int_a^b \frac{(x-a)(b-t)}{(b-a)+k(\xi-a)}g(t)\,dt - \int_a^\xi \frac{k(x-a)(\xi-t)}{(b-a)+k(\xi-a)}g(t)\,dt.
\end{aligned} \tag{14}$$

Since the right side of equation (14) is a function with respect to x, we let

$$\begin{aligned}
f_1(x) = {}& \frac{(b-x)+k(\xi-x)}{(b-a)+k(\xi-a)}\alpha + \frac{(x-a)}{(b-a)+k(\xi-a)}\beta + \int_a^x (x-t)g(t)\,dt \\
& - \int_a^b \frac{(x-a)(b-t)}{(b-a)+k(\xi-a)}g(t)\,dt - \int_a^\xi \frac{k(x-a)(\xi-t)}{(b-a)+k(\xi-a)}g(t)\,dt.
\end{aligned} \tag{15}$$

After some computations, one verifies that (14) and (15) can be rewritten as indicated in (9). □

Furthermore, one can obtain the following theorem and the proof has been omitted for saving space.

Theorem 2 *Under the condition of* $(b-a)+k(\xi-a) \neq 0$, *the three-point boundary value problem*

$$\begin{cases} \varphi''(x) + \psi(x,\varphi(x)) = g(x), & x \in [a,b], \\ \varphi(a) + k\varphi(\xi) = \alpha, \qquad \varphi(b) = \beta, & \xi \in (a,b), \end{cases} \tag{16}$$

is equivalent to the following Hammerstein integral equation:

$$\varphi(x) + \int_a^b K_2(x,t)\psi\bigl(t,\varphi(t)\bigr)\,dt = f_2(x), \tag{17}$$

where

$$K_2(x,t) = \begin{cases} \frac{(x-b)(t-a)}{(b-a)+k(b-\xi)}, & a \le t \le \min\{x,\xi\} \le b, \\ \frac{(t-b)[(x-a)+k(x-\xi)]}{(b-a)+k(b-\xi)}, & a \le \max\{x,\xi\} \le t \le b, \\ \frac{(x-b)[(t-a)+k(t-\xi)]}{(b-a)+k(b-\xi)}, & a \le \xi \le t \le x \le b, \\ \frac{(x-a)(t-b)+k(b-\xi)(t-x)}{(b-a)+k(b-\xi)}, & a \le x \le t \le \xi \le b, \end{cases}$$

$$f_2(x) = \int_a^b K_2(x,t)g(t)\,dt + \frac{(b-x)}{(b-a)+k(b-\xi)}\alpha + \frac{(x-a)+k(x-\xi)}{(b-a)+k(b-\xi)}\beta.$$

In the following, by considering the boundary value conditions in the cases of III and IV, we give Theorems 3 and 4, respectively.

Theorem 3 *In virtue of $k+(b-a)\neq 0$, the ordinary differential equation with three-point boundary value conditions*

$$\begin{cases} \varphi''(x) + \psi(x,\varphi(x)) = g(x), & x \in [a,b], \\ \varphi(a) = \alpha, \quad \varphi(b) + k\varphi'(\xi) = \beta, & \xi \in (a,b), \end{cases} \tag{18}$$

can be changed to the Hammerstein integral equation. That is, one has

$$\varphi(x) + \int_a^b K_3(x,t)\psi\bigl(t,\varphi(t)\bigr)\,dt = f_3(x), \tag{19}$$

where

$$K_3(x,t) = \begin{cases} \frac{(a-t)[k+(b-x)]}{k+(b-a)}, & a \le t \le \min\{x,\xi\} \le b, \\ \frac{(a-x)(b-t)}{k+(b-a)}, & a \le \max\{x,\xi\} \le t \le b, \\ \frac{(a-t)(b-x)+k(x-t)}{k+(b-a)}, & a \le \xi \le t \le x \le b, \\ \frac{(a-x)[k+(b-t)]}{k+(b-a)}, & a \le x \le t \le \xi \le b, \end{cases}$$

$$f_3(x) = \int_a^b K_3(x,t)g(t)\,dt + \frac{(k+b-x)}{k+(b-a)}\alpha + \frac{(x-a)}{k+(b-a)}\beta.$$

Proof Performing the integration procedures similar to those in Theorems 1 and 2, we obtain

$$\varphi(x) + \int_\xi^x (x-t)\psi\bigl(t,\varphi(t)\bigr)\,dt = \int_\xi^x (x-t)g(t)\,dt + \varphi'(\xi)(x-\xi) + \varphi(\xi). \tag{20}$$

It is assumed that $x = a$ in (20) and one arrives at

$$\varphi(\xi) = \varphi(a) + \int_\xi^a (a-t)\psi\bigl(t,\varphi(t)\bigr)\,dt - \int_\xi^a (a-t)g(t)\,dt - \varphi'(\xi)(a-\xi). \tag{21}$$

Insertion of (21) into (20) yields

$$\begin{aligned} &\varphi(x) + \int_\xi^x (x-t)\psi\bigl(t,\varphi(t)\bigr)\,dt + \int_a^\xi (a-t)\psi\bigl(t,\varphi(t)\bigr)\,dt \\ &\quad = \int_\xi^x (x-t)g(t)\,dt + \int_a^\xi (a-t)g(t)\,dt + \varphi'(\xi)(x-a) + \varphi(a). \end{aligned} \tag{22}$$

With the knowledge of $\varphi(b)$ in (12), and using the boundary value conditions in (18), one further has

$$\begin{aligned} &\varphi(x) + \int_a^x \frac{(b-a)(x-t)}{k+(b-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt - \int_a^b \frac{(x-a)(b-t)}{k+(b-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt \\ &\quad + \int_\xi^x \frac{k(x-t)}{k+(b-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt + \int_a^\xi \frac{k(a-t)}{k+(b-a)}\psi\bigl(t,\varphi(t)\bigr)\,dt \end{aligned}$$

$$
\begin{aligned}
&= \int_a^x \frac{(b-a)(x-t)}{k+(b-a)} g(t)\,dt - \int_a^b \frac{(x-a)(b-t)}{k+(b-a)} g(t)\,dt \\
&\quad + \int_\xi^x \frac{k(x-t)}{k+(b-a)} g(t)\,dt + \int_a^\xi \frac{k(a-t)}{k+(b-a)} g(t)\,dt \\
&\quad + \frac{(k+b-x)}{k+(b-a)}\alpha + \frac{(x-a)}{k+(b-a)}\beta.
\end{aligned} \tag{23}
$$

The proof is completed by rewriting (23) as (19). □

Theorem 4 *If* $(b-a)-k \neq 0$, *the following boundary value problem*:

$$
\begin{cases}
\varphi''(x) + \psi(x,\varphi(x)) = g(x), & x \in [a,b], \\
\varphi(a) + k\varphi'(\xi) = \alpha, \qquad \varphi(b) = \beta, & \xi \in (a,b),
\end{cases} \tag{24}
$$

is equivalent to the Hammerstein integral equation as follows:

$$
\varphi(x) + \int_a^b K_4(x,t)\psi\big(t,\varphi(t)\big)\,dt = f_4(x), \tag{25}
$$

where

$$
K_4(x,t) = \begin{cases}
\frac{(x-b)(t-a)}{(b-a)-k}, & a \le t \le \min\{x,\xi\} \le b, \\
\frac{(t-b)[(x-a)-k]}{(b-a)-k}, & a \le \max\{x,\xi\} \le t \le b, \\
\frac{(x-b)[(t-a)-k]}{(b-a)-k}, & a \le \xi \le t \le x \le b, \\
\frac{(x-a)(t-b)+k(x-t)}{(b-a)-k}, & a \le x \le t \le \xi \le b,
\end{cases}
$$

$$
f_4(x) = \int_a^b K_4(x,t)g(t)\,dt + \frac{(b-x)}{(b-a)-k}\alpha + \frac{(x-a-k)}{(b-a)-k}\beta.
$$

The proof can be completed similar to that of Theorem 3.

It is seen from Theorems 1-4 that the nonlinear second-order three-point boundary value problems have been transformed into Hammerstein integral equations. We remark that the integration method is uniform and enough to transform any nonlocal boundary value problem of ordinary differential equations into an integral equation [19, 31]. In the end, as a check, we consider the special case of $g(x) = \alpha = \beta = 0$ and obtain the boundary value problems as those in [32]. Based on Theorems 1-4, the solutions and Green's functions of the nonlinear three-point boundary value problems in [32] can be determined easily.

2.2 Existence and uniqueness of solution

For a nonlinear equation, the fixed point theorems are always used to address the existence and uniqueness of solutions [33–35]. Here since the considered three-point boundary value problems have been transformed into the Hammerstein integral equations, it is natural to study the existence and uniqueness of solutions for the obtained Hammerstein integral equations. Moreover, it is seen that the existence and uniqueness of solutions for Hammerstein integral equations have been investigated widely such as those in the book [36] and the recent results on $\mathbf{L}_1$ spaces [37]. In the present paper, for the obtained Hammerstein integral equations, we will use the Schauder fixed point theorem to address the existence of the solutions, and apply the Banach fixed point theorem to investigate the uniqueness of the solutions. The obtained results are related to the considered nonlinear boundary value problems of ordinary differential equations.

Now it is convenient to rewrite equations (9), (17), (19), and (25) in the following general form, namely:

$$\varphi(x) + \int_a^b K_i(x,t)\psi\big(t,\varphi(t)\big)\,dt = \int_a^b K_i(x,t)g(t)\,dt + \rho_i(x), \tag{26}$$

where $\rho_i(x)$ $(i = 1,2,3,4)$ are the linear functions with respect to x. It is easy to see that $K_i(x,t)$ and $\rho_i(x)$ are continuous on $[a,b;a,b]$ and $[a,b]$, respectively. Furthermore, we assume that

$$A_i = \max_{a\le x,t\le b}\big|K_i(x,t)\big|, \qquad D_i = \max_{a\le x\le b}\big|\rho_i(x)\big|.$$

Based on the Schauder fixed point theorem, we first give the following theorem to address the existence of the solutions in (26).

Theorem 5 *It is assumed that* $S = \{\varphi|\varphi \in \mathbf{L}_2[a,b], \|\varphi\| \le M\}$. $\forall\varphi \in S$, *one has*

$$\int_a^b \big|\psi\big(t,\varphi(t)\big) - g(t)\big|^2\,dt \le B^2.$$

$\forall\varepsilon > 0$, $\varphi_1,\varphi_2 \in S$, $\exists\delta(\varepsilon) > 0$, *when* $\|\varphi_1 - \varphi_2\| < \delta(\varepsilon)$, *it gives*

$$\int_a^b \big|\psi\big(t,\varphi_1(t)\big) - \psi\big(t,\varphi_2(t)\big)\big|^2\,dt < \varepsilon^2.$$

If $A_iB(b-a) + D_i\sqrt{b-a} < M$, *the nonlinear integral equation* (26) *at least has a solution in* S.

Proof Obviously S is a closed convex set. For convenience, we define the integral operator $\mathbf{T}$ from $\mathbf{L}_2[a,b]$ to $\mathbf{L}_2[a,b]$ such that

$$\varphi = \int_a^b K_i(x,t)\big[g(t) - \psi\big(t,\varphi(t)\big)\big]\,dt + \rho_i = \mathbf{T}\varphi.$$

First, one can see that $\mathbf{T}$ is a mapping from S to S. In fact, for $\|\varphi\| \le M$, we have

$$\begin{aligned}
\big|\mathbf{T}\varphi(x)\big| &= \left|\int_a^b K_i(x,t)\big[g(t) - \psi\big(t,\varphi(t)\big)\big]\,dt + \rho_i(x)\right| \\
&\le \int_a^b \big|K_i(x,t)\big| \cdot \big|g(t) - \psi\big(t,\varphi(t)\big)\big|\,dt + \big|\rho_i(x)\big| \\
&\le A_i\int_a^b \big|g(t) - \psi\big(t,\varphi(t)\big)\big|\,dt + D_i \\
&\le A_i\left[\int_a^b \big|g(t) - \psi\big(t,\varphi(t)\big)\big|^2\,dt \cdot \int_a^b 1^2\,dt\right]^{\frac{1}{2}} + D_i \\
&\le A_iB\sqrt{b-a} + D_i < \frac{M}{\sqrt{b-a}},
\end{aligned}$$

then

$$\|\mathbf{T}\varphi\| = \left(\int_a^b \left|\mathbf{T}\varphi(x)\right|^2 dx\right)^{1/2} < M.$$

Second, we prove that $\mathbf{T}$ is continuous. $\forall\varepsilon > 0$, $\varphi_1, \varphi_2 \in S$, $\exists\delta(\varepsilon) > 0$, such that when $\|\varphi_1 - \varphi_2\| < \delta(\varepsilon)$, it follows that

$$\begin{aligned}|\mathbf{T}\varphi_1 - \mathbf{T}\varphi_2| &= \left|\int_a^b K_i(x,t)\left[\psi\left(t,\varphi_1(t)\right) - \psi\left(t,\varphi_2(t)\right)\right] dt\right| \\ &\leq A_i\left[\int_a^b \left|\psi\left(t,\varphi_1(t)\right) - \psi\left(t,\varphi_2(t)\right)\right|^2 dt \cdot \int_a^b 1^2\, dt\right]^{\frac{1}{2}} \\ &\leq (A_i\sqrt{b-a})\varepsilon\end{aligned}$$

and

$$\|\mathbf{T}\varphi_1 - \mathbf{T}\varphi_2\| = \left(\int_a^b \left|\mathbf{T}\varphi_1(x) - \mathbf{T}\varphi_2(x)\right|^2 dx\right)^{1/2} \leq A_i(b-a)\varepsilon.$$

In the end, let us prove that $\mathbf{T}(S)$ is relatively compact. Since $K_i(x,t)$ and $\rho_i(x)$ are continuous on $[a,b;a,b]$ and $[a,b]$, respectively, they are uniformly continuous. Consequently, $\forall\varepsilon > 0$, $\varphi \in S$, $\exists\eta(\varepsilon) > 0$, for $|x_1 - x_2| < \eta(\varepsilon)$, one has

$$\begin{aligned}&\left|\mathbf{T}\varphi(x_1) - \mathbf{T}\varphi(x_2)\right| \\ &\quad= \left|\int_a^b \left[K_i(x_1,t) - K_i(x_2,t)\right] \cdot \left[g(t) - \psi\left(t,\varphi(t)\right)\right] dt + \left[\rho_i(x_1) - \rho_i(x_2)\right]\right| \\ &\quad\leq \left[\int_a^b \left|K_i(x_1,t) - K_i(x_2,t)\right|^2 dt \cdot \int_a^b \left|g(t) - \psi\left(t,\varphi(t)\right)\right|^2 dt\right]^{\frac{1}{2}} + \left|\rho_i(x_1) - \rho_i(x_2)\right| \\ &\quad< (B\sqrt{b-a} + 1)\varepsilon.\end{aligned}$$

It is seen that $\mathbf{T}(S)$ is uniformly bounded and equicontinuous. According to Ascoli-Arzela theorem [38], $\{\mathbf{T}\varphi(x)\}$ has a subsequence with uniform convergence, so $\mathbf{T}(S)$ is relatively compact.

By using the Schauder fixed point theorem, there exists at least a point $\varphi \in S$ such that $\mathbf{T}\varphi = \varphi$. □

In addition, we can change some conditions in Theorem 5 to obtain the following corollary.

Corollary 1 *Let* $S = \{\varphi | \varphi \in \mathbf{L}_2[a,b], \|\varphi\| \leq M\}$, *and*

$$E = \max_{a\leq x\leq b} \max_{u\in S} \left|\psi(x,u)\right|, \qquad F = \max_{a\leq x\leq b} \left|g(x)\right|.$$

$\forall\varepsilon > 0$, $\varphi_1, \varphi_2 \in S$, $\exists\delta(\varepsilon) > 0$, *when* $\|\varphi_1 - \varphi_2\| < \delta(\varepsilon)$, *one has*

$$\int_a^b \left|\psi\left(t,\varphi_1(t)\right) - \psi\left(t,\varphi_2(t)\right)\right| dt < \varepsilon.$$

If $A_i(E+F)(b-a)+D_i < \frac{M}{\sqrt{b-a}}$, *the nonlinear integral equation* (26) *at least has a solution in* S.

Furthermore, strengthening the conditions of the nonlinear term $\psi(x,\varphi(x))$, we have the uniqueness theorem of solution in Banach spaces.

Theorem 6 *Suppose that* $g(x)\in \mathbf{L}_2[a,b]$, *and* $\psi(x,\varphi(x))$ *satisfies the following Lipschitz condition*:

$$\big\|\psi\big(x,\varphi_1(x)\big)-\psi\big(x,\varphi_2(x)\big)\big\|\le L\big\|\varphi_1(x)-\varphi_2(x)\big\|,$$

with the Lipschitz constant $L>0$. *Moreover, one has the constraint conditions as*

$$\begin{aligned}&\left\|\int_a^b K_i(x,t)\psi\big(t,\varphi(t)\big)\,dt\right\|\le N\|\varphi\|,\\&\int_a^b\int_a^b K_i^2(x,t)\,dx\,dt=C_i^2<+\infty,\end{aligned}$$

where N *and* C_i *are positive constants. When* $LC_i<1$, *the nonlinear integral equation* (26) *has a unique solution in* $\mathbf{L}_2[a,b]$.

Proof Since $g(x)\in\mathbf{L}_2[a,b]$, we can easily get $f_i(x)\in\mathbf{L}_2[a,b]$. The kernel $K_i(x,t)$ is a polynomial function with respect to x and t, so $K_i(x,t)\in\mathbf{L}_2[a,b;a,b]$. Assume that $\mathbf{T}$ is an operator form $\mathbf{L}_2[a,b]$ to $\mathbf{L}_2[a,b]$, and $\mathbf{T}\varphi=f_i-\boldsymbol{\kappa}_i\varphi$, where

$$\boldsymbol{\kappa}_i\varphi(x)=\int_a^b K_i(x,t)\psi\big(t,\varphi(t)\big)\,dt.$$

By using

$$\left\|\int_a^b K_i(x,t)\psi\big(t,\varphi(t)\big)\,dt\right\|\le N\|\varphi\|,$$

it is seen that $\boldsymbol{\kappa}_i$ is a bounded operator form $\mathbf{L}_2[a,b]$ to $\mathbf{L}_2[a,b]$. $\forall\varphi_1,\varphi_2\in\mathbf{L}_2[a,b]$, we further have

$$\begin{aligned}|\boldsymbol{\kappa}_i\varphi_1-\boldsymbol{\kappa}_i\varphi_2|&=\left|\int_a^b K_i(x,t)\psi\big(t,\varphi_1(t)\big)\,dt-\int_a^b K_i(x,t)\psi\big(t,\varphi_2(t)\big)\,dt\right|\\&\le\int_a^b\big|K_i(x,t)\big|\cdot\big|\psi\big(t,\varphi_1(t)\big)-\psi\big(t,\varphi_2(t)\big)\big|\,dt\\&\le\left(\int_a^b\big|K_i(x,t)\big|^2\,dt\right)^{\frac12}\big\|\psi\big(t,\varphi_1(t)\big)-\psi\big(t,\varphi_2(t)\big)\big\|\\&\le L\left(\int_a^b\big|K_i(x,t)\big|^2\,dt\right)^{\frac12}\big\|\varphi_1(t)-\varphi_2(t)\big\|.\end{aligned}$$

Then it gives

$$\|\boldsymbol{\kappa}_i\varphi_1-\boldsymbol{\kappa}_i\varphi_2\|\le L\left(\int_a^b\int_a^b\big|K_i(x,t)\big|^2\,dt\,dx\right)^{\frac12}\|\varphi_1-\varphi_2\|=LC_i\|\varphi_1-\varphi_2\|$$

and

$$\|\mathbf{T}\varphi_1 - \mathbf{T}\varphi_2\| = \|\boldsymbol{\kappa}_i\varphi_1 - \boldsymbol{\kappa}_i\varphi_2\| \leq LC_i\|\varphi_1 - \varphi_2\|.$$

When $LC_i < 1$, $\mathbf{T}$ is a contraction operator. According to the fixed point theorem in Banach spaces, one can see that $\mathbf{T}\varphi = \varphi$ has a unique solution in $\mathbf{L}_2[a,b]$. □

3 Approximation of the solution

In practical applications, of much interest is how to obtain the solutions except for the existence of the solutions. However, the closed-form solutions of the Hammerstein integral equations in (9), (17), (19), and (25) cannot be determined easily due to the complexity of the kernels. Thus it is interesting to obtain numerical solutions of Hammerstein integral equations and many methods have been proposed [8, 39–46]. Moreover, it is noted that a simple Taylor-series expansion method has been proposed in [47] and modified in [48, 49] for numerically solving linear Fredholm integral equations of the second kind. Recently, by using the idea of piecewise approximation, the simple Taylor-series expansion method has been further modified in [28]. Here the proposed method in [28] is further extended and applied to solve the nonlinear integral equation of Hammerstein type. The convergence and error estimate of the approximate solution will be made. Moreover, it is seen from Theorems 5 and 6 that a solution in $\mathbf{L}^2[a,b]$ is only determined by using the given conditions. Based on the proposed numerical method, the solution $\varphi(x)$ should have more smoothing property and here it is assumed $\varphi(x) \in \mathbf{C}^{n+1}[a,b]$ $(n \geq 0)$. Indeed, the case of $\varphi(x) \in \mathbf{C}^{n+1}[a,b]$ $(n \geq 0)$ is important in practical applications. Two examples will be given in Section 4 to verify the extended numerical method by comparing a difference format.

3.1 Constructing the approximate solution

Generally, we write the Hammerstein integral equation as

$$\varphi(x) + \int_a^b K(x,t)\psi\big(t,\varphi(t)\big)\,dt = f(x), \quad x \in [a,b], \tag{27}$$

where $K(x,t)$ and $f(x)$ are known functions. It is convenient to define the integral operator $\boldsymbol{\kappa}$ as

$$(\boldsymbol{\kappa}\varphi)(x) = \int_a^b K(x,t)\psi\big(t,\varphi(t)\big)\,dt, \quad x \in [a,b], \tag{28}$$

and it is compact from $C^{(n+1)}[a,b]$ into $C^{(n+1)}[a,b]$ for $K(x,t) \in C^{(n+1)}[a,b;a,b]$.

Similar to those in [28], we choose a series of quadrature points as $a = x_0 < x_1 < \cdots < x_m = b$ for $m \geq 1$. The integral operator $\boldsymbol{\kappa}$ can be further expressed as the following sum:

$$(\boldsymbol{\kappa}\varphi)(x) = \sum_{q=0}^{m-1} \int_{x_q}^{x_{q+1}} K(x,t)\psi\big(t,\varphi(t)\big)\,dt. \tag{29}$$

For the simplified case, the equidistant quadrature points are always chosen as

$$x_q = a + qh, \quad q = 0,1,\ldots,m, h = \frac{b-a}{m}. \tag{30}$$

By letting $t = x_q + hs$, equation (29) is reexpressed as

$$(\boldsymbol{\kappa}\varphi)(x) = h\sum_{q=0}^{m-1}\int_0^1 K(x, x_q + hs)\psi\big(x_q + hs, \varphi(x_q + hs)\big)\,ds. \tag{31}$$

Now it is assumed that $\psi(x_q + hs, \varphi(x_q + hs))$ can be expanded as the following Taylor series:

$$\begin{aligned}\psi\big(x_q + hs, \varphi(x_q + hs)\big) &= \psi\big(x_q, \varphi(x_q)\big) + (hs)\cdot\frac{d\psi(y,\varphi(y))}{dy}\bigg|_{y=x_q} + \cdots \\ &\quad + \frac{(hs)^n}{n!}\cdot\frac{d^n\psi(y,\varphi(y))}{dy^n}\bigg|_{y=x_q} + R_n(\theta_q, h, s),\end{aligned} \tag{32}$$

where $R_n(\theta_q, h, s)$ denotes the Lagrange remainder,

$$R_n(\theta_q, h, s) = \frac{(hs)^{n+1}}{(n+1)!}\cdot\frac{d^{n+1}\psi(y,\varphi(y))}{dy^{n+1}}\bigg|_{y=\theta_q}, \quad x_q \le \theta_q \le x_q + hs. \tag{33}$$

By eliminating the Lagrange remainder, the operator $(\boldsymbol{\kappa}\varphi)(x)$ can be approximated by using

$$(\boldsymbol{\kappa}_n\varphi)(x) = h\sum_{q=0}^{m-1}\sum_{j=0}^{n}\frac{h^j}{j!}\cdot\frac{d^j\psi(y,\varphi(y))}{dy^j}\bigg|_{y=x_q}\int_0^1 K(x, x_q + hs)\cdot s^j\,ds. \tag{34}$$

Moreover, we suppose that

$$(\boldsymbol{\kappa}\varphi)^{(i)}(x) = h\sum_{q=0}^{m-1}\int_0^1 K_x^{(i)}(x, x_q + hs)\psi\big(x_q + hs, \varphi(x_q + hs)\big)\,ds, \tag{35}$$

where the superscript (i) denotes the ith-order differentiation with respect to x and $(\boldsymbol{\kappa}\varphi)^{(0)}(x) = (\boldsymbol{\kappa}\varphi)(x)$. Making use of (32), $(\boldsymbol{\kappa}\varphi)^{(i)}(x)$ can be approximated by

$$(\boldsymbol{\kappa}_n\varphi)^{(i)}(x) = h\sum_{q=0}^{m-1}\sum_{j=0}^{n}\frac{h^j}{j!}\cdot\frac{d^j\psi(y,\varphi(y))}{dy^j}\bigg|_{y=x_q}\int_0^1 K_x^{(i)}(x, x_q + hs)\cdot s^j\,ds. \tag{36}$$

Now we further have the following theorem.

Theorem 7 *Assume that one has the following conditions*:

$$\begin{aligned}&\big\|K_x^{(i)}(x,t)\big\|_\infty = \max_{a\le x,t\le b}\big|K_x^{(i)}(x,t)\big| = M_i < +\infty, \\ &\left\|\frac{d^{n+1}\psi(y,\varphi(y))}{dy^{n+1}}\right\|_\infty = \max_{a\le y\le b}\left|\frac{d^{n+1}\psi(y,\varphi(y))}{dy^{n+1}}\right| = N_0 < +\infty,\end{aligned}$$

where $i = 0, 1, 2, \ldots, n$. The sequence $(\boldsymbol{\kappa}_n\varphi)^{(i)}(x)$ is convergent, namely

$$(\boldsymbol{\kappa}_n\varphi)^{(i)}(x) \to (\boldsymbol{\kappa}\varphi)^{(i)}(x) = \int_a^b K_x^{(i)}(x,t)\psi\big(t, \varphi(t)\big)\,dt, \quad n \to +\infty. \tag{37}$$

Proof Applying equations (35) and (36), we get

$$\begin{aligned}
&\big\|(\boldsymbol{\kappa}_n\varphi)^{(i)}(x)-(\boldsymbol{\kappa}\varphi)^{(i)}(x)\big\|_\infty \\
&\quad=\left\|h\sum_{q=0}^{m-1}\int_0^1 K_x^{(i)}(x,x_q+hs)\cdot\frac{(hs)^{n+1}}{(n+1)!}\cdot\frac{d^{n+1}\psi(y,\varphi(y))}{dy^{n+1}}\bigg|_{y=\theta_q}ds\right\|_\infty \\
&\quad\le hmM_iN_0\int_0^1\frac{(hs)^{n+1}}{(n+1)!}\,ds=\frac{M_iN_0(b-a)}{(n+2)!}h^{n+1}.
\end{aligned} \tag{38}$$

From (38) it follows that $\|(\boldsymbol{\kappa}_n\varphi)^{(i)}(x)-(\boldsymbol{\kappa}\varphi)^{(i)}(x)\|_\infty\to 0$ with $n\to+\infty$ and the proof is completed. □

In the end, let us give the approximate solution of Hammerstein integral equations. From equations (34) and (36), the discretization format of the derivatives of the Hammerstein integral equation (27) can be expressed as

$$\varphi_l^{(i)}+h\sum_{q=0}^{m-1}\sum_{j=0}^{n}\frac{h^j}{j!}\cdot\frac{d^j\psi}{dy^j}\bigg|_{(x_q,\varphi_q^{(j)})}\int_0^1 K_x^{(i)}(x_l,x_q+hs)\cdot s^j\,ds=f^{(i)}(x_l), \tag{39}$$

with $i=0,1,\ldots,n$ and $l=0,1,\ldots,(m-1)$. Hereafter $\frac{d^j\psi}{dy^j}|_{(x_q,\varphi_q^{(j)})}$ means that the variable y equals x_q and the exact value $\varphi^{(j)}(x_q)$ is replaced by the approximate one $\varphi_q^{(j)}$. Once the solution of the nonlinear system (39) is given, the approximate solution of $\varphi(x)$ can be further constructed as

$$\varphi_{m,n}(x)=f(x)-h\sum_{q=0}^{m-1}\sum_{j=0}^{n}\frac{h^j}{j!}\cdot\frac{d^j\psi}{dy^j}\bigg|_{(x_q,\varphi_q^{(j)})}\int_0^1 K(x,x_q+hs)\cdot s^j\,ds, \tag{40}$$

where $a\le x\le b$.

As shown in [28], one can see from equation (40) that the approximate solution has two parameters. The proposed method is based on the discretization points x_q ($q=0,1,\ldots,m-1$), which is different from the simple Taylor-series expansion method in [47–49]. The effectiveness and advantage of the new method will be shown in the given numerical examples of Section 4. Furthermore, in order to give the approximate solution in (40), the convergence of the nonlinear system (39) is requisite. Under some conditions, the nonlinear system (39) is convergent and it will be proved in the next subsection about the error estimate of the approximate solution.

On the other hand, it should be pointed out that when the proposed method is applied to solve the Hammerstein integral equations in (9), (17), (19), and (25), the derivatives $K_x^{(i)}(x,t)$ ($i=1,2,\ldots,n$) for $x=t$ in (39) must be dealt with again. The reason is based on the fact that the derivatives $\partial^iK_j(x,t)/\partial x^i$ ($i=1,2,\ldots,n$; $j=1,2,3,4$) for $x=t$ are not existing. As shown in [50], for the practical computations, it is reasonable to adopt the following method:

$$\begin{aligned}
\int_a^b K_x^{(i)}(x,t)\psi\big(t,\varphi(t)\big)\,dt&=\int_a^x K_x^{(i)}(x,t)\big|_{x>t}\psi\big(t,\varphi(t)\big)\,dt \\
&\quad+\int_x^b K_x^{(i)}(x,t)\big|_{x<t}\psi\big(t,\varphi(t)\big)\,dt.
\end{aligned} \tag{41}$$

In addition, we can rewrite equation (26) as

$$\varphi(x) + \int_a^b K_i(x,t)\big[\psi\big(t,\varphi(t)\big) - g(t)\big]\,dt = \rho_i(x), \tag{42}$$

where $\rho_i(x)$ ($i = 1,2,3,4$) are the linear functions with respect to x. For generic nonlinear functions $\psi(t,\varphi(t))$ and $g(t)$, the proposed numerical methods can be used similarly for equation (42).

3.2 Convergence and error estimate

From the viewpoint of mathematical theory and practical applications, the convergence and error estimate of the approximate solution are all important. For the approximation method, we have the following theorem.

Theorem 8 *It is assumed that $d^j\psi(y,\varphi(y))/dy^j$ satisfies the Lipschitz conditions as follows*:

$$\left\| \frac{d^j\psi(y,\varphi_1(y))}{dy^j} - \frac{d^j\psi(y,\varphi_2(y))}{dy^j} \right\|_\infty \le \sum_{\nu=0}^{j} L_\nu \big\| \varphi_1^{(\nu)}(y) - \varphi_2^{(\nu)}(y) \big\|_\infty$$

with the Lipschitz constants $L_\nu > 0$ and $j = 0,1,\ldots,n$. One further has the following conditions:

$$\begin{aligned} &\big\|K_x^{(i)}(x,t)\big\|_\infty = \max_{a\le x,t\le b}\big|K_x^{(i)}(x,t)\big| = M_i < \frac{1}{\bar{L}(b-a)e^h} < +\infty, \\ &\left\| \frac{d^{n+1}\psi(y,\varphi(y))}{dy^{n+1}} \right\|_\infty = \max_{a\le y\le b}\left| \frac{d^{n+1}\psi(y,\varphi(y))}{dy^{n+1}} \right| = N_0 < +\infty, \end{aligned}$$

where $\bar{L} = \max_{\nu=0,1,\ldots,j} L_\nu$ and $i = 0,1,2,\ldots,n$. The approximate solution $\varphi_{m,n}(x)$ in (40) is convergent to the exact solution $\varphi(x)$. That is, we get

$$\lim_{n\to+\infty} \big\|\varphi_{m,n}(x) - \varphi(x)\big\|_\infty = 0$$

and

$$\lim_{h\to 0} \big\|\varphi_{m,n}(x) - \varphi(x)\big\|_\infty = 0.$$

Moreover, the following error estimate can be obtained:

$$\big\|\varphi_{m,n}(x) - \varphi(x)\big\|_\infty \le \frac{M_0 N_0 (b-a) h^{n+1}}{[1 - \bar{M}\bar{L}(b-a)e^h](n+2)!},$$

where $\bar{M} = \max_{i=0,1,\ldots,n} M_i$.

Proof Equation (39) can be further rewritten as

$$\tilde{\Phi} + W(\tilde{\Phi}) = F, \tag{43}$$

where

$$\tilde{\Phi} = \big[\varphi_l^{(j)}\big]_{m(n+1)\times 1} = \big[\varphi_0^{(0)}, \varphi_1^{(0)}, \ldots, \varphi_{m-1}^{(n)}\big]^T,$$

$$W(\tilde{\Phi}) = \left[\tilde{w}_l^{(i)}(\tilde{\Phi})\right]_{m(n+1)\times 1},$$
$$F = \left[f^{(i)}(x_l)\right]_{m(n+1)\times 1} = \left[f^{(0)}(x_0), f^{(0)}(x_1), \dots, f^{(n)}(x_{m-1})\right]^T,$$

with

$$\tilde{w}_l^{(i)}(\tilde{\Phi}) = h \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{j!} \cdot \frac{d^j \psi}{dy^j}\bigg|_{(x_q,\varphi_q^{(j)})} \int_0^1 K_x^{(i)}(x_l, x_q + hs) \cdot s^j \, ds$$

for $i = 0, 1, \dots, n$ and $l = 0, 1, \dots, (m-1)$.

On the other hand, application of equations (27) and (32) leads to

$$\Phi + \left[W(\Phi) + R\right] = F, \tag{44}$$

where

$$\Phi = \left[\varphi^{(j)}(x_q)\right]_{m(n+1)\times 1} = \left[\varphi^{(0)}(x_0), \varphi^{(0)}(x_1), \dots, \varphi^{(n)}(x_{m-1})\right]^T,$$
$$W(\Phi) = \left[w_l^{(i)}(\Phi)\right]_{m(n+1)\times 1},$$
$$R = [r_s]_{m(n+1)\times 1},$$

with

$$w_l^{(i)}(\Phi) = h \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{j!} \cdot \frac{d^j \psi(y, \varphi(y))}{dy^j}\bigg|_{y=x_q} \int_0^1 K_x^{(i)}(x_l, x_q + hs) \cdot s^j \, ds,$$
$$|r_0| \le h \sum_{q=0}^{m-1} \int_0^1 \left|K(x_0, x_q + hs) R_n(\theta_q, h, s)\right| ds,$$
$$|r_1| \le h \sum_{q=0}^{m-1} \int_0^1 \left|K(x_1, x_q + hs) R_n(\theta_q, h, s)\right| ds,$$
$$\dots,$$
$$|r_{m(n+1)-1}| \le h \sum_{q=0}^{m-1} \int_0^1 \left|K_x^{(n)}(x_{m-1}, x_q + hs) R_n(\theta_q, h, s)\right| ds.$$

Then it is further found that

$$\begin{aligned}
&\left\| W(\Phi) - W(\tilde{\Phi}) \right\|_\infty \\
&\quad \le hM_i \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{j!} \cdot \int_0^1 s^j \, ds \cdot \left\| \frac{d^j \psi(y, \varphi(y))}{dy^j}\bigg|_{y=x_q} - \frac{d^j \psi}{dy^j}\bigg|_{(x_q,\varphi_q^{(j)})} \right\|_\infty \\
&\quad \le h\bar{M} \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{(j+1)!} \cdot \sum_{\nu=0}^{j} L_\nu \left\| \varphi^{(\nu)}(x_q) - \varphi_q^{(\nu)} \right\|_\infty \\
&\quad \le h\bar{M}\bar{L} \sum_{q=0}^{m-1} \sum_{j=0}^{+\infty} \frac{h^j}{j!} \cdot \|\Phi - \tilde{\Phi}\|_\infty \le \bar{M}\bar{L}(b-a)e^h \cdot \|\Phi - \tilde{\Phi}\|_\infty,
\end{aligned} \tag{45}$$

where $\bar{M} = \max_{i=0,1,\dots,n} M_i$ and $\bar{L} = \max_{\nu=0,1,\dots,j} L_\nu$.

Now let us assume that

$$0 < \bar{M} < \frac{1}{\bar{L}(b-a)e^h}. \tag{46}$$

Based on (43) and (44), it follows that

$$\begin{aligned} \|\Phi - \tilde{\Phi}\|_\infty &\leq \left\| W(\Phi) - W(\tilde{\Phi}) \right\|_\infty + \|R\|_\infty \\ &\leq \bar{M}\bar{L}(b-a)e^h \cdot \|\Phi - \tilde{\Phi}\|_\infty + \|R\|_\infty. \end{aligned} \tag{47}$$

Furthermore, one has

$$\begin{aligned} \|\Phi - \tilde{\Phi}\|_\infty &\leq \frac{1}{1 - \bar{M}\bar{L}(b-a)e^h} \|R\|_\infty \\ &= \frac{1}{1 - \bar{M}\bar{L}(b-a)e^h} \cdot \max_{0 \leq s \leq m(n+1)} |r_s| \\ &\leq \frac{1}{1 - \bar{M}\bar{L}(b-a)e^h} \cdot h\bar{M}N_0 \sum_{q=0}^{m-1} \int_0^1 \frac{(hs)^{(n+1)}}{(n+1)!}\, ds \\ &\leq \frac{\bar{M}N_0(b-a)h^{n+1}}{[1 - \bar{M}\bar{L}(b-a)e^h](n+2)!}. \end{aligned} \tag{48}$$

One can see from (48) that the nonlinear system (39) is convergent with $n \to +\infty$ and $h \to 0$, respectively.

In the following, we define a sequence of numerical integration operators $\bar{\boldsymbol{\kappa}}_n$ such as

$$(\bar{\boldsymbol{\kappa}}_n \varphi)(x) = h \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{j!} \cdot \frac{d^j \psi}{dy^j}\bigg|_{(x_q, \varphi_q^{(j)})} \int_0^1 K(x, x_q + hs) \cdot s^j \, ds. \tag{49}$$

From Theorem 7, it is seen that

$$\begin{aligned} &\left\| \varphi(x) - \varphi_{m,n}(x) \right\|_\infty \\ &\quad= \left\| (\boldsymbol{\kappa}\varphi)(x) - (\bar{\boldsymbol{\kappa}}_n\varphi)(x) \right\|_\infty \\ &\quad= \left\| (\boldsymbol{\kappa}\varphi)(x) - (\boldsymbol{\kappa}_n\varphi)(x) + (\boldsymbol{\kappa}_n\varphi)(x) - (\bar{\boldsymbol{\kappa}}_n\varphi)(x) \right\|_\infty \\ &\quad\leq \left\| (\boldsymbol{\kappa}\varphi)(x) - (\boldsymbol{\kappa}_n\varphi)(x) \right\|_\infty + \left\| (\boldsymbol{\kappa}_n\varphi)(x) - (\bar{\boldsymbol{\kappa}}_n\varphi)(x) \right\|_\infty \\ &\quad\leq \frac{M_0 N_0 (b-a)}{(n+2)!} h^{n+1} + h M_0 \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{(j+1)!} \left\| \frac{d^j \psi}{dy^j}\bigg|_{y=x_q} - \frac{d^j \psi}{dy^j}\bigg|_{(x_q, \varphi_q^{(j)})} \right\|_\infty \\ &\quad\leq \frac{M_0 N_0 (b-a)}{(n+2)!} h^{n+1} + h M_0 \sum_{q=0}^{m-1} \sum_{j=0}^{n} \frac{h^j}{(j+1)!} \cdot \sum_{\nu=0}^{j} L_\nu \left\| \varphi^{(\nu)}(x_q) - \varphi_q^{(\nu)} \right\|_\infty \\ &\quad\leq \frac{M_0 N_0 (b-a)}{(n+2)!} h^{n+1} + M_0 \bar{L}(b-a)e^h \cdot \|\Phi - \tilde{\Phi}\|_\infty \\ &\quad\leq \frac{M_0 N_0 (b-a) h^{n+1}}{[1 - \bar{M}\bar{L}(b-a)e^h](n+2)!}. \end{aligned} \tag{50}$$

It is found from (50) that when $n \to +\infty$ or $m \to +\infty$ (*i.e.* $h \to 0$), one always has $\|\varphi(x) - \varphi_{m,n}(x)\|_\infty \to 0$. This completes the proof. □

As shown in Theorem 8, one can choose a pair of feasible values for m (*i.e.* h) and n to obtain a good approximation of the exact solution. The above observations will be further verified by using the numerical examples in the next section.

4 Numerical results

In order to show the effectiveness of the proposed methods, we give two numerical examples corresponding to cases I and IV, respectively. The existence and uniqueness of the solution will be considered, and the approximate solution will be calculated numerically. All the computations are made by using the programming language of MATLAB (R2014).

Example 1 Assume that a second-order three-point boundary value problem is given as

$$\begin{cases} \varphi''(x) + e^{-x}\varphi^2(x) = (x^2 + x + 2)e^x, & x \in [0,1], \\ \varphi(0) = 0, \qquad \varphi(1) + 2\varphi(1/2) = e + e^{1/2}, \end{cases} \tag{51}$$

where $x \in [0,1]$, $\varphi(x) \in \mathbf{C}^\infty[0,1]$ and $|\varphi(x)| < 3$. The exact solution is $\varphi(x) = xe^x$.

According to Theorem 1, the nonlinear ordinary differential equation with three-point boundary value conditions in (51) can be transformed into the following Hammerstein integral equation:

$$\varphi(x) + \int_0^1 K_1(x,t)e^{-t}\varphi^2(t)\,dt = f_1(x), \tag{52}$$

where

$$K_1(x,t) = \begin{cases} \frac{t(3x-2)}{2}, & 0 \le t \le \min\{\frac{1}{2}, x\} \le 1, \\ \frac{x(t-1)}{2}, & 0 \le \max\{\frac{1}{2}, x\} < t \le 1, \\ \frac{x+tx-2t}{2}, & \frac{1}{2} \le t \le x \le 1, \\ \frac{x(3t-2)}{2}, & 0 \le x < t < \frac{1}{2}, \end{cases}$$

$$f_1(x) = \left(x^2 - 3x + 6\right)e^x + \frac{1}{4}\left(36 - 6e - 17e^{\frac{1}{2}}\right)x - 6.$$

We first prove the uniqueness of the solution and choose the Lipschitz constant as

$$L = \left\| \frac{\partial\psi(x,\varphi(x))}{\partial\varphi(x)} \right\| = \left\|2e^{-x}\varphi(x)\right\| < 6.$$

Then one obtains

$$\begin{aligned} \int_0^1\int_0^1 \left|K_1(x,t)\right|^2 dx\,dt &= \left(\int_0^{\frac{1}{2}}\int_t^1 + \int_{\frac{1}{2}}^1\int_t^1 + \int_{\frac{1}{2}}^1\int_0^t + \int_0^{\frac{1}{2}}\int_0^t\right)\left|K_1(x,t)\right|^2 dx\,dt \\ &= \int_0^{\frac{1}{2}}\int_t^1 \frac{t^2(3x-2)^2}{4}\,dx\,dt + \int_{\frac{1}{2}}^1\int_0^t \frac{x^2(t-1)^2}{4}\,dx\,dt \\ &\quad + \int_{\frac{1}{2}}^1\int_t^1 \frac{(x+tx-2t)^2}{4}\,dx\,dt + \int_0^{\frac{1}{2}}\int_0^t \frac{x^2(3t-2)^2}{4}\,dx\,dt \\ &= 0.0050347 = C_1^2. \end{aligned}$$

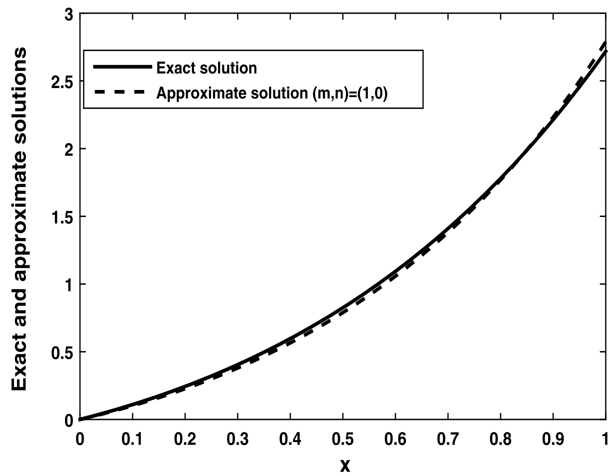

Figure 1 The exact solution $\varphi(x)$ and the approximate one $\varphi_{m,n}(x)$ with $(m,n) = (1,0)$.

Table 1 The absolute errors of the approximate and exact solutions for Example 1

x	$\varphi_{m,n}(x)$: The present method				$\varphi_h(x)$: The difference method	
	(2, 2)	(2, 4)	(4, 2)	(4, 4)	$h = 0.1$	$h = 0.05$
0.10	1.8921e–3	1.3455e–4	4.0629e–4	8.1435e–6	1.7128e–4	4.2829e–5
0.20	3.7502e–3	2.6842e–4	7.8165e–4	1.5896e–5	3.0443e–4	7.6114e–5
0.30	5.4142e–3	3.9569e–4	1.0209e–3	2.0120e–5	3.9356e–4	9.8358e–5
0.40	6.4733e–3	4.8748e–4	1.2069e–3	2.3084e–5	4.3201e–4	1.0798e–4
0.50	6.1339e–3	4.5086e–4	1.2227e–3	2.3171e–5	4.1224e–4	1.0301e–4
0.60	4.6732e–3	2.8768e–4	1.0183e–3	1.7900e–5	3.2672e–4	8.1720e–5
0.70	3.0518e–3	1.2016e–4	6.9688e–4	1.1472e–5	1.6693e–4	4.1672e–5
0.80	7.6299e–4	6.2865e–5	9.8850e–4	3.8926e–6	7.3872e–5	1.8509e–5
0.90	3.5063e–3	3.2838e–4	2.0438e–3	2.2064e–5	4.0280e–4	1.0080e–4
1.00	1.2268e–2	9.0172e–4	3.5652e–3	4.6343e–5	8.2447e–4	2.0603e–4

Since $LC_1 < 6\sqrt{0.0050347} = 0.42573 < 1$, the nonlinear three-point boundary value problem has a unique solution according to Theorem 6.

Now the proposed numerical method is applied to solve the obtained Hammerstein integral equation (52). It is seen that because $f_1(x)$ in (52) is explicit, the reformulation of the problem in (42) is unnecessary and the proposed numerical method is carried out directly. Similar to those in [28], it is convenient to write the parameters m and n in vector style (m,n). Figure 1 shows the exact solution and the approximate one with $(m,n) = (1,0)$. It is seen that the approximate solution is approached to the exact one. Moreover, the cases of $(m,n) = (2,2)$, $(m,n) = (2,4)$, $(m,n) = (4,2)$, and $(m,n) = (4,4)$ are chosen to compute, respectively. The absolute errors between the approximate and exact solutions are listed in Table 1. One can see from Table 1 that we have given a good approximation of the exact solution and the proposed numerical method is effective. On the other hand, the central difference format is used to compute the nonlinear boundary value problem and the obtained results are given in Table 1 for comparisons. A system of nonlinear algebraic equations is constructed and it is solved by using the Broyden iterative method [51]. The initial vectors are $\mathbf{x}_0 = [1,1,\ldots,1]^T_{10\times1}$ and $\mathbf{x}_0 = [1,1,\ldots,1]^T_{20\times1}$ for the step $h = 0.1$ and $h = 0.05$, respectively. The results are obtained by iterating five times. The comparisons reveal that the proposed method is effective.

Example 2 Consider a second-order three-point boundary value problem as follows:

$$\begin{cases} \varphi''(x) + \sin x \cdot e^{\varphi(x)} = e^{x^2} \sin x + 2, & x \in [0,1], \\ \varphi(0) - \varphi'(1/2) = -1, & \varphi(1) = 1. \end{cases} \tag{53}$$

where $x \in [0,1]$, $\varphi(x) \in \mathbf{C}^\infty[0,1]$ and $|\varphi(x)| \leq 1$. The exact solution is $\varphi(x) = x^2$.

Making use of Theorem 4, the boundary value problem (53) can be transformed into the following Hammerstein integral equation:

$$\varphi(x) + \int_0^1 K_4(x,t) \sin t \cdot e^{\varphi(t)}\, dt = f_4(x), \tag{54}$$

where

$$K_4(x,t) = \begin{cases} \frac{t(x-1)}{2}, & 0 \leq t \leq \min\{\frac{1}{2}, x\} \leq 1, \\ \frac{(x+1)(t-1)}{2}, & 0 \leq \max\{\frac{1}{2}, x\} < t \leq 1, \\ \frac{(x-1)(t+1)}{2}, & \frac{1}{2} \leq t \leq x \leq 1, \\ \frac{(xt-2x+t)}{2}, & 0 \leq x < t < \frac{1}{2}, \end{cases}$$

$$f_4(x) = \int_0^1 K_4(x,t)\bigl(e^{t^2} \sin t + 2\bigr)\, dt + x.$$

If we choose the Lipschitz constant as

$$L = \left\| \frac{\partial \psi(x,\varphi(x))}{\partial \varphi(x)} \right\| = \bigl\| \sin x \cdot e^{\varphi(x)} \bigr\| \leq e,$$

it follows that

$$\begin{aligned} \int_0^1 \int_0^1 \bigl|K_4(x,t)\bigr|^2\, dx\, dt &= \left(\int_0^{\frac{1}{2}} \int_t^1 + \int_{\frac{1}{2}}^1 \int_t^1 + \int_{\frac{1}{2}}^1 \int_0^t + \int_0^{\frac{1}{2}} \int_0^t \right) \bigl|K_4(x,t)\bigr|^2\, dx\, dt \\ &= \int_0^{\frac{1}{2}} \int_t^1 \frac{t^2(x-1)^2}{4}\, dx\, dt + \int_{\frac{1}{2}}^1 \int_0^t \frac{(x+1)^2(t-1)^2}{4}\, dx\, dt \\ &\quad + \int_{\frac{1}{2}}^1 \int_t^1 \frac{(x-1)^2(t+1)^2}{4}\, dx\, dt + \int_0^{\frac{1}{2}} \int_0^t \frac{(xt-2x+t)^2}{4}\, dx\, dt \\ &= 0.016840 = C_4^2. \end{aligned}$$

Under the consideration of $LC_4 \leq e\sqrt{0.016840} = 0.35275 < 1$, one can see that the three-point boundary value problem has a unique solution by using Theorem 6.

For numerical computations, as shown in (42), equation (54) should be rewritten as

$$\varphi(x) + \int_0^1 K_4(x,t)\bigl[\sin t \cdot e^{\varphi(t)} - e^{t^2} \sin t - 2\bigr]\, dt = x. \tag{55}$$

In addition, the exact solution and the approximate one for $(m,n) = (1,0)$ are depicted in Figure 2. The absolute errors of the exact and approximate solutions by using $(m,n) = (2,1)$, $(m,n) = (2,2)$, $(m,n) = (4,1)$, and $(m,n) = (4,2)$, and the central difference format are shown in Table 2. It is found from Figure 2 and Table 2 that a good approximation

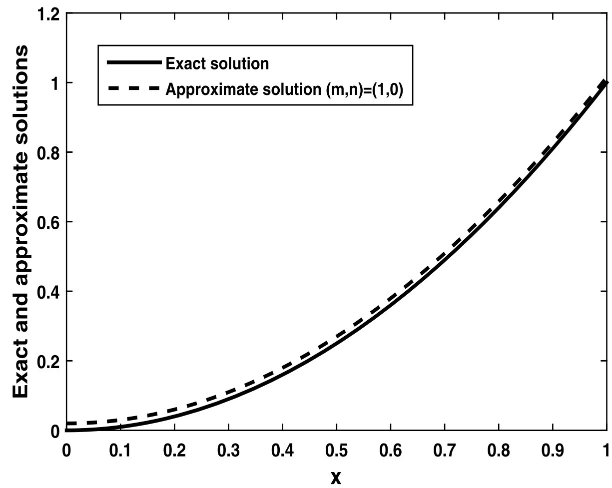

Figure 2 The exact solution $\varphi(x)$ and the approximate one $\varphi_{m,n}(x)$ with $(m,n) = (1,0)$.

Table 2 The absolute errors of the approximate and exact solutions for Example 2

x	$\varphi_{m,n}(x)$: The present method				$\varphi_h(x)$: The difference method	
	(2, 1)	(2, 2)	(4, 1)	(4, 2)	$h = 0.1$	$h = 0.05$
0.00	1.8390e–3	9.9247e–4	1.0502e–3	1.6925e–4	4.9821e–2	2.5019e–2
0.10	3.7770e–3	1.2501e–3	1.5834e–3	2.2288e–4	4.5369e–2	2.2780e–2
0.20	5.6959e–3	1.5063e–3	2.0998e–3	2.7613e–4	4.0870e–2	2.0517e–2
0.30	7.5115e–3	1.7571e–3	2.5446e–3	3.2734e–4	3.6285e–2	1.8212e–2
0.40	9.0093e–3	1.9876e–3	2.9475e–3	3.7623e–4	3.1581e–2	1.5848e–2
0.50	9.7699e–3	2.1512e–3	3.2074e–3	4.1161e–4	2.6730e–2	1.3411e–2
0.60	9.9119e–3	2.2428e–3	3.2926e–3	4.2675e–4	2.1712e–2	1.0891e–2
0.70	9.8280e–3	2.3012e–3	3.2046e–3	4.2358e–4	1.6516e–2	8.2834e–3
0.80	9.0100e–3	2.2273e–3	2.5986e–3	3.3970e–4	1.1146e–2	5.5885e–3
0.90	6.3775e–3	1.7078e–3	1.7315e–3	2.2813e–4	5.6224e–3	2.8184e–3

solution is determined. When m or n is increasing, the absolute error $|\varphi(x) - \varphi_{m,n}(x)|$ is decreasing. The observation is in accordance with the theoretical analysis in Theorem 8 and that in [28].

5 Conclusions

Four cases of nonlinear second-order three-point boundary value problems have been investigated and they are transformed into the Hammerstein integral equations by using the integration method. Based on the Schauder fixed point theorem, the sufficient conditions for the existence of the solutions have been given. The uniqueness of the solutions has been considered by using the Banach fixed point theorem. Furthermore, we have constructed the approximate solution of Hammerstein integral equations by applying a novel numerical method, which depends on the values of two parameters. The convergence and error estimate of the approximate solution have been made, and they show that one can get a good approximation of the exact solution by choosing a pair of the parameters. Two examples have been carried out numerically and the obtained results have revealed that the proposed methods are effective. In the future, the proposed method will be extended to solve nonlinear second-order differential equations with various nonlocal boundary value conditions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XCZ and QA made equal contributions to theory, computations, and writing of the article. HMW made main contributions to the numerical computations by using a difference format. All authors read and approved the final manuscript.

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