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Existence of three solutions for equations of $p(x)$-Laplace type operators with nonlinear Neumann boundary conditions

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Abstract

In this paper, we are concerned with nonlinear elliptic equations of the $p(x)$-Laplace type operators

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ a(x,\nabla u)\frac{\partial u}{\partial n} = \lambda\theta g(x,u) & \text{on } \partial\Omega, \end{cases}$$

which are subject to nonlinear Neumann boundary conditions. Here the function $a(x,v)$ is of type $|v|^{p(x)-2}v$ with a continuous function $p:\overline{\Omega}\to(1,\infty)$ and the functions f,g satisfy a Carathéodory condition. The main purpose of this paper is to establish the existence of at least three weak solutions of the above problem by applying an abstract three critical points theorem which is inspired by the work of Ricceri (Nonlinear Anal. 74:7446-7454, 2011) Furthermore, we determine two intervals of λ's precisely such that the first is where the given problem admits only the trivial solution, and the second is where the given problem has at least two nontrivial solutions as considering the positive principal eigenvalue for the $p(x)$-Laplacian Neumann problems and an estimate of the Sobolev trace embedding's constant.

MSC: 35D30; 35J15; 35J60; 35J62

Keywords: $p(x)$-Laplace type operators; variable exponent Lebesgue-Sobolev spaces; three critical points theorem; nonlinear Neumann boundary conditions

1 Introduction

In the present paper, we are concerned with multiplicity of weak solutions of nonlinear Neumann boundary problems involving $p(x)$-Laplace type

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ a(x,\nabla u)\frac{\partial u}{\partial n} = \lambda\theta g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{N}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$, the function $a(x,v)$ is of type $|v|^{p(x)-2}v$ with a continuous function $p:\overline{\Omega}\to(1,\infty)$, the functions f,g satisfy a Carathéodory condition, and λ,θ are real parameters. Many authors have widely studied the existence of nontrivial solutions of nonlinear elliptic boundary value problems; see [2–10] and the references therein.

The study of differential equations and variational problems involving $p(x)$-growth condition have been a strong rise of interest in recent years, since there are many physical phenomena which can be modeled by such kind of equations, for instance, elastic mechanics, electro-rheological fluid dynamics, image processing, *etc.* We refer the reader to [4, 5, 7, 11–29] and the references therein.

Since the pioneer work of Ricceri [30, 31], an abstract three critical points theorem has become one of the main tools for finding the multiple solutions to elliptic equations of variational type. Afterward, many results on the existence of at least three weak solutions of nonlinear elliptic equations have been investigated; see [11, 12, 30, 32–35]. However, Ricceri's theorems in [30, 31, 35] gave no additional information on the size and location of a precise interval of λ's for an energy functional admits at least three critical points. The refinement process of Theorem 1 in [35] as giving precise information on the location and size of the three critical points interval was completed in [1]. G. Bonanno and A. Chinnì [12] obtained the existence of at least one, two or three distinct weak solutions of $p(x)$-Laplacian Dirichlet problems as applications of two recent critical point theorems in [36, 37]. The authors in [34] localized the interval on the existence of three solutions of equations of p-Laplace type with various boundary conditions (for example, homogeneous Dirichlet and inhomogeneous Robin problems) which were inspired by the study of Arcoya and Carmona [38]. Recently, Kim and Park [7] established the localization of the interval for the existence of at least three solutions for equations of $p(x)$-Laplace type with nonlinear Neumann boundary conditions which was based on [34]. In particular, they showed that the problem (N) possessed at least three weak solutions in any closed and bounded interval of the parameters contained in an unbounded open interval of positive real numbers. In the present paper, we establish more precise interval than that of [7], by applying an abstract three critical points theorem introduced by Ricceri [1]. Roughly speaking, we determine two intervals of λ's precisely such that the first is where problem (N) admits only the trivial solution, and the second is where problem (N) has at least two nontrivial solutions. To do this, we consider the existence of the positive principal eigenvalue for the $p(x)$-Laplacian Neumann problems and an estimate of the Sobolev trace embedding's constant based on J.F. Bonder and J.D. Rossi's result [39].

This paper is arranged as follows. We first introduce some basic results on the variable exponent Lebesgue-Sobolev spaces and present some properties of the corresponding integral operators. Second, we observe multiple solutions for equations of $p(x)$-Laplace type with nonlinear Neumann boundary conditions using the abstract three critical points theory introduced by Ricceri [1] (see Theorem 2.11). And finally we give the interval much more accurate than the three critical points interval in Theorem 2.11 (see Theorem 2.14).

2 Preliminaries and main results

We first introduce some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\Omega)$. For a more rigorous treatment on these spaces, we refer to [13–17, 20, 40].

Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1 \right\}$$

and, for any $h \in C_+(\overline{\Omega})$, we write

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \left\{ u : u \text{ is a measurable real-valued function, } \int_\Omega \left|u(x)\right|^{p(x)}\,dx < \infty \right\},$$

endowed with the Luxemburg norm,

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\,dx \le 1 \right\}.$$

The variable exponent Sobolev space $X := W^{1,p(\cdot)}(\Omega)$ is defined by

$$X = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_X = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Throughout the present paper, we assume that a function $p : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω if there is a constant C_0 such that

$$\left|p(x) - p(y)\right| \le \frac{C_0}{-\log|x-y|} \tag{2.1}$$

for every $x, y \in \Omega$ with $|x-y| \le 1/2$. As established in [13, 14], if Ω is a bounded domain with Lipschitz boundary and p satisfies the log-Hölder continuity condition, then smooth functions are dense in variable exponent Sobolev spaces.

Lemma 2.1 [16] *The space $L^{p(\cdot)}(\Omega)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Lemma 2.2 [16] *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx$$

for all $u \in L^{p(\cdot)}(\Omega)$. Then:

(1) $\rho(u) > 1$ $(= 1; < 1)$ *if and only if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$ $(= 1; < 1)$, *respectively*;
(2) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, *then* $\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}$;
(3) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, *then* $\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}$.

Remark 2.3 Denote

$$\rho(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx$$

for all $u \in X$. Then:

(1) $\rho(u) > 1$ $(= 1; < 1)$ if and only if $\|u\|_X > 1$ $(= 1; < 1)$, respectively;
(2) if $\|u\|_X > 1$, then $\|u\|_X^{p_-} \le \rho(u) \le \|u\|_X^{p_+}$;
(3) if $\|u\|_X < 1$, then $\|u\|_X^{p_+} \le \rho(u) \le \|u\|_X^{p_-}$.

Lemma 2.4 [14] *Let $q \in L^\infty(\Omega)$ be such that $1 \le p(x)q(x) \le \infty$ for almost all $x \in \Omega$. If $u \in L^{q(\cdot)}(\Omega)$ with $u \neq 0$, then*

(1) *if $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} > 1$, then $\|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \||u|^{q(x)}\|_{L^{p(\cdot)}(\Omega)} \le \|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\Omega)}$;*
(2) *if $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} < 1$, then $\|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \||u|^{q(x)}\|_{L^{p(\cdot)}(\Omega)} \le \|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\Omega)}$.*

Lemma 2.5 [17] *Let Ω be an open, bounded set with Lipschitz boundary and let $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$. If $q \in C(\overline{\Omega})$ satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases}$$

then we have a continuous embedding

$$X \hookrightarrow L^{q(\cdot)}(\Omega)$$

and the embedding is compact if $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$.

Lemma 2.6 [15] *Let $\Omega \subset \mathbb{R}^N, N \ge 2$ be a bounded domain with smooth boundary. Suppose that $p \in C_+(\overline{\Omega})$ and $r \in C(\partial\Omega)$ satisfy the condition*

$$1 \le r(x) < p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases}$$

for all $x \in \partial\Omega$. Then there exists a compact and continuous embedding $X \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.

Now we shall give the proof of the existence of at least three weak solutions for problem (N) by applying an abstract three critical points theorem obtained by Ricceri in [1]. To do this, let us introduce the two functions

$$\chi_1(r) = \inf_{u\in\Psi^{-1}((-\infty,r))} \frac{\inf_{v\in\Psi^{-1}(r)} A(v) - A(u)}{\Psi(u) - r},$$
$$\chi_2(r) = \sup_{u\in\Psi^{-1}((r,+\infty))} \frac{\inf_{v\in\Psi^{-1}(r)} A(v) - A(u)}{\Psi(u) - r},$$

for every $r \in (\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u))$.

The following lemma is the main tool of this section.

Lemma 2.7 [1] *Let X be a reflexive real Banach space; $A : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1-functional, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ is C^1-functional with compact derivative. Moreover, assume that*

$$\text{there exists } r \in \Big(\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u)\Big) \quad \text{such that } \chi_1(r) < \chi_2(r), \tag{2.2}$$

and that, for each $\lambda \in (\chi_1(r), \chi_2(r))$, the functional $A + \lambda\Psi$ is coercive. Then, for each compact interval $[a_0, b_0] \subset (\chi_1(r), \chi_2(r))$, there exists $R > 0$ with the following property: for every $\lambda \in [a_0, b_0]$ and every C^1-functional $\Gamma : X \to \mathbb{R}$ with compact derivative, there exists a number $\tau > 0$ such that, for each $\theta \in [0, \tau]$, the equation

$$A'(u) + \lambda\Psi'(u) + \theta\Gamma'(u) = 0$$

has at least three distinct solutions whose norms are less than R.

To begin with, we assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $A_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}, A_0 = A_0(x, v)$, that is, $a(x, v) = \frac{d}{dv}A_0(x, v)$. Suppose that a and A_0 satisfy the following assumptions: for $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$,

(A1) the equality

$$A_0(x, \mathbf{0}) = 0$$

holds for all $x \in \Omega$;

(A2) there is a nonnegative constant b such that

$$\big|a(x, v)\big| \le b|v|^{p(x)-1}$$

holds for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$;

(A3) $A_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$;

(A4) there exists a positive constant c_* such that the relations

$$c_*|v|^{p(x)} \le a(x, v) \cdot v \quad \text{and} \quad c_*|v|^{p(x)} \le p_+A_0(x, v)$$

hold for all $x \in \Omega$ and $v \in \mathbb{R}^N$.

Let us define the functional $A : X \to \mathbb{R}$ by

$$A(u) = \int_\Omega A_0(x, \nabla u)\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx$$

for any $u \in X$. Under the assumptions (A1), (A2) and (A4), it is easy to check that the functional A is well defined on X, by similar calculations as in [23]. And then we can modify the proof of Lemma 3.2 in [19] to get $A \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\big\langle A'(u), \varphi\big\rangle = \int_\Omega a(x, \nabla u) \cdot \nabla\varphi\,dx + \int_\Omega |u|^{p(x)-2}u\varphi\,dx$$

for any $\varphi \in X$ where $\langle\cdot,\cdot\rangle$ denotes the pairing of X and its dual X^*.

The fact that the operator A' is a mapping of type (S_+) plays an important role in obtaining our main results. The proof is essentially the same as the one in [21]; see also [22].

Lemma 2.8 *Assume that* (A1)-(A4) *hold. Then the functional* $A : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* A' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$ *and* $\limsup_{n\to\infty}\langle A'(u_n) - A'(u), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* X *as* $n \to \infty$.

Corollary 2.9 *Assume that* (A1)-(A4) *hold. Then the operator* $A' : X \to X^*$ *is strictly monotone, coercive, and hemicontinuous on* X. *Furthermore, the operator* A' *is a homeomorphism onto* X^*.

Proof It is obvious that the operator A' is strictly monotone, coercive, and hemicontinuous on X. By the Browder-Minty theorem, the inverse operator $(A')^{-1}$ exists (see Theorem 26.A in [41]). If we apply Lemma 2.8, then the proof of the continuity of the inverse operator $(A')^{-1}$ is similar to that in the case of a constant exponent and is omitted here. □

In order to deal with our main results, we need the following assumptions for nonlinear terms f and g. Denoting $F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$, then we assume that:

(H1) $p \in C_+(\overline{\Omega})$ and $1 < p_- \le p_+ < p^*(x)$ for all $x \in \Omega$.

(H2) $m \in L^\infty(\Omega)$ and $m(x) > 0$ for almost all $x \in \Omega$.

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two nonnegative functions $\rho_1, \sigma_1 \in L^\infty(\Omega)$ such that

$$\bigl|f(x,s)\bigr| \le \rho_1(x) + \sigma_1(x)|s|^{\gamma_1(x)-1},$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where $\gamma_1 \in C_+(\overline{\Omega})$ and $(\gamma_1)_+ < p_-$.

(F2) There exist a real number s_0, an element $x_0 \in \mathbb{R}^N$, and a positive constant r_0 so small that

$$\int_{B_N(x_0,r_0)} F\bigl(x,|s_0|\bigr)\,dx > 0 \quad \text{and} \quad F(x,t) \ge 0$$

for almost all $x \in B_N(x_0,r_0) \setminus B_N(x_0,\sigma r_0)$ with $\sigma \in (0,1)$ and for all $0 \le t \le |s_0|$, where $B_N(x_0,r_0) = \{x \in \Omega : |x - x_0| \le r_0\} \subseteq \Omega$.

(F3) $\limsup_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{\xi_1(x)-1}} < +\infty$ uniformly for almost all $x \in \Omega$, where $\xi_1 \in C_+(\overline{\Omega})$ with $p(x) < \xi_1(x) < p^*(x)$ for all $x \in \Omega$.

(G1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two nonnegative functions $\rho_2, \sigma_2 \in L^\infty(\partial\Omega)$

$$\bigl|g(x,s)\bigr| \le \rho_2(x) + \sigma_2(x)|s|^{\gamma_2(x)-1}$$

for all $(x,s) \in \partial\Omega \times \mathbb{R}$, where $\gamma_2 \in C_+(\partial\Omega)$ and $(\gamma_2)_+ < p_-$.

(G2) $\limsup_{s\to 0} \frac{|g(x,s)|}{|s|^{\xi_2(x)-1}} < +\infty$ uniformly for almost all $x \in \partial\Omega$, where $\xi_2 \in C_+(\partial\Omega)$ with $p(x) < \xi_2(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

Define the functionals $\Psi, H : X \to \mathbb{R}$ by

$$\Psi(u) = -\int_\Omega F(x,u)\,dx \quad \text{and} \quad H(u) = -\int_{\partial\Omega} G(x,u)\,dS$$

for any $u \in X$, where dS is the measure on the boundary. Then we obtain $\Psi, H \in C^1(X,\mathbb{R})$ and these Fréchet derivatives are

$$\langle \Psi'(u), v\rangle = -\int_\Omega f(x,u)v\,dx \quad \text{and} \quad \langle H'(u), v\rangle = -\int_{\partial\Omega} g(x,u)v\,dS$$

for any $u, v \in X$, respectively.

Definition 2.10 We say that $u \in X$ is a weak solution of the problem (N) if

$$\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx = \lambda\int_\Omega f(x,u)v\,dx + \lambda\theta\int_{\partial\Omega} g(x,u)v\,dS$$

for all $v \in X$.

As applications of Lemma 2.7, we give two consequences about the localization of the intervals of λ's for the existence of at least three solutions of problem (N); see Theorems 2.11 and 2.14. In particular, the three critical points interval in Theorem 2.14 is finer than that of Theorem 2.11.

Theorem 2.11 *Assume that* (A1)-(A4), (H1)-(H2), (F1)-(F3), *and* (G1)-(G2) *hold. Then, there exists a constant* $\hat{\ell}$ *with the following property: for each compact interval* $[a_0, b_0] \subset (\hat{\ell}, +\infty)$, *there exists a number* $\tau > 0$ *such that, for each* $\theta \in [0,\tau]$, *the problem* (N) *has at least three distinct solutions for every* $\lambda \in [a_0, b_0]$.

Proof By Lemma 2.8, the functional $A : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous C^1-functional. Moreover, it is bounded on each bounded subset of X. Using Corollary 2.9, the operator A' is a homeomorphism onto X^*, that is, there exists a continuous inverse operator $(A')^{-1} : X^* \to X$. Moreover, the modification of the proof of Proposition 3.1 in [4] shows that the operators $\Psi', H' : X \to X^*$ are compact. Using the assumptions (A4) and (F1), we know

$$\lim_{\|u\|_X\to\infty}\left\{A(u) + \lambda\Psi(u)\right\} = +\infty$$

for all $u \in X$ and all $\lambda \in \mathbb{R}$. In fact, for $\|u\|_X$ large enough and for all $\lambda \in \mathbb{R}$, it follows from Lemmas 2.1, 2.2, 2.4 and 2.5 that

$$\begin{aligned}
&A(u) + \lambda\Psi(u) \\
&\quad = \int_\Omega A_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx - \lambda\int_\Omega F(x,u)\,dx \\
&\quad \geq \frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx - |\lambda|\int_\Omega \left|\rho_1(x)\right||u|\,dx \\
&\qquad - |\lambda|\int_\Omega \frac{1}{\gamma_1(x)}\left|\sigma_1(x)\right||u|^{\gamma_1(x)}\,dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{p_+}\int_\Omega |u|^{p(x)}\,dx - |\lambda|\|\rho_1\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)} \\
&\quad - \frac{|\lambda|}{(\gamma_1)_-}\|\sigma_1\|_{L^\infty(\Omega)}\big\||u|^{\gamma_1(x)}\big\|_{L^1(\Omega)} \\
&\geq \frac{\min\{c_*,1\}}{p_+}\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx\right) - |\lambda|C_1\|u\|_X - \frac{|\lambda|C_2}{(\gamma_1)_-}\|u\|^{(\gamma_1)_+}_{L^{\gamma_1(\cdot)}(\Omega)} \\
&\geq \frac{\min\{c_*,1\}}{p_+}\|u\|_X^{p_-} - |\lambda|C_1\|u\|_X - \frac{|\lambda|C_3}{(\gamma_1)_-}\|u\|_X^{(\gamma_1)_+}
\end{aligned}
$$

for some positive constants C_i $(i = 1,2,3)$. Since $p_- > (\gamma_1)_+ > 1$, we deduce

$$
\lim_{\|u\|_X\to\infty}\bigl\{A(u) + \lambda\Psi(u)\bigr\} = +\infty
$$

for all $\lambda \in \mathbb{R}$.

Let $s_0 \neq 0$ be from (F2). Fix $\varrho \in (0,1)$, define

$$
u_\varrho(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B_N(x_0,r_0), \\ |s_0| & \text{if } x \in B_N(x_0,\varrho r_0), \\ \frac{|s_0|}{r_0(1-\varrho)}(r_0 - |x-x_0|) & \text{if } x \in B_N(x_0,r_0)\setminus B_N(x_0,\varrho r_0). \end{cases}
$$

It is clear that $0 \leq u_\varrho(x) \leq |s_0|$ for all $x \in \Omega$, and so $u_\varrho \in X$. Moreover, the continuous embedding $L^{p(x)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ (see Theorem 2.8 of [20]) implies

$$
\|u_\varrho\|_X^\alpha \geq \|\nabla u_\varrho\|^\alpha_{L^{p(x)}(\Omega)} \geq C_4\int_\Omega |\nabla u_\varrho|^{p_-}\,dx = \frac{C_4|s_0|^{p_-}(1-\varrho^N)}{(1-\varrho)^{p_-}}r_0^{N-p_-}\omega_N > 0
$$

for a positive constant C_4, where α is either p_+ or p_- and ω_N is the volume of $B_N(0,1)$. Also, by using the assumption (F2), we get

$$
\begin{aligned}
-\Psi(u_\varrho) &= \int_{B_N(x_0,\varrho r_0)} F\bigl(x,|s_0|\bigr)\,dx + \int_{B_N(x_0,r_0)\setminus B_N(x_0,\varrho r_0)} F\left(x, \frac{|s_0|}{r_0(1-\varrho)}\bigl(r_0 - |x-x_0|\bigr)\right)dx \\
&> 0.
\end{aligned}
$$

Then the crucial number

$$
\hat{\ell} := \chi_1(0) = \inf_{u\in\Psi^{-1}((-\infty,0))}\left(-\frac{A(u)}{\Psi(u)}\right)
$$

is well defined. It remains to show that the condition (2.2) holds. For all $u \in \Psi^{-1}((-\infty,0))$, we get

$$
\chi_1(r) = \inf_{u\in\Psi^{-1}((-\infty,r))}\frac{\inf_{v\in\Psi^{-1}(r)}A(v) - A(u)}{\Psi(u) - r} \leq \frac{\inf_{v\in\Psi^{-1}(r)}A(v) - A(u)}{\Psi(u) - r} \leq \frac{A(u)}{r - \Psi(u)}
$$

for all $r \in (\Psi(u),0)$. Then we have

$$
\limsup_{r\to 0-}\chi_1(r) \leq -\frac{A(u)}{\Psi(u)}
$$

for all $u \in \Psi^{-1}((-\infty, 0))$. Hence we assert that

$$\limsup_{r\to 0-} \chi_1(r) \le \chi_1(0).$$

Now, we claim that there exists a positive real number K_* such that

$$\left|F(x,s)\right| \le K_* m(x)|s|^{\xi_1(x)} \tag{2.3}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$. First of all, the assumptions (F1) and (F3) imply that $f(x,0) = 0$ for almost all $x \in \Omega$. Indeed, if there exists $A \subset \Omega$, $|A| > 0$ such that $|f(x,0)| > 0$ for all $x \in A$, then $\lim_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{\xi_1(x)-1}} = \infty$ for all $x \in A$, contradicting with (F3). Thus, we get $\limsup_{s\to 0} \frac{|F(x,s)|}{m(x)|s|^{\xi_1(x)}} < \infty$ uniformly almost everywhere in Ω by the L'Hôpital rule. We denote

$$K = \limsup_{s\to 0} \frac{|F(x,s)|}{m(x)|s|^{\xi_1(x)}}$$

for almost all $x \in \Omega$. Then there exists $\delta > 0$ such that $|F(x,s)| \le (K+1)m(x)|s|^{\xi_1(x)}$ for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| < \delta$. Next, let s be fixed with $|s| \ge \delta$. Then it follows from (F1) that

$$\begin{aligned}
\left|F(x,s)\right| &\le \left(\frac{\rho_1(x)}{m(x)}|s|^{1-\xi_1(x)} + \frac{\sigma_1(x)}{\gamma_1(x)m(x)}|s|^{\gamma_1(x)-\xi_1(x)}\right)m(x)|s|^{\xi_1(x)} \\
&\le \left(K_1|s|^{1-\xi_1(x)} + \frac{K_2}{(\gamma_1)_-}|s|^{\gamma_1(x)-\xi_1(x)}\right)m(x)|s|^{\xi_1(x)} \\
&\le \left(K_1\left(\delta^{1-(\xi_1)_+} + \delta^{1-(\xi_1)_-}\right) + \frac{K_2}{(\gamma_1)_-}\left(\delta^{(\gamma_1)_- -(\xi_1)_+} + \delta^{(\gamma_1)_+ -(\xi_1)_-}\right)\right)m(x)|s|^{\xi_1(x)}
\end{aligned}$$

for almost all $x \in \Omega$ and for positive constants K_1 and K_2. Hence equation (2.3) holds, where $K_* = \max\{K+1, K_1(\delta^{1-(\xi_1)_+} + \delta^{1-(\xi_1)_-}) + K_2/(\gamma_1)_-(\delta^{(\gamma_1)_- -(\xi_1)_+} + \delta^{(\gamma_1)_+ -(\xi_1)_-})\}$.

By (2.3), it follows that

$$\left|\Psi(u)\right| \le \int_\Omega K_* m(x)|u|^{\xi_1(x)}\,dx \le 2C_5K_*\|m\|_{L^\infty(\Omega)}\|u\|_X^\beta$$

for a positive constant C_5 and for any $u \in X$, where β is either $(\xi_1)_+$ or $(\xi_1)_-$. If $r < 0$ and $v \in \Psi^{-1}(r)$, then we obtain by (A4)

$$\begin{aligned}
r = \Psi(v) &\ge -2C_5K_*\|m\|_{L^\infty(\Omega)}\|v\|_X^\beta \\
&\ge -2C_5K_*\|m\|_{L^\infty(\Omega)}\left(\frac{p_+}{\min\{c_*,1\}}A(v)\right)^{\frac{\beta}{\alpha}},
\end{aligned}$$

where α is either p_+ or p_-. Since $u = 0 \in \Psi^{-1}((r,+\infty))$, by the definition of χ_2, we have

$$\chi_2(r) \ge \frac{1}{|r|}\inf_{v\in\Psi^{-1}((r,+\infty))} A(v) \ge \frac{|r|^{\frac{\alpha}{\beta}-1}}{(2C_5K_*)^{\frac{\alpha}{\beta}}\|m\|_{L^\infty(\Omega)}^{\frac{\alpha}{\beta}}}\frac{\min\{c_*,1\}}{p_+}$$

and hence $\lim_{r\to 0-} \chi_2(r) = +\infty$ because $\beta > \alpha$. Then we deduce

$$\limsup_{r\to 0-} \chi_1(r) \le \chi_1(0) = \hat{\ell} < \lim_{r\to 0-} \chi_2(r) = +\infty.$$

This confirms that, for all integers $n \ge \hat{n} = 2 + [\hat{\ell}]$, there exists a negative sequence $\{r_n\}$ converging to 0 with

$$\chi_1(r_n) < \hat{\ell} + 1/n < n < \chi_2(r_n),$$

and thus

$$\bigcup_{n=\hat{n}}^{\infty} \bigl(\chi_1(r_n), \chi_2(r_n)\bigr) \supset \bigcup_{n=\hat{n}}^{\infty} \left[\hat{\ell} + \frac{1}{n}, n\right] = (\hat{\ell}, +\infty).$$

Due to Lemma 2.7 with $\Gamma = \lambda H$, we conclude that for each compact interval $[a_0, b_0] \subset (\hat{\ell}, +\infty)$, there exists a number $\tau > 0$ such that, for each $\theta \in [0, \tau]$, the problem (N) has at least three distinct solutions for every $\lambda \in [a_0, b_0]$. This completes the proof. □

In the remainder of this paper, we give much more accurate interval than the three critical points interval in Theorem 2.11. Roughly speaking, we determine the intervals of λ's for which problem (N) admits only the trivial solution and for which problem (N) has at least two nontrivial solutions. To do this, we consider an estimate of the Sobolev trace embedding's constant and the positive principal eigenvalue for $p(x)$-Laplacian Neumann problem. To begin with, we observe the constant of the embedding $X \hookrightarrow L^{\ell(x)}(\partial\Omega)$. Thanks to Bonder and Rossi's result (see for instance [39]), it is possible to obtain the estimate of the embedding constant $\tilde{c}_\ell$.

Remark 2.12 For any $1 < r < N$ and $1 \le \ell \le r^\partial := r(N-1)/(N-r)$, we have $W^{1,r}(\Omega) \hookrightarrow L^{r^\partial}(\partial\Omega)$ and hence the following inequality holds:

$$\tilde{c}_{r,r^\partial} \|u\|^r_{L^{r^\partial}(\partial\Omega)} \le \|u\|^r_{W^{1,r}(\Omega)} \tag{2.4}$$

for all $u \in W^{1,r}(\Omega)$. This is known as the Sobolev trace embedding theorem. The best constant for this embedding is the largest $\tilde{c}_{r,r^\partial}$ such that the above inequality holds, that is,

$$\tilde{c}_{r,r^\partial} = \inf_{u\in W^{1,r}(\Omega)\setminus\{0\}} \frac{\|u\|^r_{W^{1,r}(\Omega)}}{\|u\|^r_{L^{r^\partial}(\partial\Omega)}}.$$

From Lemma 2.6, we have the continuous embedding $W^{1,r}(\Omega) \hookrightarrow L^\ell(\partial\Omega)$ for $1 < r < N$ and $1 \le \ell \le r^\partial$. Denote $d = r^\partial/\ell$ and $d' = r^\partial/(r^\partial - \ell)$. Since $|u|^\ell \in L^{\frac{r^\partial}{\ell}}(\partial\Omega)$, by the Hölder inequality, we have $|u|^\ell \in L^1(\partial\Omega)$ and

$$\begin{aligned}\int_{\partial\Omega} |u(x)|^\ell \, ds &= \|u^\ell\|_{L^1(\partial\Omega)} \le \|u^\ell\|_{L^d(\partial\Omega)} \|1\|_{L^{d'}(\partial\Omega)} \\ &= \left(\int_{\partial\Omega} |u(x)|^{r^\partial} \, ds\right)^{\frac{\ell}{r^\partial}} |\partial\Omega|^{\frac{1}{d'}} = |\partial\Omega|^{\frac{r^\partial-\ell}{r^\partial\ell}} \|u\|^\ell_{L^{r^\partial}(\partial\Omega)}\end{aligned}$$

and so

$$\|u\|_{L^{\ell}(\partial\Omega)} = \|u^{\ell}\|_{L^1(\partial\Omega)}^{\frac{1}{\ell}} \le |\partial\Omega|^{\frac{r^{\partial}-\ell}{r^{\partial}}} \|u\|_{L^{r^{\partial}}(\partial\Omega)}.$$

From (2.4), we get

$$\|u\|_{L^{\ell}(\partial\Omega)} \le \|u\|_{L^{r^{\partial}}(\partial\Omega)} |\partial\Omega|^{\frac{r^{\partial}-\ell}{r^{\partial}}} \le \left(\frac{1}{\tilde{c}_{r,r^{\partial}}}\right)^{\frac{1}{r}} |\partial\Omega|^{\frac{r^{\partial}-\ell}{r^{\partial}}} \|u\|_{W^{1,r}(\Omega)}. \tag{2.5}$$

Now let $\ell \in C(\partial\Omega)$ and $1 < \ell(x) \le \ell_+ \le (r_-)^{\partial} \le r^{\partial}(x)$ for each $x \in \partial\Omega$. By using (2.5) for $r = p_-$ and $\ell = \ell_+$, we obtain

$$\|u\|_{L^{\ell_+}(\partial\Omega)} \le \left(\frac{1}{\tilde{c}_{p_-,p_-^{\partial}}}\right)^{\frac{1}{p_-}} |\partial\Omega|^{\frac{p_-^{\partial}-\ell_+}{p_-^{\partial}}} \|u\|_{W^{1,p_-}(\Omega)} \tag{2.6}$$

for each $u \in X$. Taking into account that (see Theorem 2.8 in [20]) $L^{p(x)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ and $L^{\ell_+}(\partial\Omega) \hookrightarrow L^{\ell(x)}(\partial\Omega)$ with continuous embeddings and that the constants of such embeddings do not exceed $|\Omega| + 1$ and $|\partial\Omega| + 1$, respectively, we have

$$\|u\|_{W^{1,p_-}(\Omega)} = \|u\|_{L^{p_-}(\Omega)} + \|\nabla u\|_{L^{p_-}(\Omega)} \le \big(|\Omega| + 1\big)\|u\|_X \tag{2.7}$$

and

$$\|u\|_{L^{\ell(x)}(\partial\Omega)} \le \big(|\partial\Omega| + 1\big)\|u\|_{L^{\ell_+}(\partial\Omega)}. \tag{2.8}$$

It follows from (2.6), (2.7), and (2.8) that

$$\|u\|_{L^{\ell(x)}(\partial\Omega)} \le \left(\frac{1}{\tilde{c}_{p_-,p_-^{\partial}}}\right)^{\frac{1}{p_-}} |\partial\Omega|^{\frac{p_-^{\partial}-\ell_+}{p_-^{\partial}\ell_+}} \big(|\Omega| + 1\big)\big(|\partial\Omega| + 1\big)\|u\|_X$$

and so denoting by $\tilde{c}_{\ell}$ the embedding constant of $X \hookrightarrow L^{\ell(x)}(\partial\Omega)$, this implies that

$$\tilde{c}_{\ell} \le \left(\frac{1}{\tilde{c}_{p_-,p_-^{\partial}}}\right)^{\frac{1}{p_-}} |\partial\Omega|^{\frac{p_-^{\partial}-\ell_+}{p_-^{\partial}\ell_+}} \big(|\Omega| + 1\big)\big(|\partial\Omega| + 1\big). \tag{2.9}$$

Next, we consider the positive principal eigenvalue for the $p(x)$-Laplacian Neumann problems. From the analogous argument to the proof of Proposition 3.8 in [7], we deduce the following proposition.

Proposition 2.13 *Let us consider the eigenvalue problem*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda m(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{E}$$

Assume that (H1)-(H2) *hold. Denote the quantity*

$$\lambda_* = \inf_{u \in X\setminus\{0\}} \frac{\int_{\Omega} |\nabla u|^{p(x)}\,dx + \int_{\Omega} |u|^{p(x)}\,dx}{\int_{\Omega} m(x)|u|^{p(x)}\,dx}. \tag{2.10}$$

Then λ_* *is a positive eigenvalue of problem* (E), *that is, there is* $u_1 \in X$ *with*

$$\int_\Omega m(x)|u_1|^{p(x)}\,dx = 1$$

such that realizes the infimum in (2.10) *and represents an eigenfunction for* λ_*. *In particular,*

$$\lambda_* \int_\Omega m(x)|u|^{p(x)}\,dx \le \int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx$$

for every $u \in X$.

Denote the crucial values

$$\mathcal{C}_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{m(x)|s|^{p(x)-1}} \quad \text{and} \quad \mathcal{C}_g = \operatorname*{ess\,sup}_{s\neq 0, x\in\partial\Omega} \frac{|g(x,s)|}{|s|^{p(x)-1}}.$$

Then the same arguments as in [34] imply that $\mathcal{C}_f$ and $\mathcal{C}_g$ are well defined and positive constants. Furthermore, as mentioned in the proof of Theorem 2.11, we get $\limsup_{s\to 0} \frac{|F(x,s)|}{m(x)|s|^{\xi_1(x)}} < \infty$ uniformly almost everywhere in Ω. Finally,

$$\mathcal{C}_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{m(x)|s|^{p(x)-1}} \in \mathbb{R}^+.$$

Indeed, $\mathcal{C}_f > 0$ having $f \not\equiv 0$ and $\mathcal{C}_f < \infty$, since, first by (F3),

$$\lim_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{p(x)-1}} = \lim_{s\to 0} \left(\frac{|f(x,s)|}{m(x)|s|^{\xi_1(x)-1}} \right) |s|^{\xi_1(x)-p(x)} = 0$$

uniformly almost everywhere in Ω, where $p(x) < \xi_1(x)$. Moreover,

$$\frac{|f(x,s)|}{m(x)|s|^{p(x)-1}} \le \frac{\rho_1(x) + \sigma_1(x)|s|^{\gamma_1(x)-1}}{m(x)|s|^{p(x)-1}} \le \frac{(\bar{C}_1 + \bar{C}_2)|s|^{\gamma_1(x)-1}}{m(x)|s|^{p(x)-1}} = \frac{(\bar{C}_1 + \bar{C}_2)|s|^{\gamma_1(x)-p(x)}}{m(x)}$$

for almost all $x \in \Omega$ and all $|s| \ge 1$ by (F1), namely $\lim_{s\to\infty} \frac{|f(x,s)|}{m(x)|s|^{p(x)-1}} = 0$ uniformly almost everywhere in Ω, since $\gamma_1(x) < p(x)$. In the same way, using the assumptions (G1) and (G2), we can show that $\mathcal{C}_g$ is positive and constant. Furthermore, the following relations hold:

$$\operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{m(x)|s|^{p(x)}} = \frac{\mathcal{C}_f}{p_-} \quad \text{and} \quad \operatorname*{ess\,sup}_{s\neq 0, x\in\partial\Omega} \frac{|G(x,s)|}{|s|^{p(x)}} = \frac{\mathcal{C}_g}{p_-}. \tag{2.11}$$

By applying Lemma 2.7, we can obtain the following assertion which is our main result in this paper.

Theorem 2.14 *Assume* (A1)-(A4), (H1)-(H2), (F1), (F3), *and* (G1)-(G2) *hold. Then we have the following*:

(i) *For every* $\theta \in \mathbb{R}$, *there exists* $\ell_* = (\lambda_* \min\{1, c_*\}p_-)/(\mathcal{C}_f + \lambda_*|\theta|\mathcal{C}_g\tilde{c}_p)p_+$ *such that problem* (N) *has only the trivial solution for all* $\lambda \in [0, \ell_*)$, *where* c_* *is the positive constant from* (A4) *and* λ_* *is the positive real number in* (2.10).

(ii) *If furthermore* f *satisfies the following assumption*:

(F4) *$\int_\Omega F(x,u_1(x))\,dx > b/c_*p_-$ holds, where u_1 is the eigenfunction corresponding to the principal eigenvalue of problem* (E) *satisfying $\int_\Omega m(x)|u_1|^{p(x)}\,dx = 1$ and $b, c_* > 0$ are constants given in* (A2) *and* (A4), *respectively,*

then for each compact interval $[a_0,b_0] \subset (\ell^*,\ c_*\lambda_*\max\{1,b\}/b)$, with $\ell^* \geq \ell_*$ and $\ell^* = \chi_1(0) < c_*\lambda_*\max\{1,b\}/b$, there exists $\tau > 0$ such that problem (N) *has at least two distinct nontrivial solutions for every $\lambda \in [a_0,b_0]$ and $\theta \in (-\tau,\tau)$.*

Proof Under the assumptions (A1)-(A4), (H1), (F1), and (G1), all of the assumptions in Lemma 2.7 except the condition (2.2) are satisfied.

Now we prove the assertion (i). Let $u \in X$ be a nontrivial weak solution of problem (N). Then it is clear that

$$\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx = \lambda\int_\Omega f(x,u)v\,dx + \lambda\theta\int_{\partial\Omega} g(x,u)v\,dS$$

for any $v \in X$. If we put $v = u$, then it follows from (A4), (2.9), (2.10), the definitions of $\mathcal{C}_f$ and $\mathcal{C}_g$, and Lemma 2.6 that

$$\begin{aligned}
&\frac{\lambda_*\min\{1,c_*\}p_-}{p_+}\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx\right)\\
&\quad\leq \lambda_*\left(\int_\Omega a(x,\nabla u)\cdot\nabla u\,dx + \int_\Omega |u|^{p(x)}\,dx\right)\\
&\quad= \lambda_*\left(\lambda\int_\Omega f(x,u)u\,dx + \lambda\theta\int_{\partial\Omega} g(x,u)u\,dS\right)\\
&\quad\leq \lambda_*\left(\lambda\int_\Omega \frac{f(x,u)}{m(x)|u|^{p(x)-1}}m(x)|u|^{p(x)}\,dx + \lambda\theta\int_{\partial\Omega}\frac{g(x,u)}{|u|^{p(x)-1}}|u|^{p(x)}\,dS\right)\\
&\quad\leq \lambda\mathcal{C}_f\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx\right) + \lambda_*\lambda|\theta|\mathcal{C}_g\int_{\partial\Omega}|u|^{p(x)}\,dS\\
&\quad\leq \lambda\left(\mathcal{C}_f + \lambda_*|\theta|\mathcal{C}_g\tilde{c}_p\right)\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx\right).
\end{aligned}$$

Thus if u is a nontrivial weak solution of problem (N), then necessarily $\lambda \geq \ell_* = (\lambda_*\min\{1,c_*\}p_-)/(\mathcal{C}_f + \lambda_*|\theta|\mathcal{C}_g\tilde{c}_p)p_+$, as claimed.

Next, we show the assertion (ii). From (F4), it is clear that the crucial positive number

$$\ell^* = \chi_1(0) = \inf_{u\in\Psi^{-1}((-\infty,0))}\left(-\frac{A(u)}{\Psi(u)}\right)$$

is well defined. Hence, by the definition of u_1 and assumption (F4), we have

$$\begin{aligned}
\ell^* = \chi_1(0) &= \inf_{u\in\Psi^{-1}((-\infty,0))}\left(-\frac{A(u)}{\Psi(u)}\right) \leq -\frac{A(u_1)}{\Psi(u_1)}\\
&= \frac{\int_\Omega A_0(x,\nabla u_1)\,dx + \int_\Omega \frac{1}{p(x)}|u_1|^{p(x)}\,dx}{\int_\Omega F(x,u_1)\,dx}\\
&< \frac{c_*p_-}{b}\left(\int_\Omega \frac{b}{p(x)}|\nabla u_1|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}|u_1|^{p(x)}\,dx\right)\\
&\leq \frac{c_*\max\{1,b\}}{b}\left(\int_\Omega |\nabla u_1|^{p(x)}\,dx + \int_\Omega |u_1|^{p(x)}\,dx\right) = \frac{c_*\lambda_*\max\{1,b\}}{b}.
\end{aligned}$$

In addition, to assert $\ell^* \geq \ell_*$, let u be in X with $u \not\equiv 0$. From (A4) and (2.11), we obtain

$$\begin{aligned}
\frac{A(u)}{|\Psi(u)|} &= \frac{\int_\Omega A_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx}{|\int_\Omega F(x,u)\,dx|} \geq \frac{\frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{p_+}\int_\Omega |u|^{p(x)}\,dx}{\int_\Omega \frac{|F(x,u)|}{m(x)|u|^{p(x)}} m(x)|u|^{p(x)}\,dx} \\
&\geq \frac{\frac{\min\{1,c_*\}}{p_+}(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx)}{\frac{\mathcal{C}_f}{p_-}\int_\Omega m(x)|u|^{p(x)}\,dx} \geq \frac{\min\{1,c_*\}p_-}{\mathcal{C}_f p_+}\lambda_* \\
&\geq \frac{\lambda_* \min\{1,c_*\}p_-}{(\mathcal{C}_f + \lambda_*|\theta|\mathcal{C}_g\tilde{c}_p)p_+} = \ell_*.
\end{aligned}$$

Hence we have $\ell^* \geq \ell_*$. Now we claim that there exists a real number r satisfying condition (2.2). For any $u \in \Psi^{-1}((-\infty,0))$, we deduce that

$$\chi_1(r) \leq \frac{A(u)}{r - \Psi(u)}$$

for all $r \in (\Psi(u),0)$. This implies that

$$\limsup_{r\to 0-} \chi_1(r) \leq -\frac{A(u)}{\Psi(u)}$$

for all $u \in \Psi^{-1}((-\infty,0))$. Hence we have

$$\limsup_{r\to 0-} \chi_1(r) \leq \chi_1(0) = \ell^*.$$

As we already mentioned in Theorem 2.11, it follows from (F1) and (F3) that there exists a positive real number K_* such that

$$\left|F(x,s)\right| \leq K_* m(x)|s|^{\xi_1(x)}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$. Then it follows that

$$\begin{aligned}
\left|\Psi(u)\right| &\leq \int_\Omega K_* m(x)|u|^{\xi_1(x)}\,dx \\
&\leq 2C_6 K_* \|m\|_{L^\infty(\Omega)}\|u\|_X^\alpha + \frac{b\min\{1,c_*\}}{c_*\lambda_* p_+ \max\{1,b\}}\|u\|_X^\beta
\end{aligned}$$

for a positive constant C_6 and for all $u \in X$, where α is either $(\xi_1)_+$ or $(\xi_1)_-$ and β is either p_+ or p_-. If $r < 0$ and $v \in \Psi^{-1}(r)$, then it follows from (A4) that

$$\begin{aligned}
\frac{c_* p_+ \max\{1,b\}}{b\min\{1,c_*\}} r &= \frac{c_* p_+ \max\{1,b\}}{b\min\{1,c_*\}}\Psi(v) \\
&\geq -2C_6 K_*\|m\|_{L^\infty(\Omega)} \frac{c_* p_+ \max\{1,b\}}{b\min\{1,c_*\}}\|v\|_X^\alpha - \frac{1}{\lambda_*}\|v\|_X^\beta \\
&\geq -2C_6 K_*\|m\|_{L^\infty(\Omega)} \frac{c_* \max\{1,b\}}{b}\left(\frac{p_+}{\min\{1,c_*\}}\right)^{\frac{\alpha}{\beta}+1} A(v)^{\frac{\alpha}{\beta}} \\
&\quad - \frac{p_+}{\lambda_* \min\{1,c_*\}}A(v). \qquad (2.12)
\end{aligned}$$

Since $u = 0 \in \Psi^{-1}((r,+\infty))$, by the definition of χ_2, we have

$$\chi_2(r) \geq \frac{1}{|r|} \inf_{v\in\Psi^{-1}((r,+\infty))} A(v),$$

and hence there exists an element $u_r \in \Psi^{-1}((r,+\infty))$ such that $A(u_r) = \inf_{v\in\Psi^{-1}((r,+\infty))} A(v)$; see Theorem 6.1.1 of [42]. According to (2.12), we get

$$\begin{aligned} \frac{c_*p_+ \max\{1,b\}}{b\min\{1,c_*\}} &\leq \hat{C}|r|^{\frac{\alpha}{\beta}-1}\left(\frac{A(u_0)}{|r|}\right)^{\frac{\alpha}{\beta}} + \frac{p_+}{\lambda_* \min\{1,c_*\}}\frac{A(u_0)}{|r|} \\ &\leq \hat{C}|r|^{\frac{\alpha}{\beta}-1}\chi_2(r)^{\frac{\alpha}{\beta}} + \frac{p_+}{\lambda_* \min\{1,c_*\}}\chi_2(r), \end{aligned} \tag{2.13}$$

where the positive constant $\hat{C}$ denotes

$$\hat{C} = 2C_6K_*\|m\|_{L^\infty(\Omega)}\frac{c_* \max\{1,b\}}{b}\left(\frac{p_+}{\min\{1,c_*\}}\right)^{\frac{\alpha}{\beta}+1}.$$

Then there are two possibilities to be considered: either χ_2 is locally bounded at $0-$, so that relation (2.13) shows $\liminf_{r\to 0-}\chi_2(r) \geq (c_*\lambda_* \max\{1,b\})/b$ because $\alpha > \beta$, or $\limsup_{r\to 0-}\chi_2(r) = \infty$.

Since the functional $A + \lambda\Psi$ is coercive for all $\lambda \in \mathbb{R}$ by Theorem 2.11. For all integers $n \geq n^* := 1 + 2/[(c_*\lambda_* \max\{1,b\}/b) - \ell^*]$, there exists a negative sequence $\{r_n\}$ converging to 0 as $n \to \infty$ such that $\chi_1(r_n) < \ell^* + 1/n < (c_*\lambda_* \max\{1,b\}/b) - 1/n < \chi_2(r_n)$. Due to Lemma 2.7 with $\Gamma = \lambda H$, we conclude that $u \equiv 0$ is a critical point of the functional $A + \lambda\Psi + \lambda\theta H$ and there exists $\tau > 0$ such that problem (N) admits at least two distinct weak solutions for each compact interval

$$\begin{aligned} [a_0,b_0] \subset \left(\ell^*, \frac{c_*\lambda_* \max\{1,b\}}{b}\right) &= \bigcup_{n=n^*}^{\infty}\left[\ell^* + \frac{1}{n}, \frac{c_*\lambda_* \max\{1,b\}}{b} - \frac{1}{n}\right] \\ &\subset \bigcup_{n=n^*}^{\infty}\big(\chi_1(r_n), \chi_2(r_n)\big) \end{aligned}$$

and for every $\lambda \in [a_0,b_0]$ and $\theta \in (-\tau,\tau)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Ricceri, B: A further refinement of a three critical points theorem. Nonlinear Anal. **74**, 7446-7454 (2011)
2. Barletta, G, Chinnì, A, O'Regan, D: Existence results for a Neumann problem involving the $p(x)$-Laplacian with discontinuous nonlinearities. Nonlinear Anal., Real World Appl. **27**, 312-325 (2016)
3. Bonanno, G, Molica Bisci, G, Rădulescu, V: Existence of three solutions for a non-homogeneous Neumann problem through Orlicz-Sobolev spaces. Nonlinear Anal. **74**, 4785-4795 (2011)
4. Boureanu, M-M, Preda, F: Infinitely many solutions for elliptic problems with variable exponent and nonlinear boundary conditions. Nonlinear Differ. Equ. Appl. **19**, 235-251 (2012)
5. Boureanu, M-M, Udrea, D-N: Existence and multiplicity result for elliptic problems with $p(\cdot)$-growth conditions. Nonlinear Anal., Real World Appl. **14**, 1829-1844 (2013)
6. D'Aguì, G, Sciammetta, A: Infnitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions. Nonlinear Anal. **75**, 5612-5619 (2012)
7. Kim, Y-H, Park, K: Multiple solutions for equations of $p(x)$-Laplace type with nonlinear Neumann boundary condition. Bull. Korean Math. Soc. in press
8. Rădulescu, V: Nonlinear elliptic equations with variable exponent: old and new. Nonlinear Anal. **121**, 336-369 (2015)
9. Yao, J: Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators. Nonlinear Anal. **68**, 1271-1283 (2008)
10. Zhao, J-H, Zhao, P-H: Existence of infinitely many weak solutions for the p-Laplacian with nonlinear boundary conditions. Nonlinear Anal. **69**, 1343-1355 (2008)
11. Aouaoui, S: On some degenerate quasilinear equations involving variable exponents. Nonlinear Anal. **75**, 1843-1858 (2012)
12. Bonanno, G, Chinnì, A: Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent. J. Math. Anal. Appl. **418**, 812-827 (2014)
13. Diening, L, Harjulehto, P, Hästö, P, Rùžička, M: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Berlin (2011)
14. Edmunds, DE, Rákosník, J: Sobolev embeddings with variable exponent. Stud. Math. **143**, 267-293 (2000)
15. Fan, X: Boundary trace embedding theorems for variable exponent Sobolev spaces. J. Math. Anal. Appl. **339**, 1395-1412 (2008)
16. Fan, X, Zhao, D: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424-446 (2001)
17. Fan, X, Shen, J, Zhao, D: Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. **262**, 749-760 (2001)
18. Fu, Y, Shan, Y: On the removability of isolated singular points for elliptic equations involving variable exponent. Adv. Nonlinear Anal. **5**, 121-132 (2016)
19. Kim, IH, Kim, Y-H: Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents. Manuscr. Math. **147**, 169-191 (2015)
20. Kovacik, O, Rakosnik, J: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J. **41**, 592-618 (1991)
21. Le, VK: On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces. Nonlinear Anal. **71**, 3305-3321 (2009)
22. Lee, SD, Park, K, Kim, Y-H: Existence and multipicity of solutions for equations involving nonhomogeneous operators of $p(x)$-Laplace type in $\mathbb{R}^N$. Bound. Value Probl. **2014**, 261 (2014)
23. Mihăilescu, M, Rădulescu, V: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. Lond. Ser. A **462**, 2625-2641 (2006)
24. Ouaro, S, Ouedraogo, A, Soma, S: Multivalued problem with Robin boundary condition involving diffuse measure data and variable exponent. Adv. Nonlinear Anal. **3**, 209-235 (2014)
25. Repovš, D: Stationary waves of Schrödinger-type equations with variable exponent. Anal. Appl. (Singap.) **13**, 645-661 (2015)
26. Rădulescu, V, Repovš, D: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis. CRC Press, Taylor & Francis Group, Boca Raton (2015)
27. Rajagopal, KR, Rùžička, M: Mathematical modeling of electrorheological materials. Contin. Mech. Thermodyn. **13**, 59-78 (2001)
28. Rùžička, M: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
29. Yücedağ, Z: Solutions of nonlinear problems involving $p(x)$-Laplacian operator. Adv. Nonlinear Anal. **4**, 285-293 (2015)
30. Ricceri, B: Existence of three solutions for a class of elliptic eigenvalue problems. Math. Comput. Model. **32**, 1485-1494 (2000)
31. Ricceri, B: On a three critical points theorem. Arch. Math. (Basel) **75**, 220-226 (2000)
32. Boureanu, M-M: A new class of nonhomogeneous differential operator and applications to anisotropic systems. Complex Var. Elliptic Equ. **61**, 712-730 (2016)
33. Choi, EB, Kim, Y-H: Three solutions for equations involving nonhomogeneous operators of p-Laplace type in $\mathbb{R}^N$. J. Inequal. Appl. **2014**, 427 (2014)
34. Colasuonno, F, Pucci, P, Varga, C: Multiple solutions for an eigenvalue problem involving p-Laplacian type operators. Nonlinear Anal. **75**, 4496-4512 (2012)
35. Ricceri, B: A three critical points theorem revisited. Nonlinear Anal. **70**, 3084-3089 (2009)
36. Bonanno, G: Relations between the mountain pass theorem and local minima. Adv. Nonlinear Anal. **1**, 205-220 (2012)
37. Bonanno, G, Marano, SA: On the structure of the critical set of non-differentiable functions with a weak compactness condition. Appl. Anal. **89**, 1-10 (2010)
38. Arcoya, D, Carmona, J: A nondifferentiable extension of a theorem of Pucci and Serrin and applications. J. Differ. Equ. **235**, 683-700 (2007)
39. Bonder, JF, Rossi, JD: Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains. Commun. Pure Appl. Anal. **1**, 359-378 (2002)
40. Cruz-Uribe, D, Fiorenza, A: Variable Lebesgue Spaces: Foundations and Harmonic Analysis. Springer, Basel (2013)
41. Zeidler, E: Nonlinear Functional Analysis and Its Applications II/B. Springer, New York (1990)
42. Berger, MS: Nonlinearity and Functional Analysis. Lectures on Nonlinear Problems in Mathematical Analysis. Pure and Applied Mathematics. Academic Press, New York (1977)