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General decay of solutions for Kirchhoff type containing Balakrishnan-Taylor damping with a delay and acoustic boundary conditions

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Abstract
In this paper, we are concerned with the general decay result of the quasi-linear wave equation for Kirchhoff type containing Balakrishnan-Taylor damping with a delay in the boundary feedback and acoustic boundary conditions.

MSC: 35L70; 35B40; 93D15

Keywords: general decay; Kirchhoff type; Balakrishnan-damping; boundary feedback; acoustic boundary condition

1 Introduction

Let Ω be a bounded domain of R^n, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ represents the unit outward normal to Γ. In this paper, we are concerned with the general decay of solutions of the quasi-linear wave equation for Kirchhoff type containing Balakrishnan-Taylor damping with a delay and acoustic boundary condition,

$$|u_t|^\rho u_{tt} - \left(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^q u_t = |u|^p u \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \tag{1.2}$$

$$\left(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\frac{\partial u}{\partial \nu} + \frac{\partial u_{tt}}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,ds + \mu_0 u_t(x,t) + \mu_1 u_t\left(x, t-\tau(t)\right) = h(x) y_t \tag{1.3}$$

on $\Gamma_1 \times (0, \infty)$,

$$u_t + f(x) y_t + m(x) y = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.5}$$

$$y(x,0) = y_0(x) \quad \text{in } \Gamma_1, \tag{1.6}$$

$$u_t(x, t - \tau(t)) = f_0(x, t) \quad \text{in } \Gamma_1, -\tau(0) \le t \le 0, \tag{1.7}$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and $\Omega \subset R^n$, $n \ge 1$ is a bounded domain with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here $a, b, \sigma, \rho, p, q > 0$, the functions $f, m, h : \Gamma_1 \to R$ are essential bounded, g represents the kernel of the memory term, μ_0, μ_1 are real numbers with $\mu_0 > 0, \mu_1 \neq 0, \tau(t) > 0$ represents the time-varying delay and initial datum (u_0, u_1, f_0, y_0) belongs to a suitable space.

System (1.1)-(1.7) represents a nonlinear viscoelastic equation for Kirchhoff type containing Balakrishnan-Taylor damping with a time-varying delay and acoustic boundary conditions. The physical applications of the above system is related to the problem of noise control and suppression in practical applications. The noise sound propagates through some acoustic medium, for example, through air, in a room which is characterized by a bounded domain Ω whose walls, ceiling, and floor are described by the boundary conditions. This is the description of Wu in [1]. For a more physical explanation of the viscoelastic wave equations with acoustic boundary conditions, we refer the reader to [2–5]. The acoustic boundary conditions were introduced by Beale and Rosencrans in [6, 7], where the authors proved the global existence and regularity of the linear problem. Time delays so often arise in many physical, chemical, biological, thermal, and economical phenomena because this phenomena depend not only on the present state but also on the past history of the system in a more complicated way. In recent years, differential equations with time delay effects have become an active area of research; see for example [8] and the references therein. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary; see [9–11]. For instance in [9], the authors proved the boundary stabilization of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback. For the model at hand, with Balakrishnan-Taylor damping ($\sigma \neq 0$) and $g = 0$, equation (1.1) is used to study the flutter panel equation and to the spillover problem, which was initially proposed by Balakrishnan and Taylor in 1989 [12], and Bass and Zes [13]. The related problems also were of concerned to You [14], Clark [15], Tatar and Zarai [16, 17], Mu *et al.* [18] and Lee *et al.* [19]. In particular, Wu [20] considered the following with Balakrishnan-Taylor damping and boundary conditions:

$$\begin{aligned}
&u_{tt} - \left(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = |u|^{p-1}u \quad \text{in } \Omega \times (0, +\infty),\\
&u = 0 \quad \text{on } \Gamma_0 \times (0, \infty),\\
&\left(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\right)\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,ds + \alpha u_t = |u|^{k-1}u \quad \text{on } \Gamma_1 \times (0, \infty),\\
&u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \quad \text{in } \Omega.
\end{aligned}$$

The author studied the general decay of solutions for a viscoelastic equation with Balakrishnan-Taylor damping. Zhang *et al.* [21], studied the global existence and asymptotic behavior of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback as follows:

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + a u_t(x, t - \tau(t)) = 0, \quad x \in \Omega, t > 0,$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,\infty),$$
$$\frac{\partial u}{\partial \nu} + g\bigl(u_t(x,t)\bigr) = 0 \quad \text{on } \Gamma_1 \times [0,\infty),$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega,$$
$$u_t\bigl(x, t-\tau(t)\bigr) = f_0(x,t), \quad x \in \Omega, -\tau(0) \le t \le 0.$$

Recently, Boukhatem and Benyatton in [22] were concerned with the local, uniqueness, global solution, and the decay of energy solution of the following model:

$$\begin{aligned}
&u_{tt} + Lu - \int_0^t g(t-s)Lu(s)\,ds = |u|^{p-2}u \quad \text{in } \Omega \times (0,\infty),\\
&u = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
&\frac{\partial u}{\partial \nu_L} - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu_L}\,ds + \mu_1 k_1\bigl(u_t(x,t)\bigr)\\
&\quad + \mu_2 k_2\bigl(u_t\bigl(x,t-\tau(t)\bigr)\bigr) = h(x)z_t \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u_t + f(x)z_t + m(x)z = 0 \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega,\\
&z(x,0) = z_0(x) \quad \text{on } \Gamma_1.
\end{aligned}$$

Motivated by previous work, in this paper, we study the general decay of solutions for Kirchhoff type containing Balakrishnan-Taylor damping with a time-varying delay and acoustic boundary conditions. This is done by applying the idea presented in [23] with some necessary modification due to the nature of the problem treated here. To the best of our knowledge, there are no results for Kirchhoff type equations containing Balakrishnan-Taylor damping with a delay and acoustic boundary conditions. Thus this work is meaningful. The plan of this paper is as following. In Section 2, we give some notation and material for our work. In Section 3, we prove the main result.

2 Preliminaries

In this section, we present some material that we shall use in order to present our result. Let $(\cdot,\cdot)$ be the scalar product in $L^2(\Omega)$, *i.e.*,

$$(u,v) = \int_\Omega u(x)v(x)\,dx,$$

and the corresponding norm $\|\cdot\|$, *i.e.*, $\|u\|^2 = (u,u)$. Also, we mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$ and by $\|\cdot\|_{q,\Gamma_1} : \|\cdot\|_{\Gamma_1}$ the $L^q(\Gamma_1)$ norm. We denote by

$$V = \bigl\{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\bigr\}$$

the closed subspace of $H^1(\Omega)$ equipped with the norm equivalent to the usual norm in $H^1(\Omega)$. The Poincaré inequality holds in V, *i.e.*, $\forall u \in V$, there exists a constant C_* such that

$$\|u\|_p \le C_* \|\nabla u\|, \quad 2 \le p \le \frac{2N}{N-2}, \tag{2.1}$$

and there exists a constant $\tilde{C}_* > 0$ such that

$$\|u\|_{\Gamma_1} \leq \tilde{C}_* \|\nabla u\|, \quad \forall u \in V. \tag{2.2}$$

For our study of problem (1.1)-(1.7) we will need the following assumptions. First, we assume that ρ and q satisfy

$$0 < \rho, q \leq \frac{2}{N-2} \quad \text{if } N \geq 3, \qquad \rho, q > 0 \quad \text{if } N = 1, 2, \tag{2.3}$$

and p satisfies

$$0 < p \leq \frac{4}{N-2} \quad \text{if } N \geq 3, \qquad 2 < p \quad \text{if } N = 1, 2. \tag{2.4}$$

With regard to the relation function $g(t)$, we assume that it verifies:

(H1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \quad a - \int_0^\infty g(s)\,ds := l > 0, \tag{2.5}$$

and there exist a positive non-increasing C^1 function $\xi : [0,\infty) \to [0,\infty)$ and a positive constant k such that

$$\begin{aligned} &g'(t) \leq -\zeta(t) g(t), \qquad \left|\frac{\zeta'(t)}{\zeta(t)}\right| \leq k, \qquad \zeta'(t) \leq 0, \\ &\zeta(t) > 0, \qquad \int_0^\infty \zeta(s)\,ds = \infty, \quad \forall t \geq 0. \end{aligned} \tag{2.6}$$

(H2) There exist three positive constants f_{0*}, m_0, and h_0 such that

$$f \geq f_{0*}, \qquad m \geq m_0, \quad \text{and} \quad h \geq h_0. \tag{2.7}$$

(H3) For the time-varying delay, we assume as in [9] that there exist positive constants $\tau_0, \bar{\tau}$ such that

$$0 < \tau_0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t > 0. \tag{2.8}$$

Furthermore, we assume that the delay satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0, \tag{2.9}$$

that

$$\tau \in W^{2,\infty}\big([0,T]\big), \quad \forall T > 0, \tag{2.10}$$

and that μ_0, μ_1 satisfy

$$|\mu_1| < \sqrt{1-d}\,\mu_0. \tag{2.11}$$

As in [9], let us introduce the function

$$z(x,\rho,t) = u_t\bigl(x,t-\tau(t)\rho\bigr), \quad x \in \Omega, \rho \in (0,1), t > 0. \tag{2.12}$$

Then problem (1.1)-(1.7) is equivalent to

$$\begin{aligned}
&|u_t|^\rho u_{tt} - \bigl(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\bigr)\Delta u - \Delta u_{tt} \\
&\quad + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^q u_t = |u|^p u \quad \text{in } \Omega \times (0,+\infty), && (2.13)\\
&u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), && (2.14)\\
&\bigl(a + b\|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)\bigr)\frac{\partial u}{\partial \nu} + \frac{\partial u_{tt}}{\partial \nu} \\
&\quad - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,ds + \mu_0 u_t(x,t) + \mu_1 z(x,1,t) = h(x)y_t \quad \text{on } \Gamma_1 \times (0,\infty), && (2.15)\\
&\tau(t)z_t(x,\rho,t) + \bigl(1-\tau'(t)\rho\bigr)z_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_1 \times (0,1) \times (0,\infty), && (2.16)\\
&u_t + f(x)y_t + m(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), && (2.17)\\
&u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, && (2.18)\\
&y(x,0) = y_0(x) \quad \text{on } \Gamma_1, && (2.19)\\
&z(x,\rho,0) = f_0\bigl(x,-\rho\tau(0)\bigr) \quad \text{on } \Gamma_1 \times (0,1). && (2.20)
\end{aligned}$$

We now state the local existence result of problem (2.13)-(2.20), which can be established by combining with the argument of [6].

Theorem 2.1 *Suppose that* (2.3), (2.4), (H1)-(H3) *hold and that* $(u_0,u_1) \in H^2(\Omega) \cap V \times V, y_0 \in L^2(\Gamma_1)$, *and* $f_0 \in L^2(\Gamma_1 \times (0,1))$.

Then, for any $T > 0$, *there exists a unique solution* (u,y,z) *of problem* (2.13)-(2.20) *on* $[0,T]$ *such that*

$$\begin{aligned}
&u \in L^\infty\bigl(0,T;H^2(\Omega)\cap V\bigr), \qquad u_t \in L^\infty(0,T;V) \cap L^{q+2}\bigl(\Omega \times (0,T)\bigr) \cap L^2\bigl(\Gamma_1 \times (0,1)\bigr),\\
&h^{1/2}y \in L^\infty\bigl(0,T;L^2(\Gamma_1)\bigr), \qquad h^{1/2}y_t \in L^2\bigl(0,T;L^2(\Gamma_1)\bigr).
\end{aligned}$$

3 Main result

In this section, we shall state and prove our main result. For this purpose, we define

$$\begin{aligned}
J(t) = {}& \frac{1}{2}\left(a - \int_0^t g(s)\,ds\right)\bigl\|\nabla u(t)\bigr\|^2 + \frac{b}{4}\bigl\|\nabla u(t)\bigr\|^4 + \frac{1}{2}\bigl\|\nabla u_t(t)\bigr\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\
&+ \frac{\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds + \frac{1}{2}\int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma \\
&- \frac{1}{p+2}\bigl\|u(t)\bigr\|_{p+2}^{p+2}
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned} I(t) &= \left(a - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{b}{2}\|\nabla u(t)\|^4 + \|\nabla u_t(t)\|^2 + (g \circ \nabla u)(t) \\ &\quad + \xi \int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds + \int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma - \|u(t)\|_{p+2}^{p+2}, \end{aligned} \tag{3.2}$$

where $(g \circ \nabla u)(t) = \int_\Omega \int_0^t g(t-s)|u(t)-u(s)|^2\,ds\,dx$. We denote the modified energy functional $E(t)$ associated with problem (2.13)-(2.20) by

$$\begin{aligned} E(t) &= \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(a - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{b}{4}\|\nabla u(t)\|^4 \\ &\quad + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds \\ &\quad + \frac{1}{2}\int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma - \frac{1}{p+2}\|u(t)\|_{p+2}^{p+2} \\ &= \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + J(t), \end{aligned} \tag{3.3}$$

where ξ, λ are suitable positive constants.

Next, we will fix ξ such that

$$\begin{aligned} &2\mu_0 - \frac{|\mu_1|}{\sqrt{1-d}} - \xi > 0, \qquad \xi - \frac{|\mu_1|}{\sqrt{1-d}} > 0, \\ &\lambda < \frac{1}{\bar{\tau}}\left|\log \frac{|\mu_1|}{\xi\sqrt{1-d}}\right|. \end{aligned} \tag{3.4}$$

Lemma 3.1 *Let* (2.8)-(2.11) *be satisfied and g satisfy* (2.5). *Then for the solution of problem* (2.13)-(2.20), *the energy functional defined by* (3.3) *satisfies*

$$\begin{aligned} E'(t) &\le -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 \\ &\quad - \int_\Omega |u_t(t)|^{q+2}\,dx - C_1 \int_{\Gamma_1}\left[u_t^2(x,t) + u_t^2\big(x, t-\tau(t)\big)\right] d\Gamma \\ &\quad - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds - \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma, \end{aligned} \tag{3.5}$$

for some positive constant C_1.

Proof Differentiating (3.3) and using (2.14), we have

$$\begin{aligned} E'(t) &= \int_\Omega |u_t(t)|^{\rho+1} u_{tt}(t)\,dx - \frac{1}{2}g(t)\|\nabla u(t)\|^2 \\ &\quad + \left(a - \int_0^t g(s)\,ds\right)\int_\Omega \nabla u(t)\nabla u_t(t)\,dx + \frac{b}{2}\|\nabla u(t)\|^2 \int_\Omega \nabla u(t)\nabla u_t(t)\,dx \\ &\quad + \int_\Omega \nabla u_t(t)\nabla u_{tt}(t)\,dx + \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\big(\nabla u(t) - \nabla u(s)\big)\,dx\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega \bigl|\nabla u(t)-\nabla u(s)\bigr|^2\,dx\,ds+\frac{\xi}{2}\int_{\Gamma_1} u_t^2(x,t)\,d\Gamma \\
&-\frac{\xi}{2}\int_{\Gamma_1} e^{-\lambda\tau(t)}u_t^2\bigl(x,t-\tau(t)\bigr)\bigl(1-\tau'(t)\bigr)\,d\Gamma-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds \\
&+\int_{\Gamma_1} h(x)m(x)y(t)y_t(t)\,d\Gamma-\int_\Omega \bigl|u(t)\bigr|^{p+1}u_t(t)\,dx \\
={}&\int_\Omega u_t(t)\biggl[\bigl(a+b\bigl\|\nabla u(t)\bigr\|^2+\sigma\bigl(\nabla u(t),\nabla u_t(t)\bigr)\bigr)\Delta u(t)+\Delta u_{tt}(t) \\
&-\int_0^t g(t-s)\Delta u(s)\,ds-\int_\Omega \bigl|u_t(t)\bigr|^q u_t(t)\,dx+\int_\Omega \bigl|u(t)\bigr|^p u(t)\biggr]\,dx \\
&-\frac{1}{2}g(t)\bigl\|\nabla u(t)\bigr\|^2+\biggl(a-\int_0^t g(s)\,ds\biggr)\int_\Omega \nabla u(t)\nabla u_t(t)\,dx \\
&+\frac{b}{2}\bigl\|\nabla u(t)\bigr\|^2\int_\Omega \nabla u(t)\nabla u_t(t)\,dx+\int_\Omega \nabla u_t(t)\nabla u_{tt}(t)\,dx \\
&+\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\bigl(\nabla u(t)-\nabla u(s)\bigr)\,dx\,ds \\
&+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega \bigl|\nabla u(t)-\nabla u(s)\bigr|^2\,dx\,ds+\frac{\xi}{2}\int_{\Gamma_1} u_t^2(t)\,d\Gamma \\
&-\frac{\xi}{2}\int_{\Gamma_1} e^{-\lambda\tau(t)}u_t^2\bigl(x,t-\tau(t)\bigr)\bigl(1-\tau'(t)\bigr)\,d\Gamma-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds \\
&+\int_{\Gamma_1} h(x)m(x)y(t)y_t(t)\,d\Gamma-\int_\Omega \bigl|u(t)\bigr|^{p+1}u_t(t)\,dx \\
={}&\int_{\Gamma_1} h(x)y_t(t)u_t(t)\,d\Gamma-\mu_0\int_{\Gamma_1} u_t(x,t)u_t(t)\,d\Gamma-\mu_1\int_{\Gamma_1} u_t\bigl(x,t-\tau(t)\bigr)u_t(t)\,d\Gamma \\
&-\sigma\biggl(\frac{1}{2}\frac{d}{dt}\bigl\|\nabla u(t)\bigr\|^2\biggr)^2-\int_\Omega \bigl|u_t(t)\bigr|^{q+2}\,dx-\frac{1}{2}g(t)\int_\Omega \bigl|\nabla u(t)\bigr|^2\,dx \\
&+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega \bigl|\nabla u(t)-\nabla u(s)\bigr|^2\,dx\,ds+\frac{\xi}{2}\int_{\Gamma_1} u_t^2(x,t)\,d\Gamma \\
&-\frac{\xi}{2}\int_{\Gamma_1} e^{-\lambda\tau(t)}u_t^2\bigl(x,t-\tau(t)\bigr)\bigl(1-\tau'(t)\bigr)\,d\Gamma-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds \\
&+\int_{\Gamma_1} h(x)m(x)y(t)y_t(t)\,d\Gamma. \qquad (3.6)
\end{aligned}
$$

Applying Young's inequality, we obtain

$$
\begin{aligned}
&-\mu_1\int_{\Gamma_1} u_t(x,t)u_t\bigl(x,t-\tau(t)\bigr)\,d\Gamma \\
&\quad\le \frac{|\mu_1|}{2\sqrt{1-d}}\int_{\Gamma_1} u_t^2(x,t)\,d\Gamma+\frac{|\mu_1|\sqrt{1-d}}{2}\int_{\Gamma_1} u_t^2\bigl(x,t-\tau(t)\bigr)\,d\Gamma, \qquad (3.7)
\end{aligned}
$$

and using (2.7), we get

$$
\int_{\Gamma_1} h(x)y_t(t)u_t(t)\,d\Gamma+\int_{\Gamma_1} f(x)h(x)y_t^2(t)\,d\Gamma=-\int_{\Gamma_1} h(x)m(x)y_t(t)y(t)\,d\Gamma. \qquad (3.8)
$$

Thus, from (3.6)-(3.8) and assumptions (2.8), (2.9), and (3.4), we arrive at

$$
\begin{aligned}
E'(t) \le{}& -\mu_0 \int_{\Gamma_1} u_t^2(t)\,d\Gamma + \frac{|\mu_1|}{2\sqrt{1-d}} \int_{\Gamma_1} u_t^2(t)\,d\Gamma + \frac{|\mu_1|\sqrt{1-d}}{2} \int_{\Gamma_1} u_t^2\bigl(x,t-\tau(t)\bigr)\,d\Gamma \\
& - \sigma\left(\frac{1}{2}\frac{d}{dt}\bigl\|\nabla u(t)\bigr\|^2\right)^2 - \int_\Omega \bigl|u_t(t)\bigr|^{q+2}\,dx + \frac{1}{2}\bigl(g'\circ\nabla u\bigr)(t) - \frac{1}{2}g(t)\bigl\|\nabla u(t)\bigr\|^2 \\
& + \frac{\xi}{2}\int_{\Gamma_1} u_t^2(t)\,d\Gamma - \frac{\xi}{2}\int_{\Gamma_1} e^{-\lambda\tau(t)} u_t^2\bigl(x,t-\tau(t)\bigr)\bigl(1-\tau'(t)\bigr)\,d\Gamma \\
& - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds - \int_{\Gamma_1} f(x)h(x)y_t^2(t)\,d\Gamma \\
\le{}& -\sigma\left(\frac{1}{2}\frac{d}{dt}\bigl\|\nabla u(t)\bigr\|^2\right)^2 - \int_\Omega \bigl|u_t(t)\bigr|^{q+2}\,dx + \frac{1}{2}\bigl(g'\circ\nabla u\bigr)(t) - \frac{1}{2}g(t)\bigl\|\nabla u(t)\bigr\|^2 \\
& - \left(\mu_0 - \frac{|\mu_1|}{2\sqrt{1-d}} - \frac{\xi}{2}\right)\int_{\Gamma_1} u_t^2(t)\,d\Gamma \\
& - \left(\frac{\xi}{2}(1-d)e^{-\lambda\bar{\tau}} - \frac{|\mu_1|\sqrt{1-d}}{2}\right)\int_{\Gamma_1} u_t^2\bigl(x,t-\tau(t)\bigr)\,d\Gamma \\
& - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds - \int_{\Gamma_1} f(x)h(x)y_t^2(t)\,d\Gamma \\
\le{}& -\sigma\left(\frac{1}{2}\frac{d}{dt}\bigl\|\nabla u(t)\bigr\|^2\right)^2 - \int_\Omega \bigl|u_t(t)\bigr|^{q+2}\,dx + \frac{1}{2}\bigl(g'\circ\nabla u\bigr)(t) - \frac{1}{2}g(t)\bigl\|\nabla u(t)\bigr\|^2 \\
& - C_1\int_{\Gamma_1}\bigl[u_t(x,t) + u_t^2\bigl(x,t-\tau(t)\bigr)\bigr]\,d\Gamma - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds \\
& - \int_{\Gamma_1} f(x)h(x)y_t^2(t)\,d\Gamma,
\end{aligned}
$$

for some positive constant C_1. □

Lemma 3.2 *Let* $(u_0,u_1) \in (H^2(\Omega)\cap V)\times V, y_0 \in L^2(\Gamma_1), f_0 \in L^2(\Gamma_1 \times [-\tau(0),0])$, *and* $(u(t),y(t),z(t))$ *be the solution of* (2.13)-(2.20). *If* $I(0) > 0$ *and*

$$
\alpha = \frac{C_*^{p+2}}{l}\left(\frac{2(p+2)}{lp}E(0)\right)^{p/2} < 1 \tag{3.9}
$$

then $I(t) > 0$ *for* $t \in [0,T]$, *where* $I(t)$ *is defined in* (3.2).

Proof Since $I(0) > 0$, there exists (by continuity of $u(t)$) $T^* < T$ such that

$$
I(t) \ge 0, \tag{3.10}
$$

for all $t \in [0,T^*]$. Then (3.1), (3.2), and (3.10) give

$$
\begin{aligned}
J(t) ={}& \frac{p}{2(p+2)}\biggl[\left(a - \int_0^t g(s)\,ds\right)\bigl\|\nabla u(t)\bigr\|^2 + \frac{b}{2}\bigl\|\nabla u(t)\bigr\|^4 + \bigl\|\nabla u_t(t)\bigr\|^2 + (g\circ\nabla u)(t) \\
& + \xi\int_{t-\tau(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u_t^2(s)\,d\Gamma\,ds + \int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma\biggr] + \frac{1}{p+2}I(t)
\end{aligned}
$$

$$\geq \frac{p}{2(p+2)}\left[\left(a-\int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2+\frac{b}{2}\left\|\nabla u(t)\right\|^4+\left\|\nabla u_t(t)\right\|^2+(g\circ\nabla u)(t)\right.$$
$$\left.+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\right]. \tag{3.11}$$

Hence from (2.5), (3.3), (3.11), and Lemma 3.1, we can deduce that

$$l\left\|\nabla u(t)\right\|^2\leq\left(a-\int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2\leq\frac{2(p+2)}{p}J(t)$$
$$\leq\frac{2(p+2)}{p}E(t)\leq\frac{2(p+2)}{p}E(0),\quad\forall t\in\left[0,T^*\right]. \tag{3.12}$$

Exploiting (2.1), (3.9), and (3.12), we obtain

$$\left\|u(t)\right\|_{p+2}^{p+2}\leq C_*^{p+2}\left\|\nabla u(t)\right\|^{p+2}\leq\frac{C_*^{p+2}}{l}\left(\frac{2(p+2)}{lp}E(0)\right)^{\frac{p}{2}}l\left\|\nabla u(t)\right\|^2$$
$$\leq\alpha l\left\|\nabla u(t)\right\|^2\leq\left(a-\int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2,\quad\forall t\in\left[0,T^*\right].$$

Consequently, we get

$$I(t)=\left(a-\int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|^2+\frac{b}{2}\left\|\nabla u(t)\right\|^4+\left\|\nabla u_t(t)\right\|^2+(g\circ\nabla u)(t)$$
$$+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma$$
$$-\left\|u(t)\right\|_{p+2}^{p+2}>0,\quad\forall t\in\left[0,T^*\right]. \tag{3.13}$$

Repeat this procedure and use the fact that

$$\lim_{t\to T^*}\frac{C_*^{p+2}}{l}\left[\frac{2(p+2)}{lp}E(t)\right]^{\frac{p}{2}}\leq\alpha<1.$$

We can take $T^*=T$. Thus the proof is complete. □

Theorem 3.1 *Suppose that* (2.1)-(2.4), (2.9)-(2.11), *and* (H1)-(H3) *hold. If* $(u_0,u_1)\in(H^2(\Omega)\cap V)\times V, y_0\in L^2(\Gamma_1), f_0\in L^2(\Gamma_1\times[-\tau(0),0])$ *and* (3.9) *is satisfied, then the solution* $(u(t),y(t),z(t))$ *of* (2.13)-(2.20) *is bounded and global in time.*

Proof It suffices to show that

$$\left\|\nabla u(t)\right\|^2+\left\|\nabla u_t(t)\right\|^2+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma$$

is bounded independent of t. Under the hypotheses in Theorem 3.1, we see from Lemma 3.2 that $I(t)>0$ for all $t\geq 0$. Therefore

$$\begin{aligned}
J(t) &= \frac{p}{2(p+2)}\bigg[\bigg(a-\int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|^2+\frac{b}{2}\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2+(g\circ\nabla u)(t)\\
&\quad+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\bigg]+\frac{1}{p+2}I(t)\\
&\ge \frac{p}{2(p+2)}\bigg[\bigg(a-\int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|^2+\frac{b}{2}\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2\\
&\quad+(g\circ\nabla u)(t)+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\bigg].
\end{aligned}$$

Hence by (H1) and the fact that $(g\circ\nabla u)(t)>0$, we can deduce that

$$\begin{aligned}
&l\|\nabla u(t)\|^2+\frac{b}{2}\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2\\
&\qquad+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\\
&\quad\le\bigg(a-\int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|^2+\frac{b}{2}\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2\\
&\qquad+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\\
&\quad\le\frac{2(p+2)}{p}J(t),\quad\forall t\in[0,T].
\end{aligned}\tag{3.14}$$

Using Lemma 3.1 and (3.14), it follows that

$$\begin{aligned}
&\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{p}{2(p+2)}\bigg(l\|\nabla u(t)\|^2+\frac{b}{2}\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2\\
&\qquad+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\bigg)\\
&\quad\le\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+J(t)=E(t)\le E(0).
\end{aligned}$$

Thus, there exists a constant $C>0$ depending p and l such that

$$\begin{aligned}
&\|\nabla u(t)\|^2+\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2\\
&\qquad+\xi\int_{t-\tau(t)}^t\int_{\Gamma_1}e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds+\int_{\Gamma_1}m(x)h(x)y^2(t)\,d\Gamma\\
&\quad\le CE(t)\le CE(0)<+\infty.
\end{aligned}$$

This implies that the solution $(u(t),y(t),z(t))$ of (2.13)-(2.20) is bounded and global in time. □

Now, we define

$$L(t)=ME(t)+\varepsilon\Psi(t)+\Phi(t),\tag{3.15}$$

where M and ε are positive constants which will be specified later and

$$\begin{aligned}\Psi(t) &= \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t)u(t)\,dx + \frac{\sigma}{4}\left\|\nabla u(t)\right\|^4 + \int_\Omega \nabla u_t(t)\nabla u(t)\,dx \\ &\quad + \int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma + \frac{1}{2}\int_{\Gamma_1} h(x)f(x)y^2(t)\,d\Gamma, \end{aligned} \tag{3.16}$$

$$\begin{aligned}\Phi(t) &= -\frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &\quad - \int_\Omega \nabla u_t(t)\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx. \end{aligned} \tag{3.17}$$

Before we prove our main result, we need the following lemmas.

Lemma 3.3 *Let $u \in L^\infty([0,T];H_0^1(\Omega))$, then for any $\rho \geq 0$, we have*

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)^{\rho+2} dx \\ &\quad \leq (a-l)^{\rho+1}C_*^{\rho+2}\left(\frac{4(p+2)E(0)}{lp}\right)^{\rho/2}(g\circ\nabla u)(t). \end{aligned} \tag{3.18}$$

Proof By the Hölder inequality, (2.1), (2.5), and (3.12), we can deduce

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)^{\rho+2} dx \\ &\quad \leq \int_\Omega \left(\int_0^t g(t-s)\,ds\right)^{\rho+1}\left(\int_0^t g(t-s)\left|u(t)-u(s)\right|^{\rho+2}ds\right)dx \\ &\quad \leq (a-l)^{\rho+1}C_*^{\rho+2}\int_0^t g(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^{\rho+2}ds \\ &\quad \leq (a-l)^{\rho+1}C_*^{\rho+2}\left(\frac{4(p+2)E(0)}{lp}\right)^{\rho/2}(g\circ\nabla u)(t). \end{aligned}$$

□

Lemma 3.4 *Let $(u(t),y(t),z(t))$ be a solution of* (2.13)-(2.20), *then there exist two constants β_1 and β_2 such that*

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t). \tag{3.19}$$

Proof By using (2.1), (3.12), and Young's inequality, we have

$$\begin{aligned}&\left|\frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t)u(t)\,dx\right| \\ &\quad \leq \frac{1}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+2)(\rho+1)}\left\|u(t)\right\|_{\rho+2}^{\rho+2} \\ &\quad \leq \frac{1}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+2)(\rho+1)}\left\|\nabla u(t)\right\|^{\rho+2} \\ &\quad \leq \frac{1}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+2)(\rho+1)}\left(\frac{2(p+2)E(0)}{lp}\right)^{\frac{\rho}{2}}\left\|\nabla u(t)\right\|^2 \\ &\quad \leq \frac{1}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} + \frac{\alpha_1}{(\rho+2)(\rho+1)}\left\|\nabla u(t)\right\|^2, \end{aligned} \tag{3.20}$$

where $\alpha_1 = C_*^{\rho+2}(\frac{2(p+2)E(0)}{lp})^{\frac{\rho}{2}}$,

$$\left|\int_\Omega \nabla u_t(t)\nabla u(t)\,dx\right| \le \frac{1}{2}\left(\|\nabla u_t(t)\|^2 + \|\nabla u(t)\|^2\right), \tag{3.21}$$

and by (2.2) and (2.7), we can deduce

$$\begin{aligned}\left|\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right| &= \int_{\Gamma_1} \frac{m(x)h(x)u(t)}{m(x)}y(t)\,d\Gamma \\ &\le \frac{\|h\|_\infty^{1/2}\|m\|_\infty^{1/2}}{m_0}\left(\int_{\Gamma_1} h(x)m(x)y^2(x)\,d\Gamma\right)^{1/2}\left(\int_{\Gamma_1}|u(t)|^2\,d\Gamma\right)^{1/2} \\ &\le \frac{\|h\|_\infty\|m\|_\infty}{2m_0^2}\int_{\Gamma_1} h(x)m(x)y^2(x)\,d\Gamma + \frac{\tilde{C}_*}{2}\|\nabla u(t)\|^2. \end{aligned} \tag{3.22}$$

Similarly, we obtain

$$\begin{aligned}&\left|-\frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\ &\quad\le \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+2)(\rho+1)}(a-l)^{\rho+1}\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^{\rho+2}\,ds \\ &\quad\le \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{(a-l)^{\rho+1}2^{\frac{\rho}{2}}\alpha_1}{(\rho+1)(\rho+2)}(g\circ\nabla u)(t)\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}&\left|-\int_\Omega \nabla u_t(t)\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\right| \\ &\quad\le \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{a-l}{2}(g\circ\nabla u)(t).\end{aligned} \tag{3.24}$$

Combining (3.15)-(3.17) and (3.20)-(3.24), we arrive at

$$\begin{aligned}&\big|L(t)-ME(t)\big| \\ &\quad\le \varepsilon\big|\Psi(t)\big| + \big|\Phi(t)\big| \\ &\quad\le \frac{\varepsilon}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{\varepsilon\alpha_1}{(\rho+2)(\rho+1)}\|\nabla u(t)\|^2 + \frac{\varepsilon\sigma}{4}\|\nabla u(t)\|^4 + \frac{\varepsilon}{2}\|\nabla u_t(t)\|^2 \\ &\qquad + \frac{\varepsilon}{2}\|\nabla u(t)\|^2 + \left(\frac{\varepsilon}{2m_0}\frac{\|h\|_\infty\|m\|_\infty}{m_0} + \|f\|_\infty\right)\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\ &\qquad + \frac{\varepsilon\tilde{C}_*}{2}\|\nabla u(t)\|^2 + \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{(a-l)^{\rho+1}2^{\frac{\rho}{2}}\alpha_1}{(\rho+1)(\rho+2)}(g\circ\nabla u)(t) \\ &\qquad + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{a-l}{2}(g\circ\nabla u)(t) \\ &\quad= \frac{\varepsilon+1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \varepsilon\left(\frac{\alpha_1}{(\rho+2)(\rho+1)} + \frac{1}{2} + \frac{\tilde{C}_*}{2}\right)\|\nabla u(t)\|^2 \\ &\qquad + \frac{\varepsilon\sigma}{4b}b\|\nabla u(t)\|^4 + \frac{\varepsilon+1}{2}\|\nabla u_t(t)\|^2 + \left(\frac{(a-l)^{\rho+1}2^{\rho/2}\alpha_1}{(\rho+1)(\rho+2)} + \frac{a-l}{2}\right)(g\circ\nabla u)(t)\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\varepsilon}{2m_0}\frac{\|h\|_\infty\|m\|_\infty}{m_0}+\|f\|_\infty\right)\int_{\Gamma_1}h(x)m(x)y^2(t)\,d\Gamma \\
&\leq CE(t),
\end{aligned}
$$

where C is some positive constant. Choose $M > 0$ sufficiently large and ε small, there exist two positive constants β_1 and β_2 such that

$$
\beta_1 E(t) \leq L(t) \leq \beta_2 E(t).
$$

Thus the proof is complete. □

Now, we state our main result.

Theorem 3.2 *Suppose that* (2.1)-(2.4), (2.9)-(2.11), *and* (H1)-(H3) *hold. If* $(u_0,u_1)\in(H^2(\Omega)\cap V)\times V, y_2\in L^2(\Gamma_1), f_0\in(L^2(\Gamma_1)\times[-\tau(0),0])$ *and* (3.9) *is satisfied. Then for each* $t>0$, *there exist positive constants* K *and* ν *such that the energy of the solution for problem* (2.13)-(2.20) *satisfies*

$$
E(t)\leq Ke^{-\nu\int_{t_0}^t\zeta(s)\,ds},\quad \forall t\geq t_0. \tag{3.25}
$$

Proof In order to obtain the energy result of $E(t)$, from Lemma 3.4, it suffices to prove that we have the estimate of $L(t)$. To this end, we need the derivative of $L(t)$. For this purpose, we estimate $\Psi'(t)$. It follows from (3.16) and equations (2.13)-(2.17) that

$$
\begin{aligned}
\Psi'(t) ={}& \int_\Omega\left|u_t(t)\right|^\rho u_{tt}(t)u(t)\,dx+\frac{1}{\rho+1}\int_\Omega\left|u_t(t)\right|^{\rho+2}dx \\
&+\sigma\int_\Omega\nabla u(t)\nabla u_t(t)\int_\Omega\nabla u(t)\nabla u(t)\,dx+\int_\Omega\nabla u_{tt}(t)\nabla u(t)\,dx+\int_\Omega\left|\nabla u_t(t)\right|^2dx \\
&+\int_{\Gamma_1}h(x)u_t(t)y(t)\,d\Gamma+\int_{\Gamma_1}h(x)u(t)y_t(t)\,d\Gamma+\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma \\
={}& \int_\Omega\left[\left|u_t(t)\right|^\rho u_{tt}(t)-\Delta u_{tt}(t)-\sigma\left(\nabla u_t(t),\nabla u(t)\right)\Delta u(t)\right]u(t)\,dx \\
&+\int_{\Gamma_1}\left(\frac{\partial u_{tt}(t)}{\partial\nu}+\sigma\left(\nabla u_t(t),\nabla u(t)\right)\frac{\partial u(t)}{\partial\nu}\right)u(t)\,d\Gamma \\
&+\frac{1}{\rho+1}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2}+\int_\Omega\left|\nabla u_t(t)\right|^2dx+\int_{\Gamma_1}h(x)u_t(t)y(t)\,d\Gamma \\
&+\int_{\Gamma_1}h(x)u(t)y_t(t)\,d\Gamma+\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma \\
&+\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma \\
={}& -\left(a+b\left\|\nabla u(t)\right\|^2\right)\int_\Omega\left|\nabla u(t)\right|^2dx+\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\
&-\int_\Omega\left|u_t(t)\right|^q u_t(t)u(t)\,dx+\int_\Omega\left|u(t)\right|^{p+2}dx \\
&+\int_{\Gamma_1}\left[\left(a+b\left\|\nabla u(t)\right\|^2+\sigma\left(\nabla u_t(t),\nabla u(t)\right)\right)\frac{\partial u(t)}{\partial\nu}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,dx + \frac{\partial u_{tt}(t)}{\partial \nu}\biggr] u(t)\,d\Gamma \\
&+ \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} + \int_\Omega |\nabla u_t(t)|^2\,dx + \int_{\Gamma_1} h(x)u_t(t)y(t)\,d\Gamma \\
&+ \int_{\Gamma_1} h(x)u(t)y_t(t)\,d\Gamma + \int_{\Gamma_1} h(x)f(x)y(t)y_t(t)\,d\Gamma \\
={}& -\bigl(a+b\|\nabla u(t)\|^2\bigr)\int_\Omega |\nabla u(t)|^2\,dx + \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\
&- \int_\Omega |u_t(t)|^q u_t(t)u(t)\,dx + \int_\Omega |u(t)|^{p+2}\,dx - \mu_0\int_{\Gamma_1} u_t(x,t)u(t)\,d\Gamma \\
&- \mu_1\int_{\Gamma_1} u_t\bigl(x,t-\tau(t)\bigr)u(t)\,d\Gamma + \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} + \int_\Omega |\nabla u_t(t)|^2\,dx \\
&+ 2\int_{\Gamma_1} h(x)u(t)y_t(t)\,d\Gamma - \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma.
\end{aligned}
\tag{3.26}
$$

Now, we estimate the right hand side of (3.26). By using (2.1), (2.2), (2.7), and Young's inequality, for any $\eta > 0$, we obtain

$$
\begin{aligned}
&\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\right| \\
&\quad= \int_\Omega\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\nabla u(t)\,ds\,dx + \int_0^t g(s)\,ds\int_\Omega |\nabla u(t)|^2\,dx \\
&\quad\le (1+\eta)\int_0^t g(s)\,ds\|\nabla u(t)\|^2 + \frac{1}{4\eta}(g\circ\nabla u)(t) \\
&\quad\le (1+\eta)(a-l)\|\nabla u(t)\|^2 + \frac{1}{4\eta}(g\circ\nabla u)(t),
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&\left|-\int_\Omega |u_t(t)|^q u_t(t)u(t)\,dx\right| \\
&\quad\le \eta C_*^2\int_\Omega |\nabla u(t)|^2\,dx + \frac{1}{4\eta}|u_t(t)|^{2(q+1)}\,dx \\
&\quad\le \eta C_*^2\int_\Omega |\nabla u(t)|^2\,dx + \frac{\alpha_2}{4\eta}\int_\Omega |\nabla u_t(t)|^2\,dx,
\end{aligned}
\tag{3.28}
$$

where $\alpha_2 = C_*^{2(q+1)}(\frac{2(p+2)E(0)}{p})^q$,

$$
\begin{aligned}
&\left|-\mu_0\int_{\Gamma_1} u_t(x,t)u(t)\,d\Gamma\right| \\
&\quad\le \eta\mu_0\tilde{C}_*^2\int_\Omega |\nabla u(t)|^2\,dx + \frac{\mu_0}{4\eta}\int_{\Gamma_1} u_t^2(t)\,d\Gamma,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
&\left|-\mu_1\int_{\Gamma_1} u_t\bigl(x,t-\tau(t)\bigr)u(t)\,d\Gamma\right| \\
&\quad\le \eta|\mu_1|\tilde{C}_*^2\int_\Omega |\nabla u(t)|^2\,dx + \frac{|\mu_1|}{4\eta}\int_{\Gamma_1} u_t^2\bigl(x,t-\tau(t)\bigr)\,d\Gamma,
\end{aligned}
\tag{3.30}
$$

and

$$\left|\int_{\Gamma_1} h(x)u(t)y_t(t)\,d\Gamma\right| \le \|h\|_\infty^{1/2}\left(\int_{\Gamma_1} h(x)y_t^2(t)\,d\Gamma\right)^{1/2}\left(\int_{\Gamma_1}\left|u(t)\right|^2 d\Gamma\right)^{1/2}$$
$$\le \eta\tilde{C}_*^2\int_\Omega\left|\nabla u(t)\right|^2 dx + \frac{\|h\|_\infty}{4\eta f_{0*}}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma. \tag{3.31}$$

Substitution of (3.27)-(3.31) into (3.26) yields

$$\begin{aligned}
\Psi'(t) \le{}& -\left(a + b\left\|\nabla u(t)\right\|^2\right)\int_\Omega\left|\nabla u(t)\right|^2 dx + (1+\eta)(a-l)\int_\Omega\left|\nabla u(t)\right|^2 dx \\
&+ \frac{1}{4\eta}(g\circ\nabla u)(t) + \eta C_*^2\int_\Omega\left|\nabla u(t)\right|^2 dx + \frac{\alpha_2}{4\eta}\int_\Omega\left|\nabla u_t(t)\right|^2 dx \\
&+ \int_\Omega\left|u(t)\right|^{p+2} dx + \eta\mu_0\tilde{C}_*^2\int_\Omega\left|\nabla u(t)\right|^2 dx + \frac{\mu_0}{4\eta}\int_{\Gamma_1} u_t^2(t)\,d\Gamma \\
&+ \eta|\mu_1|\tilde{C}_*^2\int_\Omega\left|\nabla u(t)\right|^2 dx + \frac{|\mu_1|}{4\eta}\int_{\Gamma_1} u_t^2\left(x,t-\tau(t)\right)d\Gamma \\
&+ \frac{1}{\rho+1}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} + \int_\Omega\left|\nabla u_t(t)\right|^2 dx + \eta\tilde{C}_*^2\int_\Omega\left|\nabla u(t)\right|^2 dx \\
&+ \frac{\|h\|_\infty}{2\eta f_{0*}}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma - \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\
={}& \frac{1}{\rho+1}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} - \left[\left(a + b\left\|\nabla u(t)\right\|^2\right) - (1+\eta)(a-l) - \eta C_*^2 - \eta\mu_0\tilde{C}_*^2\right. \\
&\left.- \eta|\mu_1|\tilde{C}_*^2 - \eta\tilde{C}_*^2\right]\left\|\nabla u(t)\right\|^2 + \left(\frac{\alpha_2}{4\eta}+1\right)\left\|\nabla u_t(t)\right\|^2 + \frac{1}{4\eta}(g\circ\nabla u)(t) \\
&+ \left\|u(t)\right\|_{p+2}^{p+2} + \frac{\mu_0}{4\eta}\int_{\Gamma_1} u_t^2(t)\,d\Gamma + \frac{|\mu_1|}{4\eta}\int_{\Gamma_1} u_t^2\left(x,t-\tau(t)\right)d\Gamma \\
&+ \frac{\|h\|_\infty}{2\eta f_{0*}}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma - \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma.
\end{aligned} \tag{3.32}$$

Next, we would like to estimate $\Phi'(t)$. Taking the derivative of $\Phi(t)$ in (3.17) and using (2.13)-(2.17), we can deduce that

$$\begin{aligned}
\Phi'(t) ={}& -\int_\Omega\left|u_t(t)\right|^\rho u_{tt}(t)\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&- \frac{1}{\rho+1}\int_\Omega\left|u_t(t)\right|^\rho u_t(t)\int_0^t g'(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&- \frac{1}{\rho+1}\int_\Omega\left|u_t(t)\right|^\rho u_t(t)\int_0^t g(t-s)u_t(t)\,ds\,dx \\
&- \int_\Omega \nabla u_{tt}(t)\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\,dx \\
&- \int_\Omega \nabla u_t(t)\int_0^t g'(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\,dx \\
&- \int_\Omega \nabla u_t(t)\int_0^t g(t-s)\nabla u_t(t)\,ds\,dx
\end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega \left[\left(a + b\left\|\nabla u(t)\right\|^2\right)\Delta u(t) + \sigma\left(\nabla u(t), \nabla u_t(t)\right)\Delta u(t) - \int_0^t g(t-s)u(s)\,ds \right.\\
&\quad \left. - \left|u_t(t)\right|^q u_t(t) + \left|u(t)\right|^p u(t)\right] \int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&\quad - \int_{\Gamma_1} \frac{\partial u_{tt}(t)}{\partial \nu} \int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,d\Gamma \\
&\quad - \int_\Omega \nabla u_t(t) \int_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \\
&\quad - \int_\Omega \nabla u_t(t) \int_0^t g(t-s)\nabla u_t(t)\,ds\,dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t) \int_0^t g'(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t) \int_0^t g(t-s)u_t(t)\,ds\,dx \\
&= \left(a + b\left\|\nabla u(t)\right\|^2\right)\int_\Omega \nabla u(t) \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \\
&\quad + \sigma \int_\Omega \nabla u(t)\nabla u_t(t) \int_\Omega \nabla u(t) \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \\
&\quad - \int_\Omega \int_0^t g(t-s)\nabla u(s)\,ds \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \\
&\quad - \int_{\Gamma_1} \left(h(x)y_t(t) - \mu_0 u_t(x,t) - \mu_1 u_t\left(x, t-\tau(t)\right)\right) \int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,d\Gamma \\
&\quad + \int_\Omega \left|u_t(t)\right|^q u_t(t) \int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&\quad - \int_\Omega \left|u(t)\right|^p u(t) \int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&\quad - \int_\Omega \nabla u_t(t) \int_0^t g'(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \\
&\quad - \int_\Omega \nabla u_t(t) \int_0^t g(t-s)\nabla u_t(t)\,ds\,dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t) \int_0^t g'(t-s)\left(u(t)-u(s)\right)ds\,dx \\
&\quad - \frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t) \int_0^t g(t-s)u_t(t)\,ds\,dx. \qquad (3.33)
\end{aligned}
$$

Now, we will estimate the right hand side of (3.33). From Lemma 3.3, (2.1), (2.2), (2.5), (3.5), (3.12), and Young's inequality, for any $\eta > 0$, we have the following inequalities:

$$
\begin{aligned}
&\left| \int_\Omega \left(a + b\left\|\nabla u(t)\right\|^2\right)\nabla u(t) \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \right| \\
&\quad \le \left| \int_\Omega \left(a + b\frac{2(p+2)}{lp}E(0)\right)\nabla u(t) \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right)ds\,dx \right| \\
&\quad \le \eta\left\|\nabla u(t)\right\|^2 + \frac{a-l}{4\eta}\left(a + b\frac{2(p+2)}{lp}E(0)\right)^2 (g \circ \nabla u)(t), \qquad (3.34)
\end{aligned}
$$

$$\begin{aligned}
&\left|\sigma \int_\Omega \nabla u(t)\nabla u_t(t)\,dx \int_\Omega \nabla u(t) \int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\right| \\
&\quad\le \sigma^2\left(\int_\Omega \nabla u(t)\nabla u_t(t)\,dx\right)^2 \eta\big\|\nabla u(t)\big\|^2 \\
&\qquad + \frac{1}{4\eta}\int_\Omega\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)^2 dx \\
&\quad\le \frac{2(p+2)\sigma^2}{p}E(0)\left(\int_\Omega \nabla u(t)\nabla u_t(t)\,dx\right)^2 + \frac{a-l}{4\eta}(g\circ\nabla u)(t) \\
&\quad\le -\frac{2(p+2)\sigma}{p}E(0)E'(t) + \frac{a-l}{4\eta}(g\circ\nabla u)(t), \qquad (3.35)
\end{aligned}$$

$$\begin{aligned}
&\left|-\int_\Omega\int_0^t g(t-s)\nabla u(s)\,ds \int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\right| \\
&\quad\le \left(2\eta+\frac{1}{4\eta}\right)(a-l)(g\circ\nabla u)(t) + 2\eta(a-l)^2\big\|\nabla u(t)\big\|^2, \qquad (3.36)
\end{aligned}$$

$$\begin{aligned}
&\left|-\int_{\Gamma_1} h(x)y_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma\right| \\
&\quad\le \frac{\eta\|h\|_\infty}{f_{0*}}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \frac{(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (3.37)
\end{aligned}$$

$$\begin{aligned}
&\left|-\int_{\Gamma_1} \mu_0 u_t(x,t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma\right| \\
&\quad\le \eta\mu_0\int_{\Gamma_1}\big|u_t(x,t)\big|^2\,d\Gamma + \frac{\mu_0(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (3.38)
\end{aligned}$$

$$\begin{aligned}
&\left|-\int_{\Gamma_1} \mu_1 u_t\big(x,t-\tau(t)\big)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma\right| \\
&\quad\le \eta|\mu_1|\int_{\Gamma_1} u_t^2\big(x,t-\tau(t)\big)\,d\Gamma + \frac{|\mu_1|(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (3.39)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_\Omega \big|u_t(t)\big|^q u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad\le \eta\alpha_2\big\|\nabla u_t(t)\big\|^2 + \frac{(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (3.40)
\end{aligned}$$

where $\alpha_2 = C_*^{2(q+1)}(\frac{2(p+2)E(0)}{p})^q$,

$$\begin{aligned}
&\left|-\int_\Omega \big|u_t(t)\big|^p u(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad\le \eta\alpha_3\big\|\nabla u(t)\big\|^2 + \frac{(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (3.41)
\end{aligned}$$

where $\alpha_3 = C_*^{2(p+1)}(\frac{2(p+2)E(0)}{lp})^p$,

$$\begin{aligned}
&\left|-\int_\Omega \nabla u_t(t)\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\right| \\
&\quad\le \eta\big\|\nabla u_t(t)\big\|^2 - \frac{g(0)}{4\eta}\big(g'\circ\nabla u\big)(t), \qquad (3.42)
\end{aligned}$$

$$\begin{aligned}
&\left|-\frac{1}{\rho+1}\int_\Omega \left|u_t(t)\right|^\rho u_t(t)\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad \le \frac{\eta}{\rho+1}C_*^{2(\rho+1)}\left(\frac{2(p+2)E(0)}{p}\right)^\rho \left\|\nabla u_t(t)\right\|^2-\frac{g(0)C_*^2}{4\eta(\rho+1)}\big(g'\circ\nabla u\big)(t) \\
&\quad =\frac{\eta\alpha_4}{\rho+1}\left\|\nabla u_t(t)\right\|^2-\frac{g(0)C_*^2}{4\eta(\rho+1)}\big(g'\circ\nabla u\big)(t),
\end{aligned}\tag{3.43}$$

where $\alpha_4=C_*^{2(\rho+1)}(\frac{2(p+2)E(0)}{p})^\rho$. Thus from (3.33)-(3.43) we arrive at

$$\begin{aligned}
\Phi'(t) \le{}& \eta\left\|\nabla u(t)\right\|^2+\frac{a-l}{4\eta}\left(a+b\frac{2(p+2)}{lp}E(0)\right)^2(g\circ\nabla u)(t) \\
&-\frac{2(p+2)\sigma}{p}E(0)E'(t)+\frac{a-l}{4\eta}(g\circ\nabla u)(t) \\
&+\left(2\eta+\frac{1}{4\eta}\right)(a-l)(g\circ\nabla u)(t)+2\eta(a-l)^2\left\|\nabla u(t)\right\|^2 \\
&+\frac{\eta\|h\|_\infty}{f_0}\int_{\Gamma_1}h(x)f(x)y_t^2(t)\,d\Gamma+\frac{(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t) \\
&+\eta\mu_0\left\|u_t(x,t)\right\|^2+\frac{\mu_0(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t) \\
&+\eta|\mu_1|\int_{\Gamma_1}u_t^2\big(x,t-\tau(t)\big)\,d\Gamma+\frac{|\mu_1|(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t) \\
&+\eta\alpha_2\left\|\nabla u_t(t)\right\|^2+\frac{(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t)+\eta\alpha_3\left\|\nabla u(t)\right\|^2+\frac{(a-l)\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t) \\
&+\eta\left\|\nabla u_t(t)\right\|^2-\frac{g(0)}{4\eta}\big(g'\circ\nabla u\big)(t)-\left(\int_0^t g(s)\,ds\right)\left\|\nabla u_t(t)\right\|^2 \\
&+\frac{\eta\alpha_4}{\rho+1}\left\|\nabla u_t(t)\right\|^2-\frac{g(0)C_*^2}{4\eta(\rho+1)}\big(g'\circ\nabla u\big)(t)-\left(\int_0^t g(s)\,ds\right)\frac{1}{\rho+1}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2} \\
={}&-\left(\int_0^t g(s)\,ds\right)\frac{1}{\rho+1}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2}+\eta\big(1+\alpha_3+2(a-l)^2\big)\left\|\nabla u(t)\right\|^2 \\
&-\left[\int_0^t g(s)\,ds-\eta\left(\alpha_2+\frac{\alpha_4}{\rho+1}\right)\right]\left\|\nabla u_t(t)\right\|^2 \\
&+\frac{(a-l)}{4\eta}\left[\left(a+b\frac{2(p+2)}{lp}E(0)\right)^2+2+3\tilde{C}_*^2+8\eta^2+\mu_0\tilde{C}_*^2+|\mu_1|\tilde{C}_*^2\right] \\
&\times(g\circ\nabla u)(t)-\frac{g(0)}{4\eta}\left(1+\frac{C_*^2}{\rho+1}\right)\big(g'\circ\nabla u\big)(t)-\frac{2(p+2)\sigma}{p}E(0)E'(t) \\
&+\frac{\eta\|h\|_\infty}{f_{0*}}\int_{\Gamma_1}h(x)f(x)y_t^2(t)\,d\Gamma+\eta\mu_0\int_{\Gamma_1}\left|u_t(x,t)\right|^2d\Gamma \\
&+\eta|\mu_1|\int_{\Gamma_1}u_t^2\big(x,t-\tau(t)\big)\,d\Gamma.
\end{aligned}\tag{3.44}$$

Since $g(t)$ is positive and continuous, for any $t_0>0$ we see that

$$\int_0^t g(s)\,ds\ge\int_0^{t_0}g(s)\,ds\equiv g_0,\quad \forall t\ge t_0.$$

Hence from (3.5), (3.15), (3.32), and (3.44), we conclude that for any $t \geq t_0 > 0$,

$$\begin{aligned}
L'(t) &= ME'(t) + \varepsilon\Psi'(t) + \Phi'(t) \\
&\leq -\frac{1}{\rho+1}(g_0 - \varepsilon)\|u_t(t)\|_{\rho+2}^{\rho+2} \\
&\quad - \biggl[\frac{Mg(0)}{2} - \varepsilon(1+\eta)(a-l) - \varepsilon\eta\bigl(C_*^2 + \mu_0\tilde{C}_*^2 + \tilde{C}_*^2 + |\mu_1|\tilde{C}_*^2\bigr) \\
&\quad - \eta(1+\alpha_3) - 2\eta(a-l)^2\biggr]\|\nabla u(t)\|^2 - \varepsilon\bigl(a + b\|\nabla u(t)\|^2\bigr)\|\nabla u(t)\|^2 \\
&\quad - \biggl[g_0 - \eta\biggl(\alpha_2 + \frac{\alpha_4}{\rho+1}\biggr) - \varepsilon\biggl(\frac{\alpha_2}{4\eta} + 1\biggr)\biggr]\|\nabla u_t(t)\|^2 \\
&\quad + \frac{1}{4\eta}\biggl[\varepsilon + (a-l)\biggl(a + b\frac{2(p+2)}{lp}E(0)\biggr)^2 \\
&\quad + (a-l)(2 + 3\tilde{C}_*^2 + 8\eta^2 + \mu_0\tilde{C}_*^2 + |\mu_1|\tilde{C}_*^2\biggr](g \circ \nabla u)(t) \\
&\quad + \biggl[\frac{M}{2} - \frac{g(0)}{4\eta}\biggl(1 + \frac{C_*^2}{\rho+1}\biggr)\biggr](g' \circ \nabla u)(t) - M\|u_t(t)\|_{q+2}^{q+2} + \varepsilon\|u(t)\|_{p+2}^{p+2} \\
&\quad - M\sigma\biggl(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\biggr)^2 - \biggl(M - \frac{\varepsilon\|h\|_\infty}{4\eta f_0} - \frac{\eta\|h\|_\infty}{f_0}\biggr)\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma \\
&\quad - \varepsilon\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma - \biggl(MC_1 - \frac{\varepsilon|\mu_1|}{4\eta} - \eta|\mu_1|\biggr)\int_{\Gamma_1} u_t^2\bigl(x, t-\tau(t)\bigr)\,d\Gamma \\
&\quad - \frac{M\lambda\xi}{2}\int_{t-\tau(t)}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds - \biggl(MC_1 - \frac{\varepsilon\mu_0}{4\eta} - \eta\mu_0\biggr)\int_{\Gamma_1} u_t^2(x,t)\,d\Gamma \\
&\quad - \frac{2(p+2)\sigma}{p}E(0)E'(t).
\end{aligned}$$

At this point, we choose $\varepsilon > 0$ small enough and we pick $\eta > 0$ sufficiently small such that

$$K_1 = \frac{1}{\rho+1}(g_0 - \varepsilon) > 0, \qquad K_2 = g_0 - \eta\biggl(\alpha_2 + \frac{\alpha_4}{\rho+1}\biggr) - \varepsilon\biggl(\frac{\alpha_2}{4\eta} + 1\biggr) > 0,$$

and then we choose M so large that

$$\begin{aligned}
K_3 &= \frac{Mg(0)}{2} - \varepsilon(1+\eta)(a-l) - \varepsilon\eta\bigl(C_*^2 + \mu_0\tilde{C}_*^2 + \tilde{C}_*^2 + |\mu_1|\tilde{C}_*^2\bigr) \\
&\quad - \eta(1+\alpha_3) - 2\eta(a-l)^2 > 0, \\
K_4 &= \frac{M}{2} - \frac{g(0)}{4\eta}\biggl(1 + \frac{C_*^2}{\rho+1}\biggr) > 0, \\
K_5 &= M - \frac{\varepsilon\|h\|_\infty}{4\eta f_{0*}} - \frac{\eta\|h\|_\infty}{f_{0*}} > 0, \\
K_6 &= MC_1 - \frac{\varepsilon|\mu_1|}{4\eta} - \eta|\mu_1| > 0, \\
K_7 &= MC_1 - \frac{\varepsilon\mu_0}{4\eta} - \eta\mu_0 > 0.
\end{aligned}$$

Hence, for any $t \geq t_0$, we arrive at

$$\begin{aligned} L'(t) \leq& -K_1\|u_t(t)\|_{\rho+2}^{\rho+2} - K_3\|\nabla u(t)\|^2 - \varepsilon\bigl(a + b\|\nabla u(t)\|^2\bigr)\|\nabla u(t)\|^2 \\ & - K_2\|\nabla u_t(t)\|^2 + K_8(g\circ\nabla u)(t) + K_4\bigl(g'\circ\nabla u\bigr)(t) - M\|u_t(t)\|_{q+2}^{q+2} + \varepsilon\|u(t)\|_{p+2}^{p+2} \\ & - M\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 - K_5\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma - \varepsilon\int_{\Gamma_1} h(x)m(x)y_t^2(t)\,d\Gamma \\ & - K_6\int_{\Gamma_1} u_t^2\bigl(x,t-\tau(t)\bigr)\,d\Gamma - \frac{M\lambda\xi}{2}\int_{t-\tau(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u_t^2(s)\,d\Gamma\,ds \\ & - K_7\int_{\Gamma_1} u_t^2(x,t)\,d\Gamma - \frac{2(p+2)}{p}E(0)E'(t), \end{aligned}$$

where

$$K_8 = \frac{1}{4\eta}\left[\varepsilon + (a-l)\left(a + b\frac{2(p+2)}{lp}E(0)\right)^2 + (a-l)(2 + 3\tilde{C}_*^2 + 8\eta^2 + \mu_0\tilde{C}_*^2 + |\mu_1|\tilde{C}_*^2\right].$$

It follows that

$$L'(t) \leq -K_9E(t) + K_{10}(g\circ\nabla u)(t) - \frac{2(p+2)}{p}E(0)E'(t),$$

where K_9 and K_{10} are some positive constants. Multiplying the above inequality by $\zeta(t)$ and using (2.6) and (3.5), we obtain, for any $t \geq t_0$,

$$\zeta(t)L'(t) \leq -K_9\zeta(t)E(t) - \bigl(2K_{10} + K_{11}\zeta(t)\bigr)E'(t), \tag{3.45}$$

where $K_{11} = \frac{2(p+2)}{p}E(0)$.

Now, we define

$$G(t) = \zeta(t)L(t) + \bigl(2K_{10} + K_{11}\zeta(t)\bigr)E(t).$$

As ξ is non-increasing positive function, by using Lemma 3.4, the function $G(t)$ is equivalent to $E(t)$. Using the fact that $\xi'(t) \leq 0$, (3.45) implies that

$$G'(t) \leq -K_9\zeta(t)E(t) \leq -\nu\zeta(t)G(t),$$

where ν is a positive constant.

The integration of the inequality between t_0 and t gives the following estimation for the function $G(t)$:

$$G(t) \leq G(t_0)e^{-\nu\int_{t_0}^t \zeta(s)\,ds}, \quad \forall t \geq t_0.$$

Again, employing that G is equivalent to E, we get

$$E(t) \leq Ke^{-\nu\int_{t_0}^t \zeta(s)\,ds}, \quad \forall t \geq t_0,$$

where K is a positive constant. Thus the proof of Theorem 3.2 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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