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# On a discontinuous beam-type equation with deviating argument in the curvature

Rubén Figueroa*

*Correspondence:
ruben.figueroa@usc.es
Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, Spain

**Abstract**

We deal with a nonlinear fourth-order equation where the linear part is given by the second derivative of an invertible second-order operator. The equation includes a deviating argument and the boundary conditions provide information as regards the behavior of the solution in an initial and a final interval. Some discontinuities are allowed in all the variables and we prove for this problem the existence of solutions lying between lower and upper solutions. With the extra assumption that the given second-order operator is inverse positive we prove the existence of extremal solutions. An example is also included in order to show the application of our results.

**MSC:** 34B99

**Keywords:** fourth-order boundary value problems; deviating arguments; discontinuous equations; lower and upper solutions

## 1 Introduction

Let $\mathscr{L}: W^{2,1} \longrightarrow \mathscr{C}$ be a linear differential operator of second order, $L > 0$, and $r_1, r_2 > 0$. We are concerned with the following problem of fourth order with deviating argument:

$$\begin{cases} \mathscr{L}[u]''(x) = f(x, u, \mathscr{L}[u](x), \mathscr{L}[u](\tau(x))) & \text{for almost all (a.a.) } x \in I = [0, L], \\ u(x) = \phi_1(x) & \text{for all } x \in I_- = [-r_1, 0], \\ u(x) = \phi_2(x) & \text{for all } x \in I_+ = [L, L + r_2], \end{cases} \tag{1.1}$$

where $\phi_1 \in \mathscr{C}^2(I_-)$, $\phi_2 \in \mathscr{C}^2(I_+)$, and $\tau : I \longrightarrow \tilde{I} = [-r_1, L + r_2]$ is a measurable deviating argument such that

$$\operatorname{meas}\left(\tau^{-1}\{0, L\}\right) = 0. \tag{1.2}$$

We will look for solutions for problem (1.1) inside the set

$$\mathscr{X} = \left\{u \in \mathscr{C}(\tilde{I}) : u \in \mathscr{C}^2(\mathring{I}_- \cup \mathring{I}_+) \text{ and } \mathscr{L}[u] \in W^{2,1}(I)\right\}.$$

(Notice that there is an abuse of notation here, in order to simplify our notation. Of course, the regularity conditions in each subinterval of $\tilde{I}$ are understood to be referred to the correspondent restrictions of function $u$.)

Problem (1.1) has a very general form and includes as particular cases equations of fourth order of the form

$$u^{iv} = f, \qquad u^{iv} + Mu'' = f \quad \text{or} \quad \bigl(\phi\bigl(u''\bigr)\bigr)'' = f,$$

with $\phi$ an increasing homeomorphism, which were widely studied in the literature [1–6]. Frequently, these kinds of equations appear when modeling the behavior of a beam or a suspension bridge, and then different types of boundary conditions refer to different physical meanings regarding the beam or bridge. The reader is referred to [7] for more details. As a novelty in this work, the differential equation includes a deviating argument of the form $\mathscr{L}[u](\tau(x))$, which means that the equation involves the behavior of the solution (and, in particular, of its curvature $u''$) at points that can be to the right or to the left with respect to the one which is being considered. For this reason, problem (1.1) includes as boundary conditions two functions $\phi_1$ and $\phi_2$ which represent, respectively, the initial and the final state of the solution. From this point of view, problem (1.1) can model a beam or a bridge of length $L + r_1 + r_2$ which is totally rigid in two extremal intervals. On the other hand, discontinuities with respect to all the variables are allowed in the differential equation.

The way to study problem (1.1) is *via* a reduction of order, that is, we will consider an auxiliary second-order problem and then we will go back to the fourth-order one. To do this we will need some conditions on operator $\mathscr{L}$, as we will see in Section 3. This technique of reduction of order to look for solutions for fourth-order equations has been used for example in [2], where a functional equation for $\mathscr{L} = \frac{d^2}{dx^2}$ with also functional boundary conditions is considered. Coupled with the reduction of order technique we will use a generalized monotone method in the presence of lower and upper solutions. This technique is very typical in the literature devoted to discontinuous differential equations and was developed in depth in [8].

This paper is organized as follows: in Section 2 we consider an auxiliary second-order problem and so we recall some results on comparison principles and existence of solutions that were published in [9]. Then, in Section 3, we deal with the fourth-order problem (1.1): first we consider the case when the second variable is dropped, and then we deal with the whole problem. We finish our work in Section 4 with an example of application of our results.

## 2 Second-order equations

In this section we will study a second-order equation with deviating argument that will be useful for us when reducing the order in (1.1). So, now we deal with the following problem:

$$\begin{cases} u''(x) = f(x, u(x), u(\tau(x))) & \text{for a.a. } x \in I, \\ u(x) = \hat{\phi}_1(x) \quad \text{for all } x \in I_-, & u(x) = \hat{\phi}_2(x) \quad \text{for all } x \in I_+, \end{cases} \tag{2.1}$$

where $\tau : I \longrightarrow \tilde{I} = [-r_1, L + r_2]$ is a measurable deviating argument and $\hat{\phi}_1, \hat{\phi}_2$ are continuous functions.

A problem similar to (2.1) was studied in [9], with the difference that in that paper the problem was defined in $[-r_1, L]$ instead of $[-r_1, L + r_2]$, so (2.1) is a little more general than that problem. For this reason, and for the sake of completeness, we reformulate now the main results in [9] in order to apply them to problem (2.1), with a little sketch of their proofs. The reader is referred to the original paper for more details.

We will look for solutions for problem (2.1) inside the set $\hat{\mathscr{X}} = \mathscr{C}(\tilde{I}) \cap W^{2,1}(I)$.

First we include a uniqueness result for the case of Lipschitzian nonlinearities, based on the well-known Banach contraction principle.

**Lemma 2.1** *Assume that the function* $f : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *satisfies the following conditions*:

(i) *For each* $u, v \in \mathbb{R}$, *the mapping* $x \in I \longmapsto f(x,u,v)$ *is measurable*;

(ii) *for each compact subset* $K \subset \mathbb{R}^2$ *there exists* $\psi_K \in L^1(I,[0,+\infty))$ *such that*

$$\left|f(x,u,v)\right| \leq \psi_K(x) \quad \textit{for all } (u,v) \in K;$$

(iii) *there exist nonnegative functions* $M_1, M_2$ *such that*

$$\left|f(x,u,v) - f(x,\overline{u},\overline{v})\right| \leq M_1(x)|u-\overline{u}| + M_2(x)|v-\overline{v}|$$
$$\textit{for a.a. } x \in I \textit{ and all } u, v \in \mathbb{R}.$$

*Moreover, the functions* $M_1, M_2$ *satisfy one of the following*:

($C_1$) $M_1, M_2 \in L^\infty(I)$ *and* $\|M_1 + M_2\|_\infty < \frac{1}{L^2}$;
($C_2$) $M_1, M_2 \in L^2(I)$ *and* $\|M_1 + M_2\|_2 < (\frac{3}{2L^3})^{1/2}$;
($C_3$) $M_1, M_2 \in L^1(I)$ *and* $\|M_1 + M_2\|_1 < \frac{1}{2L}$.

*Under these conditions, problem* (2.1) *has a unique solution in* $\hat{\mathscr{X}}$.

*Sketch of the proof* Consider the operator $T : \mathscr{C}(\tilde{I}) \longrightarrow \mathscr{C}(\tilde{I})$ defined for each $\gamma \in \mathscr{C}(\tilde{I})$ by

$$T\gamma(x) = \begin{cases} \hat{\phi}_1(x) & \text{if } x \in I_-, \\ \hat{\phi}_1(0) + Cx + \int_0^x (x-t) f(t,\gamma(t),\gamma(\tau(t)))\,dt & \text{if } x \in I, \\ \hat{\phi}_2(x) & \text{if } x \in I_+, \end{cases}$$

where

$$C = \frac{1}{L}\left(\hat{\phi}_2(L) - \hat{\phi}_1(0) - \int_0^L (L-t) f\bigl(t,\gamma(t),\gamma\bigl(\tau(t)\bigr)\bigr)\,dt\right).$$

Conditions (i)-(iii) imply that $T$ is a well-defined contractive mapping, and so it has a unique fixed point which corresponds with the unique solution of problem (2.1). □

The following comparison principle will be essential in order to apply the generalized monotone method to the second-order problem (2.1).

**Lemma 2.2** *Let* $w \in \hat{\mathscr{X}}$ *be a function and assume that there exist nonnegative functions* $M_1, M_2$ *satisfying the following*:

(i) $w''(x) \leq M_1(x)w(x) + M_2(x)w(\tau(x))$ *for a.a.* $x \in [0,L]$;

(ii) $w(0) \geq 0$, $w(L) \geq 0$;

(iii) $0 \leq w(x) \leq w(0)$ *for all* $x \in [-r_1, 0]$ *and* $0 \leq w(x) \leq w(L)$ *for all* $x \in [L, L+r_2]$;

(iv) *the functions* $M_1, M_2$ *satisfy one of the following conditions*:

($\hat{\mathrm{C}}_1$) *$M_1, M_2 \in L^\infty(I)$ and $\|M_1 + M_2\|_\infty < \frac{2}{L^2}$;*
($\hat{\mathrm{C}}_2$) *$M_1, M_2 \in L^2(I)$ and $\|M_1 + M_2\|_2 < \frac{\sqrt{2}}{L}$;*
($\hat{\mathrm{C}}_3$) *$M_1, M_2 \in L^1(I)$ and $\|M_1 + M_2\|_1 < \frac{1}{L}$.*

*Then $w(x) \ge 0$ for all $x \in \tilde{I}$.*

*Proof* It is the same proof as in [9], Lemma 3.1. □

**Remark 2.3** Notice that for $\frac{3}{4} \le L$ each condition ($\mathrm{C}_i$) implies ($\hat{\mathrm{C}}_i$), $i = 1, 2, 3$, and for $0 < L \le \frac{3}{4}$ we have ($\mathrm{C}_1$) $\Rightarrow$ ($\hat{\mathrm{C}}_1$), ($\mathrm{C}_3$) $\Rightarrow$ ($\hat{\mathrm{C}}_3$), and ($\hat{\mathrm{C}}_2$) $\Rightarrow$ ($\mathrm{C}_2$).

Now we are ready to look for solutions for problem (2.1). We begin by introducing the concept of lower and upper solutions.

**Definition 2.4** We say that a function $\xi \in \hat{\mathscr{X}}$ is a lower solution for problem (2.1) if the composition $x \in I \longmapsto f(x, \xi(x), \xi(\tau(x)))$ is measurable and the following inequalities hold:

$$\begin{cases} \xi''(x) \ge f(x, \xi(x), \xi(\tau(x))) \quad \text{for a.a. } x \in I, \\ \xi(x) \le \hat{\phi}_1(x) \text{ and } \hat{\phi}_1(x) - \xi(x) \le \hat{\phi}_1(0) - \xi(0) \quad \text{for all } x \in I_-, \\ \xi(x) \le \hat{\phi}_2(x) \text{ and } \hat{\phi}_2(x) - \xi(x) \le \hat{\phi}_2(L) - \xi(L) \quad \text{for all } x \in I_+. \end{cases}$$

In an analogous way, we say that $\eta \in \hat{\mathscr{X}}$ is an upper solution for problem (2.1) if the composition $x \in I \longmapsto f(x, \eta(x), \eta(\tau(x)))$ is measurable and the following inequalities hold:

$$\begin{cases} \eta''(x) \le f(x, \eta(x), \eta(\tau(x))) \quad \text{for a.a. } x \in I, \\ \eta(x) \ge \hat{\phi}_1(x) \text{ and } \eta(x) - \hat{\phi}_1(x) \le \eta(0) - \hat{\phi}_1(0) \quad \text{for all } x \in I_-, \\ \eta(x) \ge \hat{\phi}_2(x) \text{ and } \eta(x) - \hat{\phi}_2(x) \le \eta(L) - \hat{\phi}_2(L) \quad \text{for all } x \in I_+. \end{cases}$$

The main result in this section establishes the existence of extremal solutions for problem (2.1) in the presence of well-ordered lower and upper solutions. We recall that two solutions $u^*$, $u_*$ are said to be extremal in a set $Y$ if any solution $u \in Y$ satisfies $u_* \le u \le u^*$.

The proof of this result uses the generalized monotone method and, in particular, the following lemma on the existence of fixed points of nondecreasing operators.

**Lemma 2.5** ([8], Theorem 1.2.2) *Let $Y$ be a subset of an ordered metric space, $[a, b]$ a nonempty interval in $Y$ and $T : [a, b] \longrightarrow [a, b]$ a nondecreasing operator. If $\{Tu_n\}_{n \in \mathbb{N}}$ converges whenever $\{u_n\}_{n \in \mathbb{N}}$ is a monotone sequence in $[a, b]$ then the operator $T$ has the greatest, $u^*$, and the least, $u_*$, fixed point in $[a, b]$, which, moreover, satisfy*

$$u^* = \max\{u : u \le Tu\}, \qquad u_* = \min\{u : Tu \le u\}. \tag{2.2}$$

**Theorem 2.6** *Assume that there exist $\xi, \eta \in \hat{\mathscr{X}}$ which are, respectively, a lower and an upper solution for problem* (2.1), *with $\xi(x) \le \eta(x)$ for all $x \in [-r_1, L + r_2]$, and put*

$$[\xi, \eta] = \{\gamma \in \mathscr{C}(\tilde{I}) : \xi(x) \le \gamma(x) \le \eta(x) \textit{ for all } x \in [-r_1, L + r_2]\}.$$

*Assume, moreover, that the following conditions hold*:

($H_1$) *For each $\gamma \in [\xi, \eta]$ the composition $x \in I \longmapsto f(x, \gamma(x), \gamma(\tau(x)))$ is measurable*;

($H_2$) *there exists $\psi \in L^1(I, [0, \infty))$ such that for a.a. $x \in I$, all $u \in [\xi(x), \eta(x)]$ and all $v \in [\xi(\tau(x)), \eta(\tau(x))]$ we have $|f(x, u, v)| \le \psi(x)$*;

($H_3$) *there exist nonnegative functions $M_1, M_2$ such that for a.a. $t \in I$ we have*

$$f(x, \overline{u}, \overline{v}) - f(x, u, v) \le M_1(x)(\overline{u} - u) + M_2(x)(\overline{v} - v)$$

*whenever $\xi(x) \le u \le \overline{u} \le \eta(x)$, $\xi(\tau(x)) \le v \le \overline{v} \le \eta(\tau(x))$.*
*Moreover, if $L \ge \frac{3}{4}$ then functions $M_1, M_2$ satisfy one of the conditions* ($C_1$), ($C_2$), ($C_3$) *and if $0 < L < \frac{3}{4}$ then they satisfy one of the following*: ($C_1$), ($\hat{C}_2$), ($C_3$).

*Under these conditions, problem* (2.1) *has the extremal solutions in* $[\xi, \eta]$.

*Sketch of the proof* Consider an operator $T : [\xi, \eta] \longrightarrow \mathscr{C}(\tilde{I})$ defined as follows: for each $\gamma \in [\xi, \eta]$, $T\gamma$ is the unique solution of the 'linearized' problem:

$$\begin{cases} v''(x) - M_1(x)(v(x) - \gamma(x)) - M_2(x)(v(\tau(x)) - \gamma(\tau(x))) = f(x, \gamma(x), \gamma(\tau(x))), \\ \quad \text{for a.a. } x \in I, \\ v(x) = \hat{\phi}_1(x) \quad \text{for all } x \in I_-, \qquad v(x) = \hat{\phi}_2(x) \quad \text{for all } x \in I_+. \end{cases} \tag{2.3}$$

Operator $T$ is well defined by application of Lemma 2.1. Moreover, we can show by application of Lemma 2.2, in a similar way to [9], Theorem 3.5, that $T$ is a nondecreasing operator which maps interval $[\xi, \eta]$ into itself and which, moreover, maps monotone sequences into convergent ones in $\mathscr{C}(\tilde{I})$. Then Lemma 2.5 guarantees that $T$ has the extremal fixed points in $[\xi, \eta]$, which correspond with the extremal solutions of problem (2.1) in $[\xi, \eta]$. □

**Remark 2.7** Condition ($H_3$) states, roughly speaking, that the functions $f_1(u) = f(x, u, v) - M_1(x)u$ and $f_2(v) = f(x, u, v) - M_2(x)v$ must be nonincreasing. In Section 4 we will prove a lemma that can be useful in order to check this condition in practice.

## 3 Fourth order equations

Now we will use the previous results in order to deal with the original fourth-order problem. As we said in the Introduction, we begin by considering a particular case of problem (1.1) where dependence on the second variable is dropped, that is,

$$\begin{cases} \mathscr{L}[u]''(x) = f(x, \mathscr{L}[u](x), \mathscr{L}[u](\tau(x))) \quad \text{for a.a. } x \in I, \\ u(x) = \phi_1(x) \quad \text{for all } x \in I_-, \qquad u(x) = \phi_2(x) \quad \text{for all } x \in I_+. \end{cases} \tag{3.1}$$

We define lower and upper solutions for problem (3.1).

**Definition 3.1** We say that $\alpha, \beta$ are, respectively, a lower and an upper solution for problem (3.1) if $\mathscr{L}[\alpha], \mathscr{L}[\beta] \in \hat{\mathscr{X}}$, compositions $x \longmapsto f(x, \mathscr{L}[\alpha](x), \mathscr{L}[\alpha](\tau(x)))$ and $x \longmapsto f(x, \mathscr{L}[\beta](x), \mathscr{L}[\beta](\tau(x)))$ are measurable on $I$ and the following inequalities hold:

$$\begin{cases} \mathscr{L}[\alpha]''(x) \ge f(x, \mathscr{L}[\alpha](x), \mathscr{L}[\alpha](\tau(x))) \quad \text{for a.a. } x \in I, \\ \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_1](x) \text{ and } \mathscr{L}[\phi_1](x) - \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_1](0) - \mathscr{L}[\alpha](0) \quad \text{in } I_-, \\ \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_2](x) \text{ and } \mathscr{L}[\phi_2](x) - \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_2](L) - \mathscr{L}[\alpha](L) \quad \text{in } I_+; \end{cases}$$

$$\begin{cases} \mathscr{L}[\beta]''(x) \le f(x, \mathscr{L}[\beta](x), \mathscr{L}[\beta](\tau(x))) & \text{for a.a. } x \in I, \\ \mathscr{L}[\beta](x) \ge \mathscr{L}[\phi_1](x) \text{ and } \mathscr{L}[\beta](x) - \mathscr{L}[\phi_1](x) \le \mathscr{L}[\beta](0) - \mathscr{L}[\phi_1](0) & \text{in } I_-, \\ \mathscr{L}[\beta](x) \ge \mathscr{L}[\phi_2](x) \text{ and } \mathscr{L}[\beta](x) - \mathscr{L}[\phi_2](x) \le \mathscr{L}[\beta](L) - \mathscr{L}[\phi_2](L) & \text{in } I_+. \end{cases}$$

Now we prove the existence of solutions for problem (3.1). We recall that we say that the operator $\mathscr{L} : W^{2,1} \longrightarrow \mathscr{C}$ is *invertible* in a certain set $Z \subset W^{2,1}$ if there exists $\mathscr{L}^{-1} : \mathscr{C} \longrightarrow Z$ such that $v = \mathscr{L}^{-1}[\mathscr{L}[v]]$ for all $v \in Z$. This is equivalent to the fact that there exists an associated Green's function in $Z$ (see [10] for more details).

**Theorem 3.2** *Assume that there exist $\alpha, \beta$ which are, respectively, a lower and an upper solution for problem* (3.1) *with $\mathscr{L}[\alpha] \le \mathscr{L}[\beta]$ on $\tilde{I}$ and let $f$ satisfy conditions* ($H_1$), ($H_2$), *and* ($H_3$) *with $\xi = \mathscr{L}[\alpha]$ and $\eta = \mathscr{L}[\beta]$. If the operator $\mathscr{L}$ is invertible in the set*

$$Z = \{ w \in W^{2,1}(I) : w(0) = w(L) = 0 \}$$

*then problem* (3.1) *has a nonempty set of solutions.*

*Proof* It is clear that $\xi = \mathscr{L}[\alpha]$ and $\eta = \mathscr{L}[\beta]$ are, respectively, a lower and an upper solution for problem (2.1) with $\hat{\phi}_1 = \mathscr{L}[\phi_1]$ and $\hat{\phi}_2 = \mathscr{L}[\phi_2]$. Then conditions on $f$ guarantee by application of Theorem 2.6 that this problem has solutions in $[\xi, \eta]$ with least and greatest elements inside this set. Now, let $v \in [\xi, \eta]$ such a solution. As operator $\mathscr{L}$ is invertible in $Z$ there exists the associated Green's function, $G$. So, we can define a function $u \in \mathscr{X}$ as follows:

$$u(x) = \begin{cases} \phi_1(x) & \text{if } x \in I_-, \\ u_0(x) + \int_0^L G(x,t)v(t)\,dt & \text{if } x \in I, \\ \phi_2(x) & \text{if } x \in I_+, \end{cases} \tag{3.2}$$

where $u_0$ is the unique solution of $\mathscr{L}[w] = 0$ coupled with the Dirichlet conditions $w(0) = \phi_1(0)$ and $w(L) = \phi_2(L)$.

We will show that thus defined $u$ is a solution of problem (3.1). First, it is clear that $u \in \mathscr{C}^2(\mathring{I}_- \cup \mathring{I}_+)$ and $\mathscr{L}[u] = v \in W^{2,1}(I)$, therefore $u \in \mathscr{X}$. As $u$ satisfies the boundary conditions in (3.1) by construction, we only have to check that $u$ is a solution of the differential equation. First of all, notice that for $G$ being the Green's function associated to $\mathscr{L}$ in $Z$, $\mathscr{L}[u] = v$ almost everywhere in $I$. Now, as $v$ is a solution of problem (2.1) we obtain for a.a. $x \in I$

$$\mathscr{L}[u]''(x) = v''(x) = f\big(x, v(x), v\big(\tau(x)\big)\big) = f\big(x, \mathscr{L}[u](x), v\big(\tau(x)\big)\big). \tag{3.3}$$

Finally, regarding the argument with deviation, notice the following: if $\tau(x) \in I$ then we have $v(\tau(x)) = \mathscr{L}[u](\tau(x))$, if $\tau(x) \in I_-$ then $v(\tau(x)) = \hat{\phi}_1(\tau(x)) = \mathscr{L}[\phi_1](\tau(x)) = \mathscr{L}[u](\tau(x))$ and, in an analogous way, if $\tau(x) \in I_+$ then $v(\tau(x)) = \mathscr{L}[u](\tau(x))$. Therefore, we see that (3.3) actually becomes

$$\begin{aligned} &\mathscr{L}[u]''(x) = v''(x) = f\big(x, v(x), v\big(\tau(x)\big)\big) = f\big(x, \mathscr{L}[u](x), \mathscr{L}[u]\big(\tau(x)\big)\big) \\ &\quad \text{for a.a. } x \in I, \end{aligned} \tag{3.4}$$

and so $u$ solves the differential equation in (3.1). $\square$

**Remark 3.3** Notice that equation (3.2) does not imply the existence of $\mathscr{L}[u](0)$ nor $\mathscr{L}[u](L)$. However, the fact of $\mathscr{L}[u] = v$ almost everywhere, joined with condition (1.2), is enough to guarantee that (3.4) makes sense. On the other hand, we are assuming throughout this paper that $r_1, r_2 > 0$. We can relax this condition and require these constants to be only nonnegative. Therefore, if $r_1 = 0$ we could actually provide some additional condition about the value of $\mathscr{L}[u](0)$, and analogously if $r_2 = 0$.

The following result states the existence of extremal solutions under the additional hypothesis that the operator $\mathscr{L}$ is inverse positive, that is, $u \geq 0 \Rightarrow \mathscr{L}^{-1}[u] \geq 0$, in a certain set $Z'$. This condition is equivalent to saying that $\mathscr{L}$ satisfies the anti-maximum principle in $Z'$, that is, $\mathscr{L}[u] \geq 0 \Rightarrow u \geq 0$. This provides a certain *monotonicity* condition that will be essential in order to apply the monotone iterative technique. To see more examples of application of this technique in the presence of inverse positive or inverse negative operators the reader is referred to [11] or [10].

**Theorem 3.4** *In the conditions of Theorem* 3.2, *assume that* $\alpha \leq \beta$ *on* $\tilde{I}$, $\alpha_{|I_-} \leq \phi_1$, $\alpha_{|I_+} \leq \phi_2$, $\beta_{|I_-} \geq \phi_1$, *and* $\beta_{|I_+} \geq \phi_2$. *If operator* $\mathscr{L}$ *is inverse positive in the set*

$$Z' = \{w \in W^{2,1}(I) : w(0) \geq 0, w(L) \geq 0\}$$

*then problem* (3.1) *has the extremal solutions in* $[\alpha, \beta] = \{\gamma \in \mathscr{C}(\tilde{I}) : \alpha(x) \leq \gamma(x) \leq \beta(x)$ *for all* $x \in \tilde{I}\}$.

*Proof* We will divide the proof in three steps.

*Claim 1: Let* $v \in [\mathscr{L}[\alpha], \mathscr{L}[\beta]]$ *be a solution of problem* (2.1) *and u a solution of* (3.1) *obtained from v by equation* (3.2). *Then* $\alpha \leq u \leq \beta$ *on* $\tilde{I}$.- It is clear by construction of $u$ and the conditions on $\alpha, \beta$ that $\alpha(x) \leq u(x) \leq \beta(x)$ on $I_- \cup I_+$, so we only have to prove that this inequality is true in $I$.

To do this, first notice that for $\alpha$ being a lower solution we have $u(0) = \phi_1(0) \geq \alpha(0)$ and $u(L) = \phi_2(L) \geq \alpha(L)$. On the other hand, by construction of $u$ we have

$$\mathscr{L}[u] = v \geq \mathscr{L}[\alpha] \quad \text{on } I, \quad \text{that is,} \quad \mathscr{L}[u - \alpha] \geq 0 \quad \text{on } I.$$

As $\mathscr{L}$ is inverse positive in $Z'$, it satisfies the anti-maximum principle in that set and then $u \geq \alpha$ in $I$.

In a same way we prove that $u \leq \beta$ on $\tilde{I}$.

*Claim 2: If* $u \in [\alpha, \beta]$ *is a solution of problem* (3.1) *then there exists* $v \in [\mathscr{L}[\alpha], \mathscr{L}[\beta]]$, *solution of* (2.1), *such that u is obtained from v by equation* (3.2). Let $u \in [\alpha, \beta]$ be a solution of (3.1). This implies that $u = \phi_1$ in $I_-$, $u = \phi_2$ in $I_+$, and for a.a. $x \in I$ we have

$$\mathscr{L}[u]''(x) = f\bigl(x, \mathscr{L}[u](x), \mathscr{L}[u]\bigl(\tau(x)\bigr)\bigr),$$

and so $v = \mathscr{L}[u]$ is a solution of problem (2.1). Now, as $\mathscr{L}$ is invertible, it is trivial that $u$ can be obtained from $v$ by (3.2).

*Claim 3: If* $\hat{v}$ *is the greatest solution in* $[\mathscr{L}[\alpha], \mathscr{L}[\beta]]$ *of problem* (2.1) *then* $\hat{u}$ *obtained from* $\hat{v}$ *by equation* (3.2) *provides the greatest solution of problem* (3.1) *in* $[\alpha, \beta]$. Let $u \in [\alpha, \beta]$

be an arbitrary solution of problem (3.1) and $v \in [\mathscr{L}[\alpha], \mathscr{L}[\beta]]$ provided by Claim 2. Then $v(0) = \hat{v}(0)$, $v(L) = \hat{v}(L)$, and $v \le \hat{v}$ on $I$. As $\mathscr{L}$ is inverse positive in $Z$ we obtain $u = \mathscr{L}^{-1}[v] \le \mathscr{L}^{-1}[\hat{v}] = \hat{u}$ on $I$. Taking into account that $u(x) = \hat{u}(x)$ for all $x \in I_- \cup I_+$ we conclude that $u \le \hat{u}$ in $\tilde{I}$ and so $\hat{u}$ is the greatest solution of problem (3.1) in $[\alpha, \beta]$. In an analogous way we obtain the least solution of the problem in $[\alpha, \beta]$. □

Now we are going to develop a monotone method in order to look for solutions to the whole problem (1.1).

**Definition 3.5** We say that $\alpha$ is a lower solution of problem (1.1) if $\mathscr{L}[\alpha] \in \hat{\mathscr{X}}$, the composition

$$x \in I \longmapsto f\big(x, \alpha, \mathscr{L}[\alpha](x), \mathscr{L}[\alpha]\big(\tau(x)\big)\big)$$

is measurable, and the following inequalities hold:

$$\begin{cases} \mathscr{L}[\alpha]''(x) \ge f(x, \alpha, \mathscr{L}[\alpha](x), \mathscr{L}[\alpha](\tau(x))) & \text{for a.a. } x \in I, \\ \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_1](x) \text{ and } \mathscr{L}[\phi_1](x) - \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_1](0) - \mathscr{L}[\alpha](0) & \text{in } I_-, \\ \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_2](x) \text{ and } \mathscr{L}[\phi_2](x) - \mathscr{L}[\alpha](x) \le \mathscr{L}[\phi_2](L) - \mathscr{L}[\alpha](L) & \text{in } I_+. \end{cases}$$

In an analogous way, we say that $\beta$ is an upper solution for problem (1.1) if $\mathscr{L}[\beta] \in \hat{\mathscr{X}}$, the composition

$$x \in I \longmapsto f\big(x, \beta, \mathscr{L}[\beta](x), \mathscr{L}[\beta]\big(\tau(x)\big)\big)$$

is measurable, and the following inequalities hold:

$$\begin{cases} \mathscr{L}[\beta]''(x) \le f(x, \beta, \mathscr{L}[\beta](x), \mathscr{L}[\beta](\tau(x))) & \text{for a.a. } x \in I, \\ \mathscr{L}[\beta](x) \ge \mathscr{L}[\phi_1](x) \text{ and } \mathscr{L}[\beta](x) - \mathscr{L}[\phi_1](x) \le \mathscr{L}[\beta](0) - \mathscr{L}[\phi_1](0) & \text{in } I_-, \\ \mathscr{L}[\beta](x) \ge \mathscr{L}[\phi_2](x) \text{ and } \mathscr{L}[\beta](x) - \mathscr{L}[\phi_2](x) \le \mathscr{L}[\beta](L) - \mathscr{L}[\phi_2](L) & \text{in } I_+. \end{cases}$$

The following is the main result in this paper.

**Theorem 3.6** *Let $\alpha, \beta$ be a lower and an upper solution for problem* (1.1) *such that $\alpha \le \beta$ and $\mathscr{L}[\alpha] \le \mathscr{L}[\beta]$ on $\tilde{I}$, $\alpha_{|I_-} \le \phi_1$, $\alpha_{|I_+} \le \phi_2$, $\beta_{|I_-} \ge \phi_1$, and $\beta_{|I_+} \ge \phi_2$. Assume that for each*

$$\gamma \in [\alpha, \beta] = \big\{\gamma \in \mathscr{C}(\tilde{I}) : \alpha(x) \le \gamma(x) \le \beta(x) \textit{ for all } x \in \tilde{I}\big\}$$

*the function $f_\gamma := f(x, \gamma, u, v)$ satisfies conditions* ($H_1$)-($H_3$) *and*

($H_4$) *for a.a. $x \in I$, all $u \in [\mathscr{L}[\alpha](x), \mathscr{L}[\beta](x)]$ and all $v \in [\mathscr{L}[\alpha](\tau(x)), \mathscr{L}[\beta](\tau(x))]$ function $f(x, \cdot, u, v)$ is nonincreasing in $[\alpha, \beta]$.*

*If the operator $\mathscr{L}$ is inverse positive in*

$$Z' = \big\{w \in W^{2,1}(I) : w(0) \ge 0, w(L) \ge 0\big\}$$

*then problem* (1.1) *has the extremal solutions in $[\alpha, \beta]$.*

*Proof* We consider the operator $T : [\alpha,\beta] \longrightarrow [\alpha,\beta]$ defined as follows: for each $\gamma \in [\alpha,\beta]$, $T\gamma$ is the greatest solution in $[\alpha,\beta]$ of the auxiliary problem

$$(P_\gamma)\begin{cases}\mathscr{L}[u]''(x) = f_\gamma(x,\mathscr{L}[u](x),\mathscr{L}[u](\tau(x))) & \text{a.e. on } I,\\ u = \phi_1 \quad \text{in } I_-, \qquad u = \phi_2 & \text{in } I_+.\end{cases}$$

First, notice that conditions on the lower and the upper solution, coupled with ($H_1$)-($H_3$) imply by application of Theorem 3.4 that operator $T$ is well defined in $[\alpha,\beta]$. On the other hand, if we have $\gamma_1,\gamma_2 \in [\alpha,\beta]$, $\gamma_1 \le \gamma_2$ then we obtain by virtue of ($H_4$)

$$\mathscr{L}[T\gamma_1]''(x) = f_{\gamma_1}\bigl(x,\mathscr{L}[T\gamma_1](x),\mathscr{L}[T\gamma_1]\bigl(\tau(x)\bigr)\bigr) \ge f_{\gamma_2}\bigl(x,\mathscr{L}[T\gamma_1](x),\mathscr{L}[T\gamma_1]\bigl(\tau(x)\bigr)\bigr)$$

a.e. on $I$, and so $T\gamma_1 \in [\alpha,\beta]$ is a lower solution for problem $(P_{\gamma_2})$. As $\beta$ is an upper solution for this problem and $T\gamma_2$ is defined as the greatest solution in $[\alpha,\beta]$ for $(P_{\gamma_2})$ we obtain that $T\gamma_1 \le T\gamma_2$, and so operator $T$ is nondecreasing in $[\alpha,\beta]$.

Finally, we will show that $T$ maps monotone sequences into convergent ones. So, let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a monotone sequence in $[\alpha,\beta]$. As $T$ is nondecreasing and $T([\alpha,\beta]) \subset [\alpha,\beta]$ then $\{T\gamma_n\}_{n\in\mathbb{N}}$ is a monotone sequence which has its pointwise limit in $[\alpha,\beta]$, say $\Gamma$. We have to prove that $T\gamma_n \to \Gamma$ in $\mathscr{C}(I)$. As $\{T\gamma_n\}_{n\in\mathbb{N}}$ is constant in $I_- \cup I_+$, we only need to check that the convergence is uniform in $I$. To see this, put $g_n = \mathscr{L}[T\gamma_n]$ and notice that $g_n \le \mathscr{L}[\beta]$ for all $n \in \mathbb{N}$, and so $g_n$ is uniformly bounded. As the operator $\mathscr{L}^{-1}$ is compact we conclude that $\{T\gamma_n\}_{n\in\mathbb{N}}$ is precompact and so it has a convergent subsequence that coincides with $\{T\gamma_n\}_{n\in\mathbb{N}}$ because of monotonicity.

We conclude by application of Lemma 2.5 that $T$ has the greatest fixed point, $u^*$ in $[\alpha,\beta]$ and we claim that $u^*$ is the greatest solution of our problem in $[\alpha,\beta]$. Indeed, it is obvious that $u^*$ is a solution. To see that it is the greatest one, let $u \in [\alpha,\beta]$ another solution. Then $Tu = u$ and so characterization (2.2) implies that $u^* \ge u$.

To obtain the least solution of problem (1.1) in $[\alpha,\beta]$ we must redefine operator $T$, considering the least solution of auxiliar problem $(P_\gamma)$ instead of the greatest one, and reasoning like above. □

## 4 An example of application

In this section we provide an example to show how our results can be applied in practice. First we begin by introducing a lemma that will be useful for us in order to check condition ($H_3$).

**Lemma 4.1** *Let $\{x_k\}_{k=1}^{\infty}$ a strictly increasing sequence of real numbers and assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that $f_{|(x_{k-1},x_k)} \in \mathscr{C}^1(x_{k-1},x_k)$ for all $k = 1,2,\ldots$. Assume, moreover, that the following conditions hold for each $k \in \mathbb{N}$:*

(i)

$$\lim_{x\to x_k^-} f(x) \ge f(x_k) \ge \lim_{x\to x_k^+} f(x);$$

(ii) *there exist $M_k = \sup\{f'(x) : x \in (x_{k-1},x_k)\}$ and $M = \sup\{M_k : k \in \mathbb{N}\}$.*

*Under these conditions, for all $\tilde{M} \ge M$ the function $g : x \in \mathbb{R} \longmapsto g(x) = f(x) - \tilde{M}x$ is nonincreasing.*

*Proof* Fixing $k \in \mathbb{N}$ we see that the function $g$ is differentiable in $(x_{k-1}, x_k)$ and $g'(x) \leq M_k - M \leq 0$ for all $x \in (x_{k-1}, x_k)$, so $g$ is nonincreasing in that interval. On the other hand, condition (i) provides that $f$ is nonincreasing at point $x_k$, and so $g$. □

Now we consider the second-order operator $\mathscr{L}[u] = -u'' + Mu$, with $M > 0$ a parameter, with the following boundary value problem:

$$\begin{cases} \mathscr{L}[u]''(x) = \Gamma(u)\{f_1(\mathscr{L}[u](x)) + [\frac{1}{\sqrt{x}}](\mathscr{L}[u](\tau(x))^2\} \\ \quad \text{for almost all } x \in [0, \frac{\pi}{4}], \\ u(x) = \cos(x) \quad \text{for all } x \in [-\frac{\pi}{2}, 0] \cup [\frac{\pi}{4}, \frac{\pi}{2}], \end{cases} \tag{4.1}$$

where $[\cdot]$ means the integer part, $\Gamma(u) = (\int_{-\pi/2}^{\pi/2}(u(x) + \pi)\,dx)^{-1}$, $\tau(x) = 4x - \frac{\pi}{2}$, and the function $f_1$ is defined as follows: given $n \in \mathbb{N}$ we put

$$f_1(u) = \begin{cases} \cos(\frac{\pi}{2}k - \frac{n\pi}{2}u) & \text{if } u \in (\frac{(k-1)}{n}, \frac{k}{n}], k = 1, \dots, n, \\ f_1(u+1) & \text{if } u \leq 0, \\ f_1(u-1) & \text{if } u > 1. \end{cases}$$

Notice that problem (4.1) falls inside our framework with the following choices:

$$I = \left[0, \frac{\pi}{4}\right], \qquad I_- = \left[-\frac{\pi}{2}, 0\right], \qquad I_+ = \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \qquad \tilde{I} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$f(x, \gamma, u, v) = \Gamma(\gamma)\left\{f_1(u) + \left[\frac{1}{\sqrt{x}}\right]v^2\right\},$$
$$\phi_1(x) = \phi_2(x) = \cos(x).$$

Our main goal in this section is to show that problem (4.1) has the extremal solutions between suitable lower and upper solutions. We will do it in several steps and we begin by checking the measurability conditions.

**Lemma 4.2** *Given a continuous function $\lambda : \tilde{I} \longrightarrow \mathbb{R}$ and $\rho \in W^{2,1}(I)$ the composition*

$$x \in I \longmapsto f\big(x, \rho, \lambda(x), \lambda\big(\tau(x)\big)\big)$$

*is measurable.*

*Proof* We have to check that the function

$$x \in I \longrightarrow \Gamma(\rho)\left(f_1\big(\lambda(x)\big) + \left[\frac{1}{\sqrt{x}}\right]\big(\lambda\big(\tau(x)\big)\big)^2\right) \tag{4.2}$$

is measurable.

First notice that $\lambda \circ \tau$ is measurable because it is the composition of a continuous function and a measurable one. As $x \longmapsto [\frac{1}{\sqrt{x}}]$ is a monotone function it is also measurable, and so is

$$x \longmapsto \left[\frac{1}{\sqrt{x}}\right]\big(\lambda\big(\tau(x)\big)\big)^2.$$

On the other hand, the function $f_1$ is continuous almost everywhere and so the composition $f_1 \circ \lambda$ is measurable. Therefore (4.2) is a measurable function as a linear combination of measurable ones. □

Now we show the existence of well-ordered lower and upper solutions.

**Proposition 4.3** *Functions $\alpha \equiv 0$ and $\beta(x) = \cos(x)$ are, respectively, a lower and an upper solution for problem* (4.1).

*Proof* First it is trivially satisfied that $\alpha \leq \beta$ on $\tilde{I}$. On the other hand, as we have $\mathscr{L}[\alpha](x) = 0$ and $\mathscr{L}[\beta](x) = (M+1)\cos(x) \geq 0$, we also obtain $\mathscr{L}[\alpha](x) \leq \mathscr{L}[\beta](x)$ for all $x \in \tilde{I}$.

Second, as $\mathscr{L}[\phi](x) = (M+1)\cos(x)$ for all $x \in I_- \cup I_+$, the inequalities

$$\mathscr{L}[\alpha](x) \leq \mathscr{L}[\phi](x), \quad \mathscr{L}[\phi](x) - \mathscr{L}[\alpha](x) \leq \mathscr{L}[\phi](0) - \mathscr{L}[\alpha](0) \quad \text{in } I_-$$

and

$$\mathscr{L}[\alpha](x) \leq \mathscr{L}[\phi](x), \quad \mathscr{L}[\phi](x) - \mathscr{L}[\alpha](x) \leq \mathscr{L}[\phi](L) - \mathscr{L}[\alpha](L) \quad \text{in } I_+$$

hold. As $\beta = \phi$ in $I_- \cup I_+$, we conclude that analogous conditions regarding $\beta$ hold too.

On the other hand, inside the interval $I$ we have the following:

$$\mathscr{L}[\alpha]''(x) = 0 = f\big(x, \alpha, \mathscr{L}[\alpha](x), \mathscr{L}[\alpha]\big(\tau(x)\big)\big)$$

and

$$-(M+1)\cos(x) = \mathscr{L}[\beta]''(x) \leq 0 \leq f\big(x, \beta, \mathscr{L}[\beta](x), \mathscr{L}[\beta]\big(\tau(x)\big)\big)$$

in $[0, \frac{\pi}{4}]$, and so $\alpha$ and $\beta$ are a lower and an upper solution for problem (4.1). □

Finally, we prove that problem (4.1) has the extremal solutions between $\alpha$ and $\beta$.

**Proposition 4.4** *If $n \leq 7$ then problem* (4.1) *has the extremal solutions between the lower solution $\alpha \equiv 0$ and the upper solution $\beta(x) = \cos(x)$.*

We will prove the statement by application of Theorem 3.6.

We have already shown in Lemma 4.2 that condition $(\mathrm{H}_1)$ is satisfied. On the other hand, for a.a. $x \in I$, all $\gamma \in [\alpha, \beta]$, all $u \in [\mathscr{L}[\alpha](x), \mathscr{L}[\beta](x)]$, and all $v \in [\mathscr{L}[\alpha](\tau(x)), \mathscr{L}[\beta](\tau(x))]$ we have

$$\big|f(x, \gamma, u, v)\big| \leq \Gamma(0)\left(1 + \frac{1}{\sqrt{x}}\cos^2\big(\tau(x)\big)\right) \leq \frac{1}{\pi^2}\left(1 + \frac{1}{\sqrt{x}}\right),$$

and so condition $(\mathrm{H}_2)$ holds with

$$\psi(x) = \frac{1}{\pi^2}\left(1 + \frac{1}{\sqrt{x}}\right) \in L^1\big([0, \pi/4]\big).$$

Now we will check condition ($H_3$). To do that, first notice that by virtue of Lemma 4.1 we have

$$f_1(\overline{u}) - f_1(u) \leq \frac{n\pi}{2}(\overline{u} - u) \quad \text{for } x \in I \text{ and } \mathscr{L}[\alpha](x) \leq u \leq \overline{u} \leq \mathscr{L}[\beta](x)$$

and

$$\left[\frac{1}{\sqrt{x}}\right]\overline{v}^2 - \left[\frac{1}{\sqrt{x}}\right]v^2 \leq 2\left[\frac{1}{\sqrt{x}}\right](\overline{v} - v)$$
$$\text{for } x \in I \text{ and } \mathscr{L}[\alpha]\big(\tau(x)\big) \leq v \leq \overline{v} \leq \mathscr{L}[\beta]\big(\tau(x)\big).$$

Then we can conclude that condition ($H_3$) is satisfied with $M_1 \equiv \frac{n\pi}{2\pi^2} = \frac{n}{2\pi}$ and $M_2(x) = \frac{2}{\pi^2}[\frac{1}{\sqrt{x}}]$. Indeed, notice that

$$\int_0^{\pi/4} (M_1 + M_2)(x)\,dx \leq \frac{n}{8} + \frac{2}{\pi^2}\int_0^{\pi/4} \frac{1}{\sqrt{x}} = \frac{n}{8} + 2\frac{\sqrt{\pi}}{\pi^2} < \frac{4}{\pi}$$

for $n \leq 7$, and so inequality ($C_3$) holds.

Finally, for a.a. $x \in [0,\pi/4]$, all $u \in [\mathscr{L}[\alpha](x), \mathscr{L}[\beta](x)]$, and all $v \in [\mathscr{L}[\alpha](\tau(x)), \mathscr{L}[\beta](\tau(x))]$ we have

$$f(x,\cdot,u,v) = g(x,u,v)\Gamma(\cdot),$$

where $g(x,u,v) \geq 0$ and $\Gamma$ is nonincreasing in $[\alpha,\beta]$. Therefore, condition ($H_4$) is satisfied.

As the operator $\mathscr{L}$ is inverse positive in the set

$$Z' = \big\{w \in W^{2,1}\big([0,\pi/4]\big) : w(0) \geq 0, w(L) \geq 0\big\}$$

(see for instance [10]) we conclude by application of Theorem 3.6 that problem (4.1) has the extremal solutions inside the functional interval

$$\big\{\gamma \in \mathscr{C}\big([-\pi/2,\pi/2]\big) : 0 \leq \gamma(x) \leq \cos x \text{ for all } x\big\}.$$

**Competing interests**

The author declares that he has no competing interests.

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**References**

1. Bougoffa, L, Rach, R, Wazwaz, A-M: On solutions of boundary value problem for fourth-order beam equations. Math. Model. Anal. **21**(3), 304-318 (2016)
2. Cabada, A, Fialho, J, Minhós, F: Extremal solutions to fourth order discontinuous functional boundary value problems. Math. Nachr. **286**(17-18), 1744-1751 (2013)
3. Drábek, P, Holubová, G: Positive and negative solutions of one-dimensional beam equation. Appl. Math. Lett. **51**, 1-7 (2016)
4. Li, Y: Existence of positive solutions for the cantilever beam equations with fully nonlinear terms. Nonlinear Anal., Real World Appl. **27**, 221-237 (2016)
5. Minhós, F, Gyulov, T, Santos, AI: Lower and upper solutions for a fully nonlinear beam equation. Nonlinear Anal. **71**, 281-292 (2009)
6. Pei, M, Wang, L: New results for positive solutions of singular fourth-order four-point *p*-Laplacian problem. Bound. Value Probl. **2016**, 92 (2016)

7. Craig, RR: Mechanics of Materials, 3rd edn. Willey, New York (2011)
8. Heikkilä, S, Lakshmikantham, V: Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations. Dekker, New York (1994)
9. Figueroa, R: On the discontinuous second-order deviated Dirichlet problem with non-monotone conditions. Math. Nachr. **288**(2-3), 176-184 (2015)
10. Cabada, A: Green's Functions in the Theory of Ordinary Differential Equations. Springer, New York (2011)
11. Cabada, A, Lois, S: Existence results for nonlinear problems with separated boundary conditions. Nonlinear Anal. **35**, 449-456 (1999)