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Global large-data generalized solutions in a two-dimensional chemotaxis-Stokes system with singular sensitivity

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Abstract
This paper considers the following chemotaxis-Stokes system:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (\frac{n}{c}\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t = \Delta u + \nabla P + n\nabla\phi, \\ \nabla \cdot u = 0, \end{cases}$$

in two-dimensional smoothly bounded domains, which can be seen as a model to describe the migration of aerobic bacteria swimming in an incompressible fluid. It is proved that the corresponding initial-boundary value problem possesses a global generalized solution for any sufficiently regular initial data (n_0, c_0, u_0) satisfying $n_0 \geq 0$ and $c_0 > 0$. Moreover, the solution component c satisfies $c(\cdot, t) \stackrel{\star}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ as $t \to \infty$ and $c(\cdot, t) \to 0$ in $L^p(\Omega)$ as $t \to \infty$ for any $p \in [1, \infty)$.

To the best of our knowledge, this is the first result on global solvability in a chemotaxis-Stokes system with singular sensitivity and signal absorption.

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Keywords: chemotaxis; Stokes; global existence; generalized solutions

1 Introduction

In biological contexts, many simple life-forms exhibit a complex collective behavior. Chemotaxis is one particular mechanism responsible for some instances of such demeanor, where the organisms, like bacteria, adapt their movement according to the concentrations of a chemical signal (see [1–4] and the references therein).

In this paper, we consider the following chemotaxis-Stokes system with singular sensitivity:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (\frac{n}{c}\nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, t > 0, \\ u_t = \Delta u + \nabla P + n\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \\ \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0, & x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $n(x,t)$ and $c(x,t)$ denote the density of the bacteria and the concentration of the oxygen, respectively, and $u = u(x,t)$ and P represent the velocity of fluid and the associated pressure, ϕ is a given potential function.

The initial data are assumed to satisfy

$$\begin{cases} n_0 \in C^0(\bar{\Omega}), \quad n_0 \geq 0 \text{ in } \Omega \text{ and } n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega), \quad c_0 > 0 \text{ in } \bar{\Omega}, \\ u_0 \in D(A_r^\beta) \quad \text{for some } \beta \in (\frac{1}{2}, 1) \text{ and } r \in (1,\infty), \end{cases} \tag{1.2}$$

where A_r stands for the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$ (see [5]). Here $L_\sigma^r := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ for $r \in (1,\infty)$. The function ϕ is known and satisfies

$$\phi \in W^{1,\infty}(\Omega). \tag{1.3}$$

This type system arises in mathematical biology to model the evolution of oxygen-driven swimming bacteria in an impressible fluid. In the first equation of system (1.1), it is assumed that besides moving randomly and transported by the fluid, bacteria are able to adapt their swimming upwards gradients of the oxygen to survive, and that the chemotactic stimulus is perceived in accordance with the Weber-Fechner law, thus requiring the chemotactic sensitivity function $S(n,c) := \frac{n}{c}$ proportional to the reciprocal oxygen density $c(x,t)$. In the second equation of system (1.1), it is assumed that the oxygen also diffuses randomly and is transported by the fluid, and is consumed by the bacteria. In the third and fourth equation of system (1.1), the motion of the fluid is modeled by incompressible Stokes equations, and is affected by gravitational force exerted from aggregating bacteria onto the fluid. System (1.1) can be seen as a generalization of the following model, which is proposed by Tuval *et al.* [6] to model the pattern formation and the spontaneous emergence of turbulence observed experimentally when populations of aerobic bacteria are suspended in water:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0, \end{cases} \tag{1.4}$$

where $\kappa \in \mathbb{R}$, $f(c)$ and $\chi(c)$ denote the rate of consumption of the oxygen and the chemotactic sensitivity function, respectively. However, Tuval *et al.* in [6] assumed that $\chi(c)$ is unity at large c and vanishes rapidly for small c, that is, $\chi(c)$ is bounded for any c. For this type of $\chi(c)$, there have been many literatures. For example, many literatures deal with global solvability, boundedness, large time behavior of solutions to the model (1.4) for the bounded domains and the whole space (see [7–14] and the references therein for details). For the model (1.4) with nonlinear diffusion, there also exist some results on global existence, boundedness and large time behavior for the bounded domains and the whole space (see [15–21] and the references therein for details). We also remark that there are several recent works to deal with system (1.4) under the assumption that the oxygen is produced, rather than consumed, by the bacteria (see [22–26]).

However, for the model (1.1), to the best our knowledge, there is no result on global solvability. There are only few rigorous results on global existence and qualitative behavior of solutions to the following fluid-free subcase of system (1.1):

$$\begin{cases} n_t = \Delta n - \nabla \cdot (\frac{n}{c}\nabla c), & x \in \Omega, t > 0, \\ c_t = \Delta c - nc, & x \in \Omega, t > 0, \end{cases} \tag{1.5}$$

which was first proposed by Keller-Segel [2] in 1971. The model (1.5) describes that the cells (*e.g. Escherichiacoli*) are much more primitive in that they merely follow a chemical cue (*e.g.* oxygen), which they cannot produce, but which they consume as a nutrient. In [2], Keller and Segel have discussed that the model (1.5) generates wave-like solution behavior, which has attracted some scholars to study analytically on the existence and stability properties of traveling wave solutions to (1.5) (see [27–29]) and some closely related models (see [30–32]). The singular chemotactic sensitivities as in (1.5) are very important in biology, which have been underlined independently in modeling approaches (see [33–36]) and in tumor angiogenesis (see [4, 37]) and also in taxis-driven morphogen transport (see [38]). In [39, 40], the global existence for the spatially one-dimensional initial-boundary value problems for (1.5) was derived for arbitrary initial data. For the higher-dimensional case, Wang *et al.* [41] proved that the Cauchy problem for (1.5) in $\mathbb{R}^n$ ($n \in \{2,3\}$) possesses globally defined classical solutions for the appropriately small initial data. In [42], Winkler proved that spatially two-dimensional Neumann initial-boundary value problems for (1.5) possess a global generalized solution for any arbitrarily large initial data. Furthermore, some further boundedness and relaxation properties and the large time behavior of $c(x,t)$ are derived. When the second equation in (1.5) is replaced by the ODE $c_t = -nc$, the global existence is known only in one-dimensional cases, whereas in higher-dimensional cases the corresponding results have been obtained only under sufficient smallness conditions on the initial data. However, unlike model (1.5), there have been many works for the classical Keller-Segel model and its variants (see [43–49] and the references therein, for instance).

Recently, Winkler in [50] constructed large-data global generalized solutions to a two-dimensional chemotaxis system with tensor-valued sensitivities, and in [42] he also constructed large-data global generalized solutions to a two-dimensional chemotaxis system with singular sensitivity. Motivated by the above works, the goal of this paper is to deal with global solvability and the large time behavior of $c(x,t)$ in the two-dimensional version of system (1.1)-(1.2) for arbitrary large initial data in an appropriate framework. We now state the main results of this paper.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and ϕ satisfy* (1.3). *Suppose that n_0, c_0 and u_0 comply with* (1.2). *Then there exists at least one triple of functions*

$$\begin{aligned} &n \in L^1_{\mathrm{loc}}\big(\bar{\Omega} \times [0,\infty)\big), \\ &c \in L^\infty\big(\Omega \times (0,\infty)\big) \cap L^2_{\mathrm{loc}}\big([0,\infty); W^{1,2}(\Omega)\big) \quad \textit{and} \\ &u \in L^2_{\mathrm{loc}}\big(\bar{\Omega} \times [0,\infty)\big) \cap \bigcap_{p\in[1,2)} L^p_{\mathrm{loc}}\big([0,\infty); W^{1,p}_0(\Omega)\big) \end{aligned} \tag{1.6}$$

such that (n, c, u) *is a global generalized solution in the sense of Definition* 2.1 *below. The solution component c satisfies*

$$c(\cdot, t) \overset{\star}{\rightharpoonup} 0 \quad \text{in } L^\infty(\Omega) \text{ as } t \to \infty \tag{1.7}$$

and

$$c(\cdot, t) \to 0 \quad \text{in } L^p(\Omega) \text{ as } t \to \infty \tag{1.8}$$

for any $p \in [1, \infty)$. *Moreover, the solution component c has the additional property that*

$$c \in C^0_{w^\star}\bigl([0, \infty); L^\infty(\Omega)\bigr), \tag{1.9}$$

that is, c is continuous on $[0, \infty)$ *as an* $L^\infty(\Omega)$*-valued function with respect to the weak-*$\star$ *topology possibly after redefinition on a null set of times.*

To the best of our knowledge, this is the first result on global solvability in a chemotaxis-Stokes system with singular sensitivity and signal absorption of type (1.1).

The rest of this paper is arranged as follows. In Section 2, we first give the concept of global generalized solutions and then derive *a priori* estimates for the approximate solutions to the approximate problems (2.11) and (2.26). In Section 3, we complete the proof of Theorem 1.1 by an approximation procedure.

2 A generalized solution concept and *a priori* estimates

2.1 A generalized solution concept and the approximate problems

First of all, we specify our solution concept. As far as the second component c and the third u are concerned, a generalized solution of the respective sub-problem of (1.1) is straightforward. The most important part of a generalized solution concept is with respect to the first equation in (1.1). This concept is very weak due to the poor regularity of solutions. Our solution concept parallels the generalized solution concept in the fluid-free chemotaxis system which is studied in [42].

Definition 2.1 Suppose that n_0, c_0, and u_0 satisfy (1.2). Then a triple (n, c, u) of functions

$$\begin{cases} n \in L^1_{\mathrm{loc}}(\bar\Omega \times [0, \infty)), \\ c \in L^\infty_{\mathrm{loc}}(\Omega \times (0, \infty)) \cap L^2_{\mathrm{loc}}([0, \infty); W^{1,2}(\Omega)) \quad \text{and} \\ u \in L^1_{\mathrm{loc}}([0, \infty); W^{1,1}_0(\Omega)) \end{cases} \tag{2.1}$$

with

$$\begin{cases} n \geq 0 \quad \text{a.e. in } \Omega \times (0, \infty), \\ c > 0 \quad \text{a.e. in } \Omega \times (0, \infty), \\ \nabla \cdot u = 0 \quad \text{a.e. in } \Omega \times (0, \infty) \end{cases} \tag{2.2}$$

as well as

$$\nabla \ln(n+1) \in L^2_{\mathrm{loc}}\bigl(\bar\Omega \times [0, \infty)\bigr) \quad \text{and} \quad \nabla \ln c \in L^2_{\mathrm{loc}}\bigl(\bar\Omega \times [0, \infty)\bigr), \tag{2.3}$$

will be called a global generalized solution of (1.1) if n satisfies the mass conservation property

$$\int_\Omega n(x,t)\,dx = \int_\Omega n_0(x)\,dx \quad \text{for a.e. } t>0, \tag{2.4}$$

if the inequality

$$\begin{aligned}
&-\int_0^\infty \int_\Omega \ln(n+1)\varphi_t\,dx\,dt - \int_\Omega \ln(n_0+1)\varphi(x,0)\,dx \\
&\quad \geq \int_0^\infty \int_\Omega \bigl|\nabla \ln(n+1)\bigr|^2 \varphi\,dx\,dt - \int_0^\infty \int_\Omega \nabla \ln(n+1)\cdot\nabla\varphi\,dx\,dt \\
&\qquad - \int_0^\infty \int_\Omega \frac{n}{n+1}\bigl(\nabla \ln(n+1)\cdot\nabla \ln c\bigr)\varphi\,dx\,dt + \int_0^\infty \int_\Omega \frac{n}{n+1}\nabla \ln c\cdot\nabla\varphi\,dx\,dt \\
&\qquad + \int_0^\infty \int_\Omega \ln(n+1)(u\cdot\nabla\varphi)\,dx\,dt
\end{aligned} \tag{2.5}$$

holds for each nonnegative $\varphi \in C_0^\infty(\bar\Omega\times[0,\infty))$, if moreover the identity

$$\begin{aligned}
&\int_0^\infty \int_\Omega c\varphi_t\,dx\,dt + \int_\Omega c_0\varphi(x,0)\,dx \\
&\quad = \int_0^\infty \int_\Omega \nabla c\cdot\nabla\varphi\,dx\,dt + \int_0^\infty \int_\Omega nc\varphi\,dx\,dt - \int_0^\infty \int_\Omega c(u\cdot\nabla\varphi)\,dx\,dt
\end{aligned} \tag{2.6}$$

holds for any $\varphi \in L^\infty(\bar\Omega\times(0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ having compact support in $\bar\Omega \times [0,\infty)$ with $\varphi_t \in L^2(\Omega\times(0,\infty))$, and if finally the identity

$$\begin{aligned}
&-\int_0^\infty \int_\Omega u\varphi_t\,dx\,dt - \int_\Omega u_0\varphi(x,0)\,dx \\
&\quad = -\int_0^\infty \int_\Omega \nabla u\cdot\nabla\varphi\,dx\,dt + \int_0^\infty \int_\Omega n\nabla\phi\cdot\varphi\,dx\,dt
\end{aligned} \tag{2.7}$$

is valid for all $\varphi \in C_0^\infty(\Omega\times[0,\infty);\mathbb{R}^2)$ with $\nabla\cdot\varphi = 0$.

Remark

(i) The regularity requirements in (2.1), (2.2), and (2.3) along with the fact that $0 \leq \ln(n+1) \leq n$ for all $n \geq 0$ ensure that all integrals in (2.5), (2.6), and (2.7) are well defined.

(ii) Under the hypotheses in Definition 2.1, it is well known [5] that there exists a distribution P on $\Omega\times(0,\infty)$ such that $u_t = \Delta u + \nabla P + n\nabla\phi$ holds in $\mathcal{D}'(\Omega\times(0,\infty))$.

(iii) Following the proof of a statement in [50], Lemma 2.1, and in conjunction with the mass conversation identity (2.4), we can see that if $n \geq 0$ and $c > 0$ are functions from $C^0(\bar\Omega\times[0,\infty)) \cap C^{2,1}(\bar\Omega\times(0,\infty))$ and $u \in C^0(\bar\Omega\times[0,\infty);\mathbb{R}^2) \cap C^{2,1}(\bar\Omega\times(0,\infty);\mathbb{R}^2)$ such that $\nabla\cdot u \equiv 0$ and such that (n,c,u) is a global generalized solution of (1.1) in the sense that Definition 2.1, then there exists $P \in C^{1,0}(\Omega\times(0,\infty))$ such that (n,c,u,P) also is a classical solution of (1.1) in $\Omega\times(0,\infty)$.

In order to construct a global generalized solution of (1.1) in the above sense, following the approaches in [42] we fix a nonincreasing cut-off function $\rho \in C^\infty([0,\infty))$ satisfying $\rho \equiv 1$ in $[0,1]$ and $\rho \equiv 0$ in $[2,\infty)$ and define $f_\varepsilon \in C^\infty([0,\infty))$ by letting

$$f_\varepsilon(s) := \int_0^s \rho(\varepsilon\sigma)\,d\sigma, \quad s \geq 0 \tag{2.8}$$

for $\varepsilon \in (0,1)$. Then for any such ε and ρ, f_ε fulfills

$$f_\varepsilon(0) = 0 \quad \text{and} \quad 0 \leq f_\varepsilon' \leq 1 \quad \text{on } [0,\infty) \tag{2.9}$$

and

$$f_\varepsilon(s) = s \quad \text{for all } s \in \left[0, \frac{1}{\varepsilon}\right] \quad \text{and} \quad f_\varepsilon'(s) = 0 \quad \text{for all } s \geq \frac{2}{\varepsilon} \tag{2.10}$$

as well as

$$f_\varepsilon(s) \nearrow s \quad \text{and} \quad f_\varepsilon'(s) \nearrow 1 \quad \text{as } \varepsilon \searrow 0 \text{ for each } s \geq 0.$$

Thus, for any such ε, the approximate problems

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (\frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{c_\varepsilon} \nabla c_\varepsilon), \quad x \in \Omega, t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - f_\varepsilon(n_\varepsilon) c_\varepsilon, \quad x \in \Omega, t > 0, \\ u_{\varepsilon t} = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u_\varepsilon = 0, \quad x \in \Omega, t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \qquad u_\varepsilon = 0, \quad x \in \partial\Omega, t > 0, \\ n_\varepsilon(x,0) = n_0(x), \qquad c_\varepsilon(x,0) = c_0(x), \qquad u_\varepsilon(x,0) = u_0(x), \quad x \in \Omega \end{cases} \tag{2.11}$$

are indeed globally solvable in the classical sense.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and (n_0, c_0, u_0) satisfy* (1.2). *Let $\varepsilon \in (0,1)$ and $\vartheta > 2$. Then there exist functions*

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{\mathrm{loc}}([0,\infty); W^{1,\vartheta}(\Omega)), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0,\infty); \mathbb{R}^2) \cap C^{2,1}(\bar{\Omega} \times (0,\infty); \mathbb{R}^2), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0,\infty)) \end{cases}$$

such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.11) *classically in $\Omega \times (0,\infty)$, and such that $n_\varepsilon > 0$ in $\bar{\Omega} \times (0,\infty)$ and*

$$\int_\Omega n_\varepsilon(x,t)\,dx = \int_\Omega n_0(x)\,dx \quad \textit{for all } t > 0 \tag{2.12}$$

as well as

$$0 < c_\varepsilon \leq \|c_0\|_{L^\infty(\Omega)} \quad \textit{in } \bar{\Omega} \times [0,\infty). \tag{2.13}$$

Moreover, this solution is unique, up to addition of constants to P.

Proof By taking a well-known fixed point argument (see [12], Lemma 2.1, for details), one can readily verify that for each $\varepsilon \in (0,1)$ and $\vartheta > 2$ there exist $T_{\max,\varepsilon} \in (0,\infty]$ and functions

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})) \cap L^\infty_{\mathrm{loc}}([0,T_{\max,\varepsilon}); W^{1,\vartheta}(\Omega)), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon}); \mathbb{R}^2) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon}); \mathbb{R}^2), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0,T_{\max,\varepsilon})) \end{cases}$$

with $n_\varepsilon > 0$ in $\bar{\Omega} \times (0,T_{\max,\varepsilon})$ and $c_\varepsilon > 0$ in $\bar{\Omega} \times [0,T_{\max,\varepsilon})$, such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a classical solution in $\Omega \times (0,T_{\max,\varepsilon})$. This solution is unique, up to addition of constants to P. Moreover, we have

$$\begin{aligned} &\text{either} \quad T_{\max,\varepsilon} = \infty, \quad \text{or} \\ &\limsup_{t \nearrow T_{\max,\varepsilon}} \big(\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \big) \to \infty, \quad \text{or} \\ &\liminf_{t \nearrow T_{\max,\varepsilon}} \inf_{x\in\Omega} c_\varepsilon(x,t) = 0, \end{aligned} \tag{2.14}$$

where A and β are given in (1.2).

By integrating the first equation in (2.11) over Ω and applying a parabolic comparison argument to the second equation in (2.11), we obtain

$$\|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)} \quad \text{for all } t \in (0,T_{\max,\varepsilon}) \tag{2.15}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0,T_{\max,\varepsilon}). \tag{2.16}$$

To prove this lemma, we need to verify that for any fixed $\varepsilon \in (0,1)$ the corresponding maximal existence time $T_{\max,\varepsilon}$ is equal to ∞. We assume $T_{\max,\varepsilon} < \infty$ and we will show that neither the second nor the third alternative in (2.14) can occur. Since $\operatorname{supp} f'_\varepsilon \subset [0, \frac{2}{\varepsilon}]$ by (2.10), we apply the maximum principle to the first equation in (2.11) to show that

$$n_\varepsilon(x,t) \le C_1(\varepsilon) := \max\left\{ \|n_0\|_{L^\infty(\Omega)}, \frac{2}{\varepsilon} \right\} \tag{2.17}$$

for all $x \in \Omega$ and $t \in (0,T_{\max,\varepsilon})$. By applying (2.15), from [25], Lemma 2.4, we see that for any given $p \in (1,\infty)$ there exists a constant $C > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \quad \text{for all } t \in (0,T_{\max,\varepsilon}). \tag{2.18}$$

Since the Stokes operator $A = -\mathcal{P}\Delta$ is sectorial and generates a contraction semigroup $(e^{-tA})_{t\ge0}$ in $L^2(\Omega)$, where $\mathcal{P}$ represents the Helmholtz projection in $L^2(\Omega)$, for the fluid equation in (2.11) we have

$$u_\varepsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}\big(n_\varepsilon(\cdot,s)\nabla\phi\big)\,ds$$

for all $t \in (0, T_{\max,\varepsilon})$. Applying A^β ($\beta \in (\frac{1}{2}, 1)$) to the above formula, we see that there exists some $\lambda > 0$ such that

$$\begin{aligned}\|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le \|A^\beta e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\beta e^{-(t-s)A}\mathcal{P}(n_\varepsilon(\cdot,s)\nabla\phi)\|_{L^2(\Omega)}\,ds \\ &\le \|e^{-tA}A^\beta u_0\|_{L^2(\Omega)} + C_2\int_0^t (t-s)^{-\beta}e^{-\lambda(t-s)}\|\mathcal{P}(n_\varepsilon(\cdot,s)\nabla\phi)\|_{L^2(\Omega)}\,ds \\ &\le C_3 + C_4\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\int_0^t (t-s)^{-\beta}e^{-\lambda(t-s)}\,ds \le C_3 + C_5 \end{aligned} \tag{2.19}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $C_2, \dots, C_5$ are some positive constants.

For the second equation in (2.11), from the variation of constant formula we can represent c_ε by

$$c_\varepsilon(\cdot,t) = e^{t\Delta}c_0 - \int_0^t e^{(t-s)\Delta}\big(u_\varepsilon\cdot\nabla c_\varepsilon + n_\varepsilon f_\varepsilon(c_\varepsilon)\big)\,ds \quad \text{for all } t \in (0, T_{\max}),$$

where $(e^{t\Delta})_{t\ge 0}$ denotes the Neumann heat semigroup in Ω. Let $\lambda_1 > 0$ represent the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. For each $\vartheta > 2$, we follow the $L^p - L^q$ estimates for Neumann heat semigroup to obtain

$$\begin{aligned}\|\nabla c_\varepsilon(\cdot,t)\|_{L^\vartheta(\Omega)} &\le C_6\|\nabla c_0\|_{L^\vartheta(\Omega)} \\ &\quad + C_6\int_0^t (t-s)^{-1+\frac{1}{2\vartheta}}e^{-\lambda_1(t-s)}\|u_\varepsilon\cdot\nabla c_\varepsilon + n_\varepsilon f_\varepsilon(c_\varepsilon)\|_{L^{\frac{2\vartheta}{1+\vartheta}}(\Omega)}\,ds \\ &\le C_7 + C_8\int_0^t (t-s)^{-1+\frac{1}{2\vartheta}}e^{-\lambda_1(t-s)}\|u_\varepsilon\|_{L^{2\vartheta}(\Omega)}\|\nabla c_\varepsilon\|_{L^2(\Omega)}\,ds \\ &\le C_7 + C_9\int_0^t (t-s)^{-1+\frac{1}{2\vartheta}}e^{-\lambda_1(t-s)}\|\nabla c_\varepsilon\|_{L^2(\Omega)}\,ds \end{aligned} \tag{2.20}$$

with some positive constants C_6, C_7, C_8, and C_9 for all $t \in (0, T_{\max,\varepsilon})$, where we used (2.18) in the last step. We now go to estimate $\|\nabla c_\varepsilon\|_{L^2(\Omega)}$. Multiplying the second equation in (2.11) by $-\Delta c_\varepsilon$ and integrating by parts over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2\,dx + \int_\Omega |\Delta c_\varepsilon|^2\,dx \\ &\quad= \int_\Omega \Delta c_\varepsilon\nabla c_\varepsilon\cdot u_\varepsilon\,dx + \int_\Omega f_\varepsilon(n_\varepsilon)c_\varepsilon\Delta c_\varepsilon\,dx \\ &\quad\le \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2\,dx + \int_\Omega |\nabla c_\varepsilon|^2|u_\varepsilon|^2\,dx + \int_\Omega f_\varepsilon^2(n_\varepsilon)c_\varepsilon^2\,dx \end{aligned} \tag{2.21}$$

for all $t \in (0, T_{\max,\varepsilon})$. According to the estimate of n_ε in (2.17) and the definition of $f_\varepsilon(n_\varepsilon)$ in (2.8) along with the boundedness of c_ε in (2.16) and then using Hölder's inequality, we derive that there exist some positive constants C_{10} and C_{11} and $q > 2$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2\,dx + \int_\Omega |\Delta c_\varepsilon|^2\,dx \\ &\quad\le \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2\,dx + \|u_\varepsilon\|_{L^q(\Omega)}^2\|\nabla c_\varepsilon\|^2_{L^{\frac{2q}{q-2}}(\Omega)} + C_{10} \\ &\quad\le \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2\,dx + C_{11}\|\nabla c_\varepsilon\|^2_{L^{\frac{2q}{q-2}}(\Omega)} + C_{10} \end{aligned} \tag{2.22}$$

for all $t \in (0, T_{\max,\varepsilon})$. An application of the Gagliardo-Nirenberg inequality implies that

$$\begin{aligned}
\|\nabla c_\varepsilon\|^2_{L^{\frac{2q}{q-2}}(\Omega)} &\le C_{12}\|\Delta c_\varepsilon\|^{\frac{4}{q}}_{L^2(\Omega)}\|c_\varepsilon\|^{\frac{2(q-2)}{q}}_{L^\infty(\Omega)} + C_{12}\|c_\varepsilon\|^2_{L^\infty(\Omega)} \\
&\le C_{13}\|\Delta c_\varepsilon\|^{\frac{4}{q}}_{L^2(\Omega)} + C_{13} \\
&\le \frac{1}{4C_{11}}\|\Delta c_\varepsilon\|^2_{L^2(\Omega)} + C_{14} \quad \text{for all } t \in (0, T_{\max,\varepsilon})
\end{aligned}$$

with some constants $C_{12} > 0$, $C_{13} > 0$, and $C_{14} > 0$. Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2\,dx + \frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2\,dx \le C_{10} + C_{11}C_{14} \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

Therefore, we obtain $\int_\Omega |\nabla c_\varepsilon|^2\,dx \le 2(C_{10} + C_{11}C_{14})T_{\max,\varepsilon} + \int_\Omega |\nabla c_0|^2\,dx$. Inserting it into (2.20) and using the boundedness of c_ε implies that

$$\big\|c_\varepsilon(\cdot,t)\big\|_{W^{1,\vartheta}(\Omega)} \le C(T_{\max,\varepsilon}) \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.23}$$

We define $\tilde{c} := \{\min_{y\in\bar{\Omega}} c_0(y)\}e^{C_1(\varepsilon)t}$, where $C_1(\varepsilon)$ is defined in (2.17). We can easily verify that $\tilde{c}$ is a subsolution to the second equation in (2.11). Therefore, we obtain

$$c_\varepsilon(x,t) \ge \Big\{\min_{y\in\bar{\Omega}} c_0(y)\Big\}e^{C_1(\varepsilon)t} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\max,\varepsilon}). \tag{2.24}$$

Thus, (2.17), (2.19), (2.23), and (2.24) exclude the second and the third alternatives in (2.14) and we complete the proof. □

Similar to [42], we define

$$w_\varepsilon(x,t) := -\ln\left(\frac{c_\varepsilon(x,t)}{\|c_0\|_{L^\infty(\Omega)}}\right), \quad (x,t) \in \bar{\Omega}\times[0,\infty), \varepsilon \in (0,1) \tag{2.25}$$

for convenience of the following estimates. Substituting it into (2.11), we see that the corresponding system

$$\begin{cases}
n_{\varepsilon t} + u_\varepsilon\cdot\nabla n_\varepsilon = \Delta n_\varepsilon + \nabla\cdot(n_\varepsilon f_\varepsilon'(n_\varepsilon)\nabla w_\varepsilon), & x \in \Omega, t > 0, \\
w_{\varepsilon t} + u_\varepsilon\cdot\nabla w_\varepsilon = \Delta w_\varepsilon - |\nabla w_\varepsilon|^2 + f_\varepsilon(n_\varepsilon), & x \in \Omega, t > 0, \\
u_{\varepsilon t} = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon\nabla\phi, & x \in \Omega, t > 0, \\
\nabla\cdot u_\varepsilon = 0, & x \in \Omega, t > 0, \\
\frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial w_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega, t > 0, \\
n_\varepsilon(x,0) = n_0(x), \quad w_\varepsilon(x,0) = -\ln(\frac{c_0(x)}{\|c_0\|_{L^\infty(\Omega)}}), \quad u_\varepsilon(x,0) = u_0(x), & x \in \Omega
\end{cases} \tag{2.26}$$

admits a global classical solution $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ satisfying $n_\varepsilon \ge 0$ and $w_\varepsilon \ge 0$.

2.2 Some basic $\boldsymbol{\varepsilon}$-independent *a priori* estimates and compactness properties of $((n_\varepsilon, w_\varepsilon, u_\varepsilon))_{\varepsilon\in(0,1)}$

In this subsection, we derive some basic ε-independent *a priori* estimates for the solutions $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ to (2.26) and obtain some compactness properties.

Lemma 2.2 *Suppose that* $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ *is a classical solution to* (2.26). *Then for all* $\varepsilon \in (0,1)$, *we have*

$$\int_\Omega w_\varepsilon(x,t)\,dx + \int_0^t \int_\Omega |\nabla w_\varepsilon|^2\,dx\,ds \le \int_\Omega w_0\,dx + mt \quad \textit{for all } t > 0 \tag{2.27}$$

and

$$\int_0^t \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\,dx\,ds \le \int_\Omega w_0\,dx + 2m + mt \quad \textit{for all } t > 0 \tag{2.28}$$

as well as

$$\int_0^t \int_\Omega |\nabla c_\varepsilon|^2\,dx\,ds \le \int_\Omega c_0^2\,dx \quad \textit{for all } t > 0, \tag{2.29}$$

where $m := \int_\Omega n_0\,dx$.

Proof Integrating the second equation in (2.26) over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_\varepsilon\,dx + \int_\Omega |\nabla w_\varepsilon|^2 d &= \int_\Omega f_\varepsilon(n_\varepsilon)\,dx \\ &\le \int_\Omega n_\varepsilon\,dx = \int_\Omega n_0\,dx = m \quad \text{for all } t > 0,\end{aligned}$$

where we used $f_\varepsilon(n_\varepsilon) \le n_\varepsilon$ for all $t > 0$ by (2.8). Integrating the above inequality with respect to time yields (2.27). Moreover, due to the nonnegativity of w_ε, we have

$$\int_\Omega w_\varepsilon(\cdot,t)\,dx \le \int_\Omega w_0\,dx + mt \quad \text{for all } t > 0 \tag{2.30}$$

as well as

$$\int_0^t \int_\Omega |\nabla w_\varepsilon|^2\,dx\,ds \le \int_\Omega w_0\,dx + mt \quad \text{for all } t > 0. \tag{2.31}$$

Multiplying the first equation in (2.26) by $\frac{1}{n_\varepsilon+1}$ and integrating by parts over Ω, we derive that

$$\frac{d}{dt}\int_\Omega \ln(n_\varepsilon + 1)\,dx = \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\,dx + \int_\Omega n_\varepsilon f'_\varepsilon(n_\varepsilon)\nabla w_\varepsilon \cdot \frac{\nabla n_\varepsilon}{(n_\varepsilon + 1)^2}\,dx \tag{2.32}$$

for all $t > 0$. By using Young's inequality and $0 \le f'_\varepsilon \le 1$ for all $t > 0$, we have

$$\begin{aligned}\left|\int_\Omega n_\varepsilon f'_\varepsilon(n_\varepsilon)\nabla w_\varepsilon \cdot \frac{\nabla n_\varepsilon}{(n_\varepsilon + 1)^2}\,dx\right| &\le \frac{1}{2}\int_\Omega \left(\frac{n_\varepsilon}{n_\varepsilon + 1} f'_\varepsilon(n_\varepsilon)\right)^2 |\nabla w_\varepsilon|^2\,dx + \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\,dx \\ &\le \frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2\,dx + \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\,dx \quad \text{for all } t > 0.\end{aligned}$$

Inserting it into (2.32) yields

$$\frac{d}{dt}\int_\Omega \ln(n_\varepsilon + 1)\,dx \ge \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon + 1)^2}\,dx - \frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2\,dx \quad \text{for all } t > 0. \tag{2.33}$$

Integrating (2.33) in time and using $0 \le \ln(n_\varepsilon + 1) \le n_\varepsilon$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^t \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx\,ds &\le \frac{1}{2}\int_0^t \int_\Omega |\nabla w_\varepsilon|^2\,dx\,ds + \int_\Omega \bigl\{\ln(n_\varepsilon+1) - \ln(n_0+1)\bigr\}\,dx \\
&\le \frac{1}{2}\int_0^t \int_\Omega |\nabla w_\varepsilon|^2\,dx\,ds + \int_\Omega n_\varepsilon\,dx \\
&\le \frac{1}{2}\biggl(\int_\Omega w_0\,dx + mt\biggr) + m \quad \text{for all } t > 0
\end{aligned} \tag{2.34}$$

according to the mass conversation property and (2.31). This yields (2.28).

Multiplying the second equation in (2.11) by c_ε and integrating by parts over Ω, we show that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2\,dx + \int_\Omega |\nabla c_\varepsilon|^2\,dx = -\int_\Omega c_\varepsilon^2 f_\varepsilon(n_\varepsilon)\,dx \quad \text{for all } t > 0.$$

Due to the nonnegativity of $f(n_\varepsilon)$, we have $\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2\,dx + \int_\Omega |\nabla c_\varepsilon|^2\,dx \le 0$ for all $t > 0$. Integrating it in time shows that (2.29) holds. □

The following lemma is on the estimates of u_ε, which has been proved in [25], Section 2.2. Meanwhile, the scholars in [51–53] also studied some regularity results for the stationary Stokes system, the p-Laplacian in N space variables and the parabolic obstacle problems, respectively. We omit the proof of the following lemma for brevity.

Lemma 2.3 ([25])

(i) *Let* $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ *be a classical solution to* (2.26). *For any given* $p \in (1,\infty)$, *there exists a positive constant* $C = C(p, u_0, n_0, \phi)$ *such that, for any* $\varepsilon \in (0,1)$,

$$\bigl\|u_\varepsilon(\cdot,t)\bigr\|_{L^p(\Omega)} \le C \quad \textit{for all } t > 0. \tag{2.35}$$

(ii) *Let* $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ *be a classical solution to* (2.26). *For any given* $r \in (1,2)$, *there exists a positive constant* $C = C(r, u_0, n_0, \phi)$ *such that, for any* $\varepsilon \in (0,1)$,

$$\bigl\|u_\varepsilon(\cdot,t)\bigr\|_{W^{1,r}(\Omega)} \le C \quad \textit{for all } t > 0. \tag{2.36}$$

In order to obtain the compactness properties of n_ε, w_ε and c_ε, we need to derive some regularity properties of time derivatives.

Lemma 2.4 *Let* $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ *be a classical solution to* (2.26). *Then for all* $T > 0$ *and each* $p > 2$, *there exist constants* $C(T) > 0$ *and* $C(p,T) > 0$ *such that, for any* $\varepsilon \in (0,1)$,

$$\int_0^T \bigl\|\partial_t \ln\bigl(n_\varepsilon(\cdot,t)+1\bigr)\bigr\|_{(W^{2,2}(\Omega))^\star}\,dt \le C(T) \tag{2.37}$$

and

$$\int_0^T \bigl\|w_{\varepsilon t}(\cdot,t)\bigr\|_{(W^{2,2}(\Omega))^\star}\,dt \le C(T) \tag{2.38}$$

as well as

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,p}(\Omega))^\star}\,dt \le C(p,T). \tag{2.39}$$

Proof We take $\psi \in C^\infty(\bar\Omega)$ and test the first equation in (2.26) by $\frac{\psi}{n_\varepsilon(x,t)+1}$ for any fixed $t > 0$ to obtain

$$\begin{aligned}
&\int_\Omega \partial_t \ln\bigl(n_\varepsilon(x,t)+1\bigr)\psi\,dx \\
&\quad = -\int_\Omega \frac{1}{n_\varepsilon+1}\nabla n_\varepsilon\cdot\nabla\psi\,dx + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\psi\,dx \\
&\qquad - \int_\Omega \frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\nabla w_\varepsilon\cdot\nabla\psi\,dx + \int_\Omega \frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{(n_\varepsilon+1)^2}(\nabla w_\varepsilon\cdot\nabla n_\varepsilon)\psi\,dx \\
&\qquad - \int_\Omega (u_\varepsilon\cdot\nabla n_\varepsilon)\frac{\psi}{n_\varepsilon(\cdot,t)+1}\,dx.
\end{aligned} \tag{2.40}$$

Using the Cauchy-Schwarz inequality and Young's inequality several times yields

$$\begin{aligned}
&\left|\int_\Omega \partial_t \ln\bigl(n_\varepsilon(x,t)+1\bigr)\psi\,dx\right| \\
&\quad \le \left\{\left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx\right)^{\frac12} + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx + \left(\int_\Omega |\nabla w_\varepsilon|^2\,dx\right)^{\frac12}\right. \\
&\qquad + \left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx\right)^{\frac12}\left(\int_\Omega |\nabla w_\varepsilon|^2\,dx\right)^{\frac12} \\
&\qquad \left. + \left(\int_\Omega |u_\varepsilon|^2\,dx\right)^{\frac12}\left(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx\right)^{\frac12}\right\}\cdot\bigl\{\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)}\bigr\} \\
&\quad \le \left\{3\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx + \int_\Omega |\nabla w_\varepsilon|^2\,dx + \frac14\int_\Omega |u_\varepsilon|^2\,dx + 1\right\} \\
&\qquad \cdot\bigl\{\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)}\bigr\}
\end{aligned} \tag{2.41}$$

for all $\psi \in C^\infty(\bar\Omega)$, where we used the fact that $0 \le f_\varepsilon' \le 1$ for all $t > 0$. Since $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, there exists a constant $C_1 > 0$ such that $\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} \le C_1\|\psi\|_{W^{2,2}(\Omega)}$. Thus, we obtain

$$\bigl\|\partial_t \ln\bigl(n_\varepsilon(\cdot,t)+1\bigr)\bigr\|_{(W^{2,2}(\Omega))^\star} \le C_1\left\{3\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx + \int_\Omega |\nabla w_\varepsilon|^2\,dx + \frac14\int_\Omega |u_\varepsilon|^2\,dx + 1\right\}$$

for all $t > 0$. Integrating the above inequality in time and using (2.28), (2.31), and (2.35), we see that there exists a constant $C(T) > 0$ such that (2.37) holds.

Similarly, for arbitrary $\psi \in C^\infty(\bar\Omega)$ and fixed $t > 0$, we derive from the second equation in (2.26) that

$$\begin{aligned}
&\left|\int_\Omega w_{\varepsilon t}(x,t)\psi\,dx\right| \\
&\quad = \left|\int_\Omega \bigl(\Delta w_\varepsilon - u_\varepsilon\cdot\nabla w_\varepsilon - |\nabla w_\varepsilon|^2 + f_\varepsilon(n_\varepsilon)\bigr)\psi\,dx\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| -\int_\Omega \nabla w_\varepsilon \cdot \nabla \psi \,dx - \int_\Omega u_\varepsilon \cdot \nabla w_\varepsilon \psi \,dx - \int_\Omega |\nabla w_\varepsilon|^2 \psi \,dx + \int_\Omega f_\varepsilon(n_\varepsilon)\psi \,dx \right| \\
&\le \left\{ \left(\int_\Omega |\nabla w_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} + \left(\int_\Omega |u_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \int_\Omega |\nabla w_\varepsilon|^2 \,dx + \int_\Omega n_\varepsilon \,dx \right\} \cdot \left\{ \|\psi\|_{L^\infty(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \right\} \\
&\le \left\{ 2\int_\Omega |\nabla w_\varepsilon|^2 \,dx + \frac{1}{2}\int_\Omega |u_\varepsilon|^2 \,dx + \frac{1}{2} + \int_\Omega n_0 \,dx \right\} \cdot \left\{ \|\psi\|_{L^\infty(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \right\} \\
&\le \left\{ 2\int_\Omega |\nabla w_\varepsilon|^2 \,dx + \frac{1}{2}\int_\Omega |u_\varepsilon|^2 \,dx + \frac{1}{2} + \int_\Omega n_0 \,dx \right\} \cdot C_1 \|\psi\|_{W^{2,2}(\Omega)}. \qquad (2.42)
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\left\| w_{\varepsilon t}(\cdot, t) \right\|_{(W^{2,2}(\Omega))^\star} \\
&\quad \le C_1 \left\{ 2\int_\Omega |\nabla w_\varepsilon|^2 \,dx + \frac{1}{2}\int_\Omega |u_\varepsilon|^2 \,dx + \frac{1}{2} + \int_\Omega n_0 \,dx \right\} \quad \text{for all } t > 0,
\end{aligned}
$$

which implies (2.38) holds.

Finally, we derive (2.39). For fixed $p > 2$, we have $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Thus for any $\psi \in C^\infty(\bar{\Omega})$, there exists a constant $C_2 > 0$ such that $\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} \le C_2 \|\psi\|_{W^{1,p}(\Omega)}$. Similarly, we derive from the second equation in (2.11) that

$$
\begin{aligned}
&\left| \int_\Omega c_{\varepsilon t}(\cdot, t)\psi \,dx \right| \\
&\quad = \left| -\int_\Omega \nabla c_\varepsilon \cdot \nabla \psi \,dx - \int_\Omega u_\varepsilon \cdot \nabla c_\varepsilon \psi \,dx - \int_\Omega f(n_\varepsilon) c_\varepsilon \psi \,dx \right| \\
&\quad \le \left\{ \left(\int_\Omega |\nabla c_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} + \left(\int_\Omega |u_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla c_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} + \|c_0\|_{L^\infty(\Omega)} \int_\Omega n_\varepsilon \,dx \right\} \\
&\qquad \cdot \left\{ \|\psi\|_{L^\infty(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \right\} \\
&\quad \le \left\{ \left(\int_\Omega |\nabla c_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} + C^{\frac{1}{2}} \left(\int_\Omega |\nabla c_\varepsilon|^2 \,dx \right)^{\frac{1}{2}} + m\|c_0\|_{L^\infty(\Omega)} \right\} \cdot C_2 \|\psi\|_{W^{1,p}(\Omega)} \qquad (2.43)
\end{aligned}
$$

by (2.8), (2.12), (2.13), and (2.35). Thus, we have

$$
\left\| c_{\varepsilon t}(\cdot, t) \right\|^2_{(W^{1,p}(\Omega))^\star} \le 3C_2^2 \left\{ (1 + C) \int_\Omega |\nabla c_\varepsilon|^2 \,dx + m^2 \|c_0\|^2_{L^\infty(\Omega)} \right\} \quad \text{for all } t > 0.
$$

Since $|\nabla c_\varepsilon| \le |\nabla w_\varepsilon| \cdot \|c_0\|_{L^\infty(\Omega)}$, we infer from (2.31) that (2.39) holds. □

Based on Lemma 2.2-Lemma 2.4 and standard compactness arguments, we can obtain the following basic properties with regard to the solutions of (2.26).

Lemma 2.5 *Let* $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ *be the solutions to* (2.26). *Then there exist functions* n, w *and* u *defined on* $\Omega \times (0, \infty)$ *and satisfying* $n \ge 0$, $w \ge 0$ *and* $\nabla \cdot u = 0$ *a.e. on* $\Omega \times (0, \infty)$ *as well* *as a sequence* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ *such that* $\varepsilon_j \searrow 0$ *as* $j \to \infty$ *and such that as* $\varepsilon = \varepsilon_j \searrow 0$,

$$\begin{cases} n_\varepsilon \to n \quad \text{a.e. in } \Omega \times (0,\infty), \\ \ln(n_\varepsilon + 1) \rightharpoonup \ln(n+1) \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)), \\ w_\varepsilon \to w \quad \text{in } L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\ w_\varepsilon \rightharpoonup w \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)), \\ w_\varepsilon(\cdot,t) \to w(\cdot,t) \quad \text{in } L^2(\Omega) \text{ for a.e. } t>0, \\ u_\varepsilon \rightharpoonup u \quad \text{in } L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty)) \text{ and in } L^p_{\mathrm{loc}}([0,\infty); W^{1,p}_0(\Omega)) \text{ for all } p \in (1,2) \end{cases} \tag{2.44}$$

as well as

$$\begin{cases} c_\varepsilon \to c \quad \text{in } L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\ c_\varepsilon \overset{\star}{\rightharpoonup} c \quad \text{in } L^\infty(\Omega \times (0,\infty)), \\ c_\varepsilon \rightharpoonup c \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)), \\ c_\varepsilon(\cdot,t) \to c(\cdot,t) \quad \text{in } L^2(\Omega) \text{ for a.e. } t>0, \\ c_{\varepsilon t} \rightharpoonup c_t \quad \text{in } L^2_{\mathrm{loc}}([0,\infty); (W^{1,p}(\Omega))^\star) \text{ for all } p>2 \end{cases} \tag{2.45}$$

with $c := \|c_0\|_{L^\infty(\Omega)} \cdot e^{-w}$. *Moreover, the triple* (n,c,u) *has the properties* (2.1)-(2.3) *in Definition* 2.1.

Proof The properties (2.28) and (2.37) combined with the Aubin-Lions Lemma (see [5]) warrant that there exist a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j\to\infty$ and functions n and w such that $\ln(n_\varepsilon+1) \rightharpoonup \ln(n+1)$ in $L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and $\ln(n_\varepsilon+1) \to \ln(n+1)$ in $L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty))$ and a.e. in $\Omega \times (0,\infty)$ as well as $w_\varepsilon \rightharpoonup w$ in $L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and $w_\varepsilon \to w$ in $L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty))$ and a.e. in $\Omega\times(0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$. In view of Lemma 2.3, (2.35) and (2.36) imply that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact with regard to the weak topology in $L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty))$ and also in $L^p_{\mathrm{loc}}([0,\infty); W^{1,p}_0(\Omega))$ for each $p \in (1,2)$. Thus, we have obtained (2.44). Similarly, by (2.29), (2.39) and the Aubin-Lions lemma along with a standard extraction procedure, we can find a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j\to\infty$ and a function c such that $c_\varepsilon \to c$ in $L^2_{\mathrm{loc}}(\bar\Omega \times [0,\infty))$ and a.e. in $\Omega \times (0,\infty)$ and $c_\varepsilon(\cdot,t) \to c(\cdot,t)$ in $L^2(\Omega)$ for a.e. $t>0$ as well as $c_\varepsilon \rightharpoonup c$ in $L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and $c_{\varepsilon t} \rightharpoonup c_t$ in $L^2_{\mathrm{loc}}([0,\infty); (W^{1,p}(\Omega))^\star)$ for all $p>2$ as $\varepsilon = \varepsilon_j \searrow 0$. From (2.16) in Lemma 2.1, we can also obtain $c_\varepsilon \overset{\star}{\rightharpoonup} c$ in $L^\infty(\Omega\times(0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$.

The property (2.2) is from $(2.44)_1$, $(2.45)_1$, the finiteness of w a.e. in $\Omega\times(0,\infty)$ and $\nabla\cdot u_\varepsilon \equiv 0$, while the property (2.3) is straightforward from $(2.44)_2$ and $(2.44)_4$. The second and the third inclusions in (2.1) are straightforward from $(2.45)_2$, $(2.45)_3$ and $(2.44)_6$, and the first follows from Fatou's lemma, which along with (2.12) shows that

$$\int_0^T \int_\Omega n\,dx\,ds \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_0^T \int_\Omega n_\varepsilon\,dx\,ds \le mT$$

for all $T>0$. □

2.3 Strong precompactness of $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ in $L^1_{\mathrm{loc}}(\bar\Omega \times [0,\infty))$

Until now, the regularity of the functions n, c and u obtained in Lemma 2.5 does not meet the requirements of the identities (2.6) and (2.7) in Definition 2.1. Therefore, based on

(2.28), we have to derive some further compactness properties of n_ε. Since the considered space dimension is two, we can derive the strong precompactness of the sequence $(n_{\varepsilon_j})_{j\in\mathbb{N}}$ by taking a similar argument in [42], where the strong compactness is obtained by the Moser-Trudinger inequality and the Vitali convergence theorem. We first derive from (2.28) the following inequality by means of the Moser-Trudinger inequality.

Lemma 2.6 *Let $(n_\varepsilon, w_\varepsilon, u_\varepsilon)$ is a classical solution to* (2.26). *Then for all $p > 1$, there exists a constant $C(p) > 0$ such that for any given $\varepsilon \in (0,1)$ we have*

$$\int_0^t \ln\left\{\frac{1}{|\Omega|}\int_\Omega (n_\varepsilon + 1)^p dx\right\} ds \le C(p)\cdot(1+m)t + C(p)\cdot\left\{\int_\Omega w_0\,dx + m\right\} \tag{2.46}$$

for all $t > 0$, where $m := \int_\Omega n_0\,dx$.

Proof In view of the Moser-Trudinger inequality, there exist some positive constants C_1, C_2, and C_3 such that for all nonnegative function $\varphi \in W^{1,2}(\Omega)$ we have

$$\int_\Omega e^\varphi\,dx \le C_1 e^{C_2\int_\Omega |\nabla\varphi|^2\,dx + C_3\int_\Omega \varphi\,dx}.$$

Thus, for fixed $p > 1$ and $t > 0$, we obtain

$$\frac{1}{|\Omega|}\int_\Omega (n_\varepsilon+1)^p dx = \frac{1}{|\Omega|}\int_\Omega e^{p\ln(n_\varepsilon+1)}\,dx \le \frac{C_1}{|\Omega|} e^{C_2p^2\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx + C_3p\int_\Omega \ln(n_\varepsilon+1)\,dx}$$

and

$$\begin{aligned}
&\int_0^t \ln\left\{\frac{1}{|\Omega|}\int_\Omega (n_\varepsilon+1)^p dx\right\} ds \\
&\quad\le t\cdot\ln\frac{C_1}{|\Omega|} + C_2p^2\int_0^t\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx\,ds + C_3p\int_0^t\int_\Omega \ln(n_\varepsilon+1)\,dx\,ds \\
&\quad\le \left(\ln\frac{C_1}{|\Omega|} + C_3pm\right)t + C_2p^2\int_0^t\int_\Omega \frac{|\nabla n_\varepsilon|^2}{(n_\varepsilon+1)^2}\,dx\,ds \\
&\quad\le \left(\ln\frac{C_1}{|\Omega|} + C_3pm\right)t + C_2p^2\left(\int_\Omega w_0\,dx + 2m + mt\right),
\end{aligned}$$

where we used (2.28) and $\ln(n_\varepsilon + 1) \le n_\varepsilon$ for all $t > 0$. Thus, we can find some constant $C(p) > 0$ such that (2.46) holds. □

Now we derive the strong precompactness property by means of the Vitali convergence theorem. Because the proof of the following lemma is similar to the proof of Lemma 2.7 from [42], we only sketch the main steps here.

Lemma 2.7 *Let n and $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be as provided by Lemma* 2.5. *Then we have*

$$n_\varepsilon \to n \quad \textit{in } L^1_{\mathrm{loc}}\big(\bar\Omega\times[0,\infty)\big) \textit{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{2.47}$$

and

$$\int_\Omega n(x,t)\,dx = \int_\Omega n_0\,dx \quad \textit{for a.e. } t > 0. \tag{2.48}$$

Proof We fix $T > 0$ and let $C_1 := C(2) \cdot (1 + m)t + C(2) \cdot \{\int_\Omega w_0\,dx + m\}$ from Lemma 2.6 and $m := \int_\Omega n_0\,dx$. For given $\eta > 0$, we can then choose $M > 1$ large enough and thereafter $\delta > 0$ suitably small such that

$$\frac{mC_1}{\ln \frac{M}{|\Omega|}} < \frac{\eta}{2} \quad \text{and} \quad \sqrt{MT\delta} < \frac{\eta}{2}. \tag{2.49}$$

We decompose $(0, T)$ by introducing

$$\mathcal{S}_1(\varepsilon) := \left\{ t \in (0,T) \middle| \int_\Omega n_\varepsilon^2(\cdot,t)\,dx \le M \right\} \quad \text{and}$$
$$\mathcal{S}_2(\varepsilon) := \left\{ t \in (0,T) \middle| \int_\Omega n_\varepsilon^2(\cdot,t)\,dx > M \right\} \quad \text{for all } t > 0.$$

By using (2.46), we derive that

$$\begin{aligned} C_1 &\ge \int_{\mathcal{S}_2(\varepsilon)} \ln\left\{ \frac{1}{|\Omega|} \int_\Omega \bigl(n_\varepsilon(x,t) + 1\bigr)^2\,dx \right\} dt \\ &\ge \int_{\mathcal{S}_2(\varepsilon)} \ln\left\{ \frac{1}{|\Omega|} \int_\Omega n_\varepsilon(x,t)^2\,dx \right\} dt \\ &\ge \int_{\mathcal{S}_2(\varepsilon)} \ln \frac{M}{|\Omega|} = \ln \frac{M}{|\Omega|} \cdot \bigl|\mathcal{S}_2(\varepsilon)\bigr| \end{aligned}$$

and hence

$$\bigl|\mathcal{S}_2(\varepsilon)\bigr| \le \frac{C_1}{\ln \frac{M}{|\Omega|}} \quad \text{for all } M > 1 \text{ and } \varepsilon \in (0,1).$$

Assume that $E \subset \Omega \times (0,T)$ is measurable with $|E| < \delta$, and let $E(t) := \{x \in \Omega | (x,t) \in E\}$ for all $t \in (0,T)$. Then for all $\varepsilon \in (0,1)$ we derive that

$$\begin{aligned} \int\int_E n_\varepsilon\,dx\,dt &\le \int_{\mathcal{S}_1(\varepsilon)} \int_{E(t)} n_\varepsilon\,dx\,dt + \int_{\mathcal{S}_2(\varepsilon)} \int_{E(t)} n_\varepsilon\,dx\,dt \\ &\le \int_{\mathcal{S}_1(\varepsilon)} \bigl|E(t)\bigr|^{\frac{1}{2}} \left(\int_\Omega n_\varepsilon^2\,dx \right)^{\frac{1}{2}} dt + m\bigl|S_2(\varepsilon)\bigr| \\ &\le \sqrt{M} \int_{\mathcal{S}_1(\varepsilon)} \bigl|E(t)\bigr|^{\frac{1}{2}}\,dt + m\bigl|S_2(\varepsilon)\bigr| \\ &\le \sqrt{M}\sqrt{\bigl|\mathcal{S}_1(\varepsilon)\bigr|} \left(\int_{\mathcal{S}_1(\varepsilon)} \bigl|E(t)\bigr|\,dt \right)^{\frac{1}{2}} + m\bigl|S_2(\varepsilon)\bigr| \\ &\le \sqrt{MT}\sqrt{|E|} + m\bigl|S_2(\varepsilon)\bigr| \\ &\le \sqrt{MT\delta} + \frac{mC_1}{\ln \frac{M}{|\Omega|}} \\ &\le \frac{\eta}{2} + \frac{\eta}{2} = \eta. \end{aligned}$$

Because we have obtained $n_\varepsilon \to n$ a.e. in $\Omega \times (0,T)$ as $\varepsilon = \varepsilon_j \searrow 0$ in Lemma 2.5, we have $n_\varepsilon \to n$ in $L^1(\Omega \times (0,T))$ in the light of the Vitali convergence theorem. Thus, we establish (2.47). The property (2.48) is straightforward from (2.47) and (2.12). □

As a consequence thereof, we can prove the limit functions c and u indeed are weak solutions to the respective subproblems in (1.1) as required in Definition 2.1.

Lemma 2.8 *The limit functions n, c, and u obtained in Lemma* 2.5 *satisfy the identity* (2.6) *and identity* (2.7) *in Definition* 2.1 *for all test functions from the class indicated there.*

Proof First, we verify the validity of (2.6) in Definition 2.1. For each φ from the class indicated in (2.6), by using (2.47) and $(2.45)_2$ along with the definition of f_ε we derive that

$$\int_0^\infty \int_\Omega f_\varepsilon(n_\varepsilon)c_\varepsilon\varphi\,dx\,dt \to \int_0^\infty \int_\Omega nc\varphi\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{2.50}$$

where $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ is as in Lemma 2.5 (see [42], Lemma 2.8, for details). Based on $(2.45)_1$, $(2.45)_3$, and $(2.44)_6$, we obtain

$$\int_0^\infty \int_\Omega c_\varepsilon\varphi_t\,dx\,dt \to \int_0^\infty \int_\Omega c\varphi_t\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$
$$\int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi\,dx\,dt \to \int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

and

$$\int_0^\infty \int_\Omega c_\varepsilon(u_\varepsilon \cdot \nabla\varphi)\,dx\,dt \to \int_0^\infty \int_\Omega c(u \cdot \nabla\varphi)\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Therefore, the functions n, c and u obtained in Lemma 2.5 satisfy the identity (2.6).

Second, we verify (2.7). From $(2.44)_6$ in Lemma 2.5, we have

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^1_{\mathrm{loc}}\big([0,\infty); W_0^{1,1}(\Omega)\big) \text{ as } \varepsilon = \varepsilon_j \searrow 0, \tag{2.51}$$

which in conjunction with (2.47) yields for all φ from the class indicated in (2.7)

$$\int_0^\infty \int_\Omega u_\varepsilon \cdot \varphi_t\,dx\,dt \to \int_0^\infty \int_\Omega u \cdot \varphi_t\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$
$$\int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\,dx\,dt \to \int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

and

$$\int_0^\infty \int_\Omega n_\varepsilon \nabla\phi \cdot \varphi\,dx\,dt \to \int_0^\infty \int_\Omega n\nabla\phi \cdot \varphi\,dx\,dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Thus, we complete the proof. □

2.4 Strong convergence of $(\nabla w_{\varepsilon_j})_{j\in\mathbb{N}}$ in $L^2_{\mathrm{loc}}(\bar{\Omega} \times [0,\infty))$

Up to now, we only obtain the weak precompactness properties of $(\nabla \ln(n_\varepsilon + 1))_{\varepsilon\in(0,1)}$ and $(\nabla w_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{\mathrm{loc}}(\bar{\Omega} \times [0,\infty))$, which do not satisfy the strong compact requirement in this space in the cross-diffusion in $(2.11)_1$ by passing to the limit. We can prove that the family $(\nabla w_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^2_{\mathrm{loc}}(\bar{\Omega} \times [0,\infty))$ with regard to the strong topology by following a similar argument in [42], Section 2.4, for the fluid-free case $u \equiv 0$.

Lemma 2.9 *Let n, c, w, and u be given by Lemma 2.5. Then there exists a null set $N \subset (0,\infty)$ such that*

$$\int_0^{t_0}\int_\Omega |\nabla w|^2\,dx\,dt \ge \int_\Omega w_0\,dx - \int_\Omega w(x,t_0)\,dx + mt_0 \quad \textit{for all } t_0 \in (0,\infty)\backslash N, \tag{2.52}$$

where $m := \int_\Omega n_0\,dx$.

Proof We fix any sequence $(\eta_j)_{j\in\mathbb{N}} \subset (0,1)$ satisfying $\eta_j \searrow 0$ as $j \to \infty$, and for each j we can then pick a null set $N_j \subset (0,\infty)$ such that $t_0 \in (0,\infty)\backslash N_j$ is a Lebesgue point of $0 < t \mapsto \int_\Omega \ln(c(x,t)+\eta_j)\,dx$. From $(2.44)_5$ there exists a null set $N_\star \subset (0,\infty)$ such that $w(x,t_0) \in L^1(\Omega)$ for all $t_0 \in (0,\infty)\backslash N_\star$. Then given $t_0 \in (0,\infty)\backslash N$ ($N := N_\star \cup \bigcup_{j\in\mathbb{N}} N_j$), $\delta \in (0,1)$, $h \in (0,t_0)$ and $\eta \in (\eta_j)_{j\in\mathbb{N}}$, we define

$$\varphi(x,t) := \zeta_\delta(t)\cdot S_h\left[\frac{1}{c+\eta}\right](x,t), \tag{2.53}$$

where

$$\zeta_\delta(t) := \begin{cases} 1 & \text{if } t \le t_0, \\ \frac{t_0+\delta-t}{\delta} & \text{if } t \in (t_0, t_0+\delta), \\ 0 & \text{if } t \ge t_0+\delta, \end{cases}$$

and where

$$S_h\left[\frac{1}{c+\eta}\right](x,t) := \frac{1}{h}\int_{t-h}^t \frac{1}{c(x,s)+\eta}\,ds, \quad (x,t) \in \Omega\times(0,\infty),$$

where we let

$$c(x,t) := c_0(x) \quad \text{for } x \in \Omega \text{ and } t \le 0. \tag{2.54}$$

We note that the regularity properties of $c(x,t)$ in (2.45) ensure that $\varphi(x,t)$ has the regularity properties required in (2.6). Hence we take φ as a test function in (2.6), that is,

$$\begin{aligned} I_1 + I_2 &:= \int_0^\infty\int_\Omega c\varphi_t\,dx\,dt + \int_\Omega c_0\varphi(x,0)\,dx \\ &= \int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi\,dx\,dt + \int_0^\infty\int_\Omega nc\varphi\,dx\,dt \\ &\quad - \int_0^\infty\int_\Omega c(u\cdot\nabla\varphi)\,dx\,dt \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

Thus, we have

$$I_2 = \int_\Omega c_0\varphi(x,0)\,dx = \int_\Omega \frac{c_0}{c_0+\eta}\,dx \tag{2.55}$$

and

$$\begin{aligned}
I_1 &= \int_0^\infty \int_\Omega c(x,t)\cdot \biggl\{ \zeta_\delta'(t) S_h\biggl[\frac{1}{c(x,t)+\eta}\biggr] + \frac{\zeta_\delta(t)}{h}\biggl[\frac{1}{c(x,t)+\eta} - \frac{1}{c(x,t-h)+\eta}\biggr]\biggr\}\,dx\,dt \\
&= -\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega c(x,t)\cdot\frac{1}{h}\int_{t-h}^t \frac{1}{c(x,s)+\eta}\,ds\,dx\,dt \\
&\quad - \int_0^\infty\int_\Omega c(x,t)\cdot\frac{\zeta_\delta(t)}{h}\biggl[\frac{1}{c(x,t-h)+\eta} - \frac{1}{c(x,t)+\eta}\biggr]\,dx\,dt \\
&=: J_1 - J_2
\end{aligned} \tag{2.56}$$

as well as

$$I_3 = -\int_0^\infty \int_\Omega \nabla c(x,t)\cdot \zeta_\delta(t) S_h\biggl[\frac{1}{(c+\eta)^2}\nabla c\biggr](x,t)\,dx\,dt, \tag{2.57}$$

$$I_4 = \int_0^\infty \int_\Omega n(x,t)c(x,t)\zeta_\delta(t)S_h\biggl[\frac{1}{c+\eta}\biggr](x,t)\,dx\,dt \tag{2.58}$$

and

$$I_5 = \int_0^\infty \int_\Omega c(x,t)\biggl\{u(x,t)\cdot\zeta_\delta(t)S_h\biggl[\frac{1}{(c+\eta)^2}\nabla c\biggr]\biggr\}\,dx\,dt. \tag{2.59}$$

According to the concavity of $0<\xi\mapsto \ln\xi$, we derive that

$$\begin{aligned}
&\frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\bigl[\ln\bigl\{c(x,t)+\eta\bigr\} - \ln\bigl\{c(x,t-h)+\eta\bigr\}\bigr]\,dx\,dt \\
&\quad\le \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{1}{c(x,t-h)+\eta}\bigl\{c(x,t)-c(x,t-h)\bigr\}\,dx\,dt \\
&\quad= \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{c(x,t)}{c(x,t-h)+\eta}\,dx\,dt - \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{c(x,t-h)}{c(x,t-h)+\eta}\,dx\,dt \\
&\quad= \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{c(x,t)}{c(x,t-h)+\eta}\,dx\,dt - \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t+h)\frac{c(x,t)}{c(x,t)+\eta}\,dx\,dt \\
&\qquad - \frac{1}{h}\int_{-h}^0\int_\Omega \zeta_\delta(t+h)\frac{c(x,t)}{c(x,t)+\eta}\,dx\,dt \\
&\quad= -\int_0^\infty\int_\Omega \frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h}\frac{c(x,t)}{c(x,t)+\eta}\,dx\,dt \\
&\qquad + \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)c(x,t)\biggl[\frac{1}{c(x,t-h)+\eta} - \frac{1}{c(x,t)+\eta}\biggr]\,dx\,dt - \int_\Omega \frac{c_0(x)}{c_0(x)+\eta}\,dx \\
&\quad= -\int_0^\infty\int_\Omega \frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h}\frac{c(x,t)}{c(x,t)+\eta}\,dx\,dt + J_2 - I_2.
\end{aligned} \tag{2.60}$$

For the left-hand side of (2.60), we have

$$\begin{aligned}
&\frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\bigl[\ln\bigl\{c(x,t)+\eta\bigr\} - \ln\bigl\{c(x,t-h)+\eta\bigr\}\bigr]\,dx\,dt \\
&\quad= \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\ln\bigl\{c(x,t)+\eta\bigr\}\,dx\,dt - \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t+h)\ln\bigl\{c(x,t)+\eta\bigr\}\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{h}\int_{-h}^{0}\int_{\Omega}\zeta_\delta(t+h)\ln\{c(x,t)+\eta\}\,dx\,dt\\
&=-\int_0^\infty\int_\Omega\frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h}\ln\{c(x,t)+\eta\}\,dx\,dt-\int_\Omega\ln\{c_0(x)+h\}\,dx.
\end{aligned}
\tag{2.61}
$$

Inserting (2.61) and $J_2 - I_2 = J_1 - I_3 - I_4 - I_5$ into (2.60) and letting $h \searrow 0$, we obtain

$$
\begin{aligned}
&\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega\ln\{c(x,t)+\eta\}\,dx\,dt-\int_\Omega\ln\{c_0(x)+\eta\}\,dx\\
&\quad\le\int_0^\infty\int_\Omega\zeta_\delta(t)\frac{|\nabla c|^2}{(c+\eta)^2}\,dx\,dt-\int_0^\infty\int_\Omega\zeta_\delta(t)\frac{nc}{c+\eta}\,dx\,dt\\
&\qquad-\int_0^\infty\int_\Omega\zeta_\delta(t)c\frac{u\cdot\nabla c}{(c+\eta)^2}\,dx\,dt
\end{aligned}
\tag{2.62}
$$

for all $\delta \in (0,1)$. By using the monotone convergence theorem and the fact that t_0 is a Lebesgue point, we derive on taking $\delta \to 0$ that

$$
\begin{aligned}
&\int_\Omega\ln\{c(x,t_0)+\eta\}\,dx\,dt-\int_\Omega\ln\{c_0(x)+\eta\}\,dx\\
&\quad\le\int_0^{t_0}\int_\Omega\frac{|\nabla c|^2}{(c+\eta)^2}\,dx\,dt-\int_0^{t_0}\int_\Omega\frac{nc}{c+\eta}\,dx\,dt-\int_0^{t_0}\int_\Omega c\frac{u\cdot\nabla c}{(c+\eta)^2}\,dx\,dt.
\end{aligned}
\tag{2.63}
$$

By integrating by parts, we derive that

$$
\begin{aligned}
\int_0^{t_0}\int_\Omega c_\varepsilon\frac{u_\varepsilon\cdot\nabla c_\varepsilon}{(c_\varepsilon+\eta)^2}\,dx\,dt&=-\int_0^{t_0}\int_\Omega c_\varepsilon u_\varepsilon\cdot\nabla\left(\frac{1}{c_\varepsilon+\eta}\right)dx\,dt\\
&=\int_0^{t_0}\int_\Omega u_\varepsilon\cdot\frac{1}{c_\varepsilon+\eta}\nabla c_\varepsilon\,dx\,dt\\
&=-\int_0^{t_0}\int_\Omega\ln(c_\varepsilon+\eta)(\nabla\cdot u_\varepsilon)\,dx\,dt=0.
\end{aligned}
\tag{2.64}
$$

In the light of Lemma 2.5 we see that if $(\varepsilon_j)_{j\in\mathbb{N}}$ is as in Lemma 2.5 then

$$
\int_0^{t_0}\int_\Omega c_\varepsilon\frac{u_\varepsilon\cdot\nabla c_\varepsilon}{(c_\varepsilon+\eta)^2}\,dx\,dt\to\int_0^{t_0}\int_\Omega c\frac{u\cdot\nabla c}{(c+\eta)^2}\,dx\,dt\quad\text{as }\varepsilon=\varepsilon_j\searrow 0,
$$

which along with (2.64) implies that

$$
\int_0^{t_0}\int_\Omega c\frac{u\cdot\nabla c}{(c+\eta)^2}\,dx\,dt=0.
$$

We continue to use the monotone convergence theorem to show on taking $\eta = \eta_j \searrow 0$ that

$$
\int_\Omega\ln\{c(x,t_0)\}\,dx-\int_\Omega\ln c_0(x)\,dx\le\int_0^{t_0}\int_\Omega\frac{|\nabla c|^2}{c^2}\,dx\,dt-\int_0^{t_0}\int_\Omega n\,dx\,dt.
\tag{2.65}
$$

Due to $t_0 \notin N_\star$, $\int_\Omega w(x,t_0)\,dx$ is finite. Thus, (2.65) implies that (2.52) holds and thereby we complete the proof, since the measure of N is zero. □

Based on Lemma 2.5 and Lemma 2.9, we can obtain the desired convergence property of $(\nabla w_{\varepsilon_j})_{j\in\mathbb{N}}$ in $L^2_{\mathrm{loc}}(\bar{\Omega}\times[0,\infty))$.

Lemma 2.10 *Suppose that w and $(\varepsilon_j)_{j\in\mathbb{N}}$ are given by Lemma 2.5. Then for each $T>0$ we have*

$$\nabla w_\varepsilon \to \nabla w \quad \text{in } L^2\big(\Omega\times(0,T)\big) \text{ as } \varepsilon=\varepsilon_j\searrow 0. \tag{2.66}$$

Proof For given $T>0$, we can fix $t_0\geq T$ such that $\int_\Omega w_\varepsilon(x,t_0)\,dx \to \int_\Omega w(x,t_0)\,dx$ as $\varepsilon=\varepsilon_j\searrow 0$ by Lemma 2.5. From Lemmas 2.2 and 2.9, we have

$$\begin{aligned}\limsup_{\varepsilon=\varepsilon_j\searrow 0}\int_0^{t_0}\int_\Omega|\nabla w_\varepsilon|^2\,dx\,dt &\leq \limsup_{\varepsilon=\varepsilon_j\searrow 0}\left\{\int_\Omega w_0(x)\,dx-\int_\Omega w_\varepsilon(x,t_0)\,dx+mt_0\right\}\\ &= \int_\Omega w_0(x)\,dx-\int_\Omega w(x,t_0)\,dx+mt_0\\ &\leq \int_0^{t_0}\int_\Omega|\nabla w|^2\,dx\,dt.\end{aligned}$$

Therefore, we have

$$\int_0^{t_0}\int_\Omega|\nabla w_\varepsilon|^2\,dx\,dt \to \int_0^{t_0}\int_\Omega|\nabla w|^2\,dx\,dt,$$

which together with the fact $w_\varepsilon \rightharpoonup w$ in $L^2([0,t_0);W^{1,2}(\Omega))$ in Lemma 2.5 shows that $\nabla w_\varepsilon\to\nabla w$ in $L^2(\Omega\times(0,t_0))$ and hence implies (2.66) holds due to $t_0\geq T$. □

3 Proof of Theorem 1.1

Based on the above *a priori* estimates, we are now in the position to prove the main results.

Proof of Theorem 1.1 Since (2.1)-(2.4) have been proved in Lemmas 2.5-2.7, and the validity of (2.6) and (2.7) has been asserted by Lemma 2.8, we go to verify (2.5). We fix an arbitrary nonnegative function $\varphi\in C_0^\infty(\bar{\Omega}\times[0,\infty))$ and then multiply the first equation in (2.26) by $\frac{\varphi}{n_\varepsilon+1}$ to obtain

$$\begin{aligned}I_1(\varepsilon) &:= \int_0^\infty\int_\Omega\big|\nabla\ln(n_\varepsilon+1)\big|^2\varphi\,dx\,dt\\ &= -\int_0^\infty\int_\Omega\ln(n_\varepsilon+1)\varphi_t\,dx\,dt-\int_\Omega\ln(n_0+1)\varphi(x,0)\,dx\\ &\quad+\int_0^\infty\int_\Omega\nabla\ln(n_\varepsilon+1)\cdot\nabla\varphi\,dx\,dt-\int_0^\infty\int_\Omega\frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}\big(\nabla w_\varepsilon\cdot\nabla\ln(n_\varepsilon+1)\big)\varphi\,dx\,dt\\ &\quad+\int_0^\infty\int_\Omega\frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1}(\nabla w_\varepsilon\cdot\nabla\varphi)\,dx\,dt-\int_0^\infty\int_\Omega\ln(n_\varepsilon+1)(u_\varepsilon\cdot\nabla\varphi)\,dx\,dt\\ &=: I_2(\varepsilon)+I_3(\varepsilon)+I_4(\varepsilon)+I_5(\varepsilon)+I_6(\varepsilon)+I_7(\varepsilon)\end{aligned}\tag{3.1}$$

for each $\varepsilon\in(0,1)$. Here we pick T sufficiently large such that $\varphi\equiv 0$ on $\Omega\times(T,\infty)$. From $(2.44)_2$, we obtain $\ln(n_\varepsilon+1)\rightharpoonup\ln(n+1)$ in $L^2_{\mathrm{loc}}([0,\infty);W^{1,2}(\Omega))$ as $\varepsilon=\varepsilon_j\searrow 0$, which warrants that

$$\int_0^\infty \int_\Omega \left|\nabla \ln(n+1)\right|^2 \varphi \, dx \, dt \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} I_1(\varepsilon) \tag{3.2}$$

by the nonnegativity of φ and lower semicontinuity of the norm in $L^2(\Omega \times (0,T))$, and that

$$I_2(\varepsilon) \to -\int_0^\infty \int_\Omega \ln(n+1)\varphi_t \, dx \, dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{3.3}$$

and

$$I_4(\varepsilon) \to \int_0^\infty \int_\Omega \nabla \ln(n+1) \cdot \nabla \varphi \, dx \, dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.4}$$

From Lemma 2.10, we have $\nabla w_\varepsilon \to \nabla w$ in $L^2(\Omega \times (0,T))$ as $\varepsilon = \varepsilon_j \searrow 0$, which along with the observation that $0 \le \frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \le 1$ for all $\varepsilon \in (0,1)$ and $\frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{n_\varepsilon+1} \to \frac{n}{n+1}$ a.e. in $\Omega \times (0,T)$ as $\varepsilon = \varepsilon_j \searrow 0$ ensures that

$$\frac{n_\varepsilon f_\varepsilon'(n_\varepsilon)}{n_\varepsilon + 1} \nabla w_\varepsilon \to \frac{n}{n+1} \nabla w = -\frac{n}{n+1} \nabla \ln c \quad \text{in } L^2\big(\Omega \times (0,T)\big),$$

$\varepsilon = \varepsilon_j \searrow 0$. Therefore, we obtain

$$I_5(\varepsilon) \to \int_0^\infty \int_\Omega \frac{n}{n+1} \big(\nabla \ln(n+1) \cdot \nabla \ln c\big)\varphi \, dx \, dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{3.5}$$

and

$$I_6(\varepsilon) \to -\int_0^\infty \int_\Omega \frac{n}{n+1} (\nabla \ln c \cdot \nabla \varphi) \, dx \, dt \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{3.6}$$

From Lemma 2.7, we have obtained $n_\varepsilon \to n$ in $L^1(\bar{\Omega} \times (0,T))$ as $\varepsilon = \varepsilon_j \searrow 0$. This fact combines with the Lipschitz continuity of $[0,\infty) \ni \xi \mapsto \ln^2(1+\xi)$ to ensure that $\int_0^T \int_\Omega \ln^2(n_\varepsilon + 1) \, dx \, dt \to \int_0^T \int_\Omega \ln^2(n+1) \, dx \, dt$ as $\varepsilon = \varepsilon_j \searrow 0$, which together with the weak convergence property in $(2.44)_2$ entails that

$$\ln(n_\varepsilon + 1) \to \ln(n+1) \quad \text{in } L^2\big(\bar{\Omega} \times [0,\infty)\big) \tag{3.7}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Thus, (3.7) combined with the fact that $u_\varepsilon \rightharpoonup u$ in $L^2(\bar{\Omega} \times [0,\infty))$ shows that

$$I_7(\varepsilon) \to -\int_0^\infty \int_\Omega \ln(n+1)(u \cdot \nabla \varphi) \, dx \, dt \tag{3.8}$$

as $\varepsilon = \varepsilon_j \searrow 0$. By collecting (3.2)-(3.6) and (3.8), we see that (2.5) results from (3.1) and thereby we prove that (n,c,u) is a global generalized solution to (1.1). The decay (1.7)-(1.8) of the solution component c and additional property (1.9) of c can be proved in the same way as those of Theorem 1.3 in [42]. We omit the corresponding proof for brevity. Thus, we complete the proof of Theorem 1.1. □

Competing interests
The author declares that they have no competing interests.

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