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New approach for the existence and uniqueness of periodic solutions to p-Laplacian prescribed mean curvature equations

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Abstract

Using a generalized Mawhin continuation theorem, we obtain some sufficient conditions which guarantee the existence and uniqueness of periodic solutions for two types of prescribed mean curvature p-Laplacian equations.

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1 Introduction

Periodic phenomena are omnipresent in the real world. Since Hale [1] first put forward the concept of functional differential equation (FDE), a large amount of results for periodic solutions of FDEs have been obtained; see *e.g.* [2–9].

In past decades, many authors investigated the existence of periodic solutions for p-Laplacian differential equations by using Mawhin's continuation theorem, for more details, see *e.g.* [10–15] for work on this subject. Now we summarize their research methods. If we set $L = (\varphi_p(x'(t)))'$, then Mawhin's continuation theorem cannot be used directly, because the operator L in Mawhin's continuation theorem need be linear, while $(\varphi_p(x'(t)))'$ is non-linear when $p \neq 2$. In order to use Mawhin's continuation theorem, hence, we must change the form of the source equation. For example, Cheung and Ren [10] considered the following Liénard equation with p-Laplacian:

$$\left(\varphi_p\left(u'(t)\right)\right)' + f\left(t, u(t)\right)u'(t) + \beta(t)g\left(u\left(t - \tau(t)\right)\right) = e(t). \tag{1.1}$$

Transfer (1.1) into the form

$$\begin{cases} u_1'(t) = \varphi_q(u_2(t)), \\ u_2'(t) = -f(t, u_1(t))\varphi_q(u_2(t)) - \beta(t)g(u_1(t - \tau(t))) + e(t). \end{cases}$$

In the same way, in [11], one again investigated p-Laplacian Rayleigh equation as follows:

$$\left(\varphi_p\left(u'(t)\right)\right)' + f\left(t, u'(t)\right) + \beta g\left(t, u\left(t - \tau(t)\right)\right) = e(t). \tag{1.2}$$

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Transfer (1.2) into the form

$$\begin{cases} u_1'(t) = \varphi_q(u_2(t)), \\ u_2'(t) = -f(\varphi_q(u_2(t))) - \beta g(u_1(t - \tau(t))) + e(t). \end{cases}$$

Very recently, Wang [16] considered a class of prescribed mean curvature p-Laplacian Rayleigh equation with one deviating argument as follows:

$$\left(\varphi_p \left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}} \right) \right)' + f(x'(t)) + g(x(t - \tau(t))) = e(t). \tag{1.3}$$

After that, based on the work of [16], Li, An, and Ge [17] studied a class of prescribed mean curvature p-Laplacian Liénard equation with two deviating arguments,

$$\left(\varphi_p \left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}} \right) \right)' + f(x(t))x'(t) + g(x(t - \tau(t))) + h(x(t - \gamma(t))) = e(t). \tag{1.4}$$

Similar to the above approach, (1.3) and (1.4) can be changed to a two-dimensional system.

In addition, we find that the above method can be applied to studying the existence of periodic solutions to neutral equations with p-Laplacian, Zhu and Lu [7] considered the following neutral equation with p-Laplacian:

$$\left(\varphi_p \left[(u(t) - cu(t - \sigma))' \right] \right)' + g(t, u(t - \tau(t))) = e(t). \tag{1.5}$$

Transfer (1.5) into the form

$$\begin{cases} (Au_1)'(t) = \varphi_q(u_2(t)), \\ u_2'(t) = -g(t, u_1(t - \tau(t))) + e(t), \end{cases}$$

where $Au(t) = u(t) - cu(t - \sigma)$.

In short, most of the papers used this method (transfer the source equation to a two-dimensional system) for studying p-Laplacian equations. In the face of such a fact, we cannot but ask 'how can we find the different ways for studying the p-Laplacian equations?' In order to investigate BVPs with a p-Laplacian, Ge and Ren [18] extended Mawhin's continuation theorem. The above generalized theorem removes the restriction that L is a linear operator. Motivated by the work of Ge and Ren, in the present paper we will study the existence and uniqueness of periodic solutions for equations (1.6) and (1.7) by using the generalized Mawhin's continuation theorem. Here, it is worth stating that the forms of equations (1.6) and (1.7) need not to be transferred to a two-dimensional system. The major challenges are as follows: (1) in order to apply a generalized Mawhin's continuation theorem, the proof of M-compactness (see Definition 2.2) of the operators is very difficult; (2) equations (1.6) and (1.7) are of much stronger non-linearity, so how to estimate the prior bounds for periodic solutions is crucial; and (3) it is non-trivial to establish a unified framework to handle the prescribed mean curvature term and the influence of variable delays. It is, therefore, the main purpose of this paper to make the first attempt to handle the listed challenges.

On the other hand, the dynamic properties of Duffing and Rayleigh equations have been widely investigated due to the applications in many fields, such as mechanics, physics,

and the engineering technique fields; see *e.g.* [19–22]. In such applications, it is important to know the existence and uniqueness of periodic solution to the Rayleigh and Duffing equations. We note that p-Laplacian equations have been widely applied in the field of non-linear elastic mechanics and fluid mechanics, it is necessary to study the periodic solutions of FDEs involving the scalar p-Laplacian. For the problems of periodic solutions for p-Laplacian equations, see *e.g.* [7, 8, 10–13]. This paper is devoted to an investigation of the existence and uniqueness of periodic solutions for a prescribed mean curvature p-Laplacian Rayleigh equation and a p-Laplacian Duffing equation as follows:

$$\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' + f(x'(t)) + g(x(t-\tau_1(t))) = e_1(t) \tag{1.6}$$

and

$$\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' + g(x(t-\tau_2(t))) = e_2(t), \tag{1.7}$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1, \varphi_q = \varphi_p^{-1}, \frac{1}{p} + \frac{1}{q} = 1; f, g, e_i, \tau_i \in C(\mathbb{R}, \mathbb{R})$ where $e_i(t) = e_i(t+T)$ and $\tau_i(t) = \tau_i(t+T), i = 1, 2, T > 0$, is a given constant.

Remark 1.1 Equations (1.6) and (1.7) are the one-dimensional version of the elliptic Dirichlet problem,

$$-\text{div}\left(\frac{\nabla x}{\sqrt{1+\|\nabla x\|^2}}\right) = F(t,x) \quad \text{in } \Omega, x = 0 \text{ on } \partial\Omega. \tag{1.8}$$

It is well known that a solution x of (1.8) defines a Cartesian in $\mathbb{R}^{N+1}$ whose mean curvature is prescribed by the right-hand side of the equation. Such problems have attracted much attention by many researches; see *e.g.* [23–27] and the references therein.

Remark 1.2 As for equations (1.6) and (1.7), a common method for study is to transfer the forms of equations (1.6) and (1.7) into the forms as follows, respectively:

$$\begin{cases} x_1'(t) = \frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}}, \\ x_2'(t) = -f\left(\frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}}\right) - g(x_1(t-\tau_1(t))) + e_1(t), \end{cases} \tag{1.9}$$

and

$$\begin{cases} x_1'(t) = \varphi_q\left(\frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}}\right), \\ x_2'(t) = -g(x_1(t-\tau_2(t))) + e_2(t). \end{cases} \tag{1.10}$$

Let $L = (x_1', x_2')^T$, then L is a linear operator. Thus Mawhin's continuation theorem can be used to study the existence and uniqueness of periodic solutions to equation (1.9) and equation (1.10).

To the best of our knowledge, there are few results for studying the existence and uniqueness of periodic solutions to p-Laplacian prescribed mean curvature equations by using

the generalized Mawhin continuation theorem, the main purpose of this paper is to introduce a new method for studying the above equations. As to methods for studying the existence and uniqueness of periodic solutions, such as the Lyapunov second method, fixed point theorem, coincidence degree theory, the Yoshizawa type theorem, and the Massera type theorem, see [1, 28–30] and references therein.

The following sections are organized as follows: In Section 2, we give some useful lemmas and definitions. In Section 3, sufficient conditions are established for existence results of periodic solutions to equations (1.6) and (1.7). In Section 4, sufficient conditions are established for uniqueness results of periodic solutions to equations (1.6) and (1.7). In Section 5, some examples are given to show the feasibility of our results. Finally, some conclusions are given for this paper.

2 Preliminary

In this section, we give some definitions and lemmas which will be used in this paper.

Definition 2.1 ([18]) Denote X and Z two Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_Z$, respectively. A continuous operator

$$\mathcal{M} : X \cap \operatorname{dom}\mathcal{M} \to Z$$

is said to be quasi-linear if
  (i) $\operatorname{Im}\mathcal{M} := \mathcal{M}(X \cap \operatorname{dom}M)$ is a closed subset of Z;
  (ii) $\operatorname{Ker}\mathcal{M} := \{x \in X \cap \operatorname{dom}\mathcal{M} : \mathcal{M}x = 0\}$ is linearly homeomorphic to $\mathbb{R}^n, n < \infty$.

Definition 2.2 ([18]) Let $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$. $N_\lambda : \bar{\Omega} \to Z, \lambda \in [0,1]$ is said to be $\mathcal{M}$-compact in $\bar{\Omega}$ if there exists a subset Z_1 of Z satisfying $\dim Z_1 = \dim \operatorname{Ker}\mathcal{M}$ and an operator $R : \bar{\Omega} \times [0,1] \to X_2$ being continuous and compact such that, for $\lambda \in [0,1]$,
  (a) $(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im}\mathcal{M} \subset (I - Q)Z$,
  (b) $QN_\lambda x = 0, \lambda \in (0,1) \Leftrightarrow QNx = 0, \forall x \in \Omega$,
  (c) $R(\cdot, 0) \equiv 0$ and $R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$,
  (d) $\mathcal{M}[P + R(\cdot, \lambda)] = (I - Q)N_\lambda, \lambda \in [0,1]$,
where X_2 is a complement space of $\operatorname{Ker}\mathcal{M}$ in X, *i.e.*, $X = \operatorname{Ker}\mathcal{M} \oplus X_2$; P, Q are two projectors satisfying $\operatorname{Im}P = \operatorname{Ker}\mathcal{M}, \operatorname{Im}Q = Z_1, N = N_1, \Sigma_\lambda = \{x \in \bar{\Omega} : \mathcal{M}x = N_\lambda x\}$.

Lemma 2.1 ([18]) *Let X and Z be two Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_Z$, respectively, and $\Omega \subset X$ be an open and bounded nonempty set. Suppose*

$$M : X \cap \operatorname{dom}\mathcal{M} \to Z$$

is quasi-linear and $N_\lambda : \bar{\Omega} \to Z, \lambda \in [0,1]$ is M-compact in $\bar{\Omega}$. In addition, if the following conditions hold:

(H_1) $\mathcal{M}x \neq N_\lambda x, \forall(x, \lambda) \in \partial\Omega \times (0,1)$;
(H_2) $QNx \neq 0, \forall x \in \operatorname{Ker}\mathcal{M} \cap \partial\Omega$;
(H_3) $\deg\{JQN, \Omega \cap \operatorname{Ker}\mathcal{M}, 0\} \neq 0, J : \operatorname{Im}Q \to \operatorname{Ker}\mathcal{M}$ *is a homeomorphism*,

then the abstract equation $\mathcal{M}x = Nx$ has at least one solution in $\operatorname{dom}\mathcal{M} \cap \bar{\Omega}$.

Lemma 2.2 ([12]) *Let* $p \in (1, +\infty)$ *be a constant,* $s \in C(\mathbb{R}, \mathbb{R})$ *such that* $s(t + T) \equiv s(t)$, $u \in C^1(R, R)$ *with* $u(t + T) \equiv u(t)$. *Then*

$$\int_0^T \left| u(t) - u\bigl(t - s(t)\bigr) \right|^p \, dt \leq 2 \left(\max_{t \in [0,T]} |s(t)| \right)^p \int_0^T |u'(t)|^p \, dt.$$

Throughout this paper, assume that $e_i(t)(i = 1, 2)$ is not a constant function on $\mathbb{R}$. Furthermore, we suppose that $\tau_i \in C^1(\mathbb{R}, \mathbb{R})$ with $\tau_i'(t) < 1, \forall t \in \mathbb{R}, i = 1, 2$. It is clear that the function $t - \tau_i(t)$ has a unique inverse, denoted by $\gamma_i(t)$.

Now, we list the following assumptions:

[A1] there is a constant $\sigma \geq 0$ such that

$$f(u)u \geq \sigma |u|^p, \quad \forall u \in \mathbb{R}$$

or

$$f(u)u \leq -\sigma |u|^p, \quad \forall u \in \mathbb{R};$$

[A2] there is a constant $\sigma \geq 0$ such that

$$f(u) \geq \sigma |u|^p, \quad \forall u \in \mathbb{R}$$

or

$$f(u) \leq -\sigma |u|^p, \quad \forall u \in \mathbb{R};$$

[A3] there is a constant $l \geq 0$ such that

$$\left| g(v_1) - g(v_2) \right| \leq l|v_1 - v_2|, \quad \forall v_1, v_2 \in \mathbb{R};$$

[A4] there exists a constant $r \geq 0$ such that

$$\lim_{|u| \to +\infty} \frac{|g(u)|}{|u|^m} \leq r, \quad m \in \mathbb{N};$$

[A5] there is a constant $D > 0$ such that

$$g(x) < -|e|_0 - \left| f(0) \right| \quad \text{for } x > D$$

and

$$g(x) > |e|_0 + \left| f(0) \right| \quad \text{for } x < -D;$$

[A6] there is a constant $D_1 > 0$ such that

$$g(x) < -|e|_0 \quad \text{for } x > D_1$$

and

$$g(x) > |e|_0 \quad \text{for } x < -D_1;$$

[A7] there is a constant $\sigma \geq 0$ such that

$$g(x) \geq \sigma |x|^m, \quad \forall x \in \mathbb{R}, m \in \mathbb{N}$$

or

$$g(x) \leq -\sigma |x|^m, \quad \forall x \in \mathbb{R}, m \in \mathbb{N}.$$

3 Existence of periodic solutions

Denote $C_T = \{u | u \in C(\mathbb{R}, \mathbb{R}), u(t + T) = u(t)\}$, $C_T^1 = \{u | u \in C^1(\mathbb{R}, \mathbb{R}), u(t + T) = u(t)\}$, $X = C_T^1$ with the norm $\|u\| = \max\{|u|_0, |u'|_0\}$, where $|u|_0 = \max_{0 \leq t \leq T} |u(t)|$, $Z = C_T$, then the operators $\mathcal{M}, N_\lambda$ are defined by

$$\mathcal{M} : \operatorname{dom} \mathcal{M} \cap X \to Z, \quad (\mathcal{M}x)(t) = \left(\varphi_p\left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}}\right)\right)', \quad t \in \mathbb{R}, \tag{3.1}$$

$$N_\lambda : X \to Z, \quad (N_\lambda x)(t) = -\lambda f(x'(t)) - \lambda g(x(t - \tau_1(t))) + \lambda e_1(t),$$
$$t \in \mathbb{R}, \lambda \in [0, 1], \tag{3.2}$$

where $\operatorname{dom} \mathcal{M} = \{x \in X : \varphi_p(x'(t)/\sqrt{1 + x'^2(t)}) \in C_T^1\}$. Denote

$$\mathcal{F}(t, x(t), x'(t), x(t - \tau_1(t))) = -f(x'(t)) - g(x(t - \tau_1(t))) + e_1(t), \tag{3.3}$$

then $(N_\lambda x)(t) = \lambda F$. By (3.1) and (3.2), equation (1.6) is equivalent to the operator equation $Nx = \mathcal{M}x$, where $N_1 = N$. Then we have

$$\operatorname{Ker} \mathcal{M} = \{x \in \operatorname{dom} \mathcal{M} \cap X : x(t) = a, a \in \mathbb{R}, t \in \mathbb{R}\},$$

$$\operatorname{Im} \mathcal{M} = \left\{z \in Z : \int_0^T z(s)\,ds = 0\right\}.$$

Clearly, $\operatorname{Ker} M \cong \mathbb{R}$, $\operatorname{Im} M$ is a closed set in Z, we have

Lemma 3.1 $\mathcal{M}$ *is defined by* (3.1), *then* $\mathcal{M}$ *is a quasi-linear operator.*

Let

$$P : X \to \operatorname{Ker} \mathcal{M}, \quad (Px)(t) = x(0), \quad t \in \mathbb{R},$$

$$Q : Z \to \mathbb{R}, \quad (Qz)(t) = \frac{1}{T} \int_0^T z(s)\,ds.$$

Lemma 3.2 *If* $f, g, e_1, \tau_1 \in C(\mathbb{R}, \mathbb{R})$ *with* $e_1(t) = e_1(t + T)$ *and* $\tau_1(t) = \tau_1(t + T)$, *then* N_λ *is* $\mathcal{M}$-*compact.*

Proof Denote $Z_1 = \operatorname{Im} Q$. For each bounded set $\bar{\Omega} \subset X \neq \emptyset, \forall x \in \bar{\Omega}$, since $x(0) = x(T)$, there is a point $\eta \in [0, T]$ such that

$$x'(\eta) = 0. \tag{3.4}$$

Define the operator: $R : \bar{\Omega} \times [0, 1] \to \operatorname{Ker} P$,

$$
R(x, \lambda)(t) = \int_0^t \sqrt{1 + x'^2(s)} \varphi_q
$$
$$
\times \left[\int_\eta^s \lambda \big(\mathcal{F}\big(r, x(r), x'(r), x\big(r - \tau(r)\big)\big) - (Q\mathcal{F})(r) \big) dr \right] ds, \quad t \in [0, T],
$$

where $\mathcal{F}$ is defined by (3.3), η is defined by (3.4). Based on the properties of f, g, τ_1, e_1, we see that $R(\cdot, \lambda)$ is continuous and relatively compact. Then we show that N_λ is $\mathcal{M}$-compact in four steps.

Step 1. Using $Q^2 = Q$, we get $Q(I - Q)N_\lambda(\bar{\Omega}) = 0$, so $(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Ker} Q = \operatorname{Im} \mathcal{M}$. On the other hand, $\forall z \in \operatorname{Im} \mathcal{M}$, obviously $Qz = 0$, so $z = z - Qz = (I - Q)z$, then $z \in (I - Q)Z$. Thus,

$$(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} \mathcal{M} \subset (I - Q)Z.$$

Step 2. We verify that $QN_\lambda x = 0, \lambda \in (0, 1) \Leftrightarrow QNx = 0, \forall x \in \Omega$. Since $QN_\lambda x = \frac{1}{T} \int \lambda \mathcal{F} \, dr = 0$, then $\frac{1}{T} \int \mathcal{F} \, dr = 0$, *i.e.*, $QNx = 0$. The inverse is true.

Step 3. Obviously $R(\cdot, 0) = 0$. $\forall x \in \Sigma_\lambda = \{x \in \bar{\Omega} : \mathcal{M}x = N_\lambda x\}$, we have $(\varphi_p(x'/\sqrt{1 + x'^2}))' = \lambda \mathcal{F}$ and $Q\mathcal{F} = 0$. Hence

$$
R(x, \lambda)(t) = \int_0^t \sqrt{1 + x'^2(s)} \varphi_q \left[\int_\eta^s \lambda \big(\mathcal{F}\big(r, x(r), x'(r), x\big(r - \tau(r)\big)\big) - (Q\mathcal{F})(r) \big) dr \right] ds
$$
$$
= \int_0^t \sqrt{1 + x'^2(s)} \varphi_q \left[\int_\eta^s \lambda \mathcal{F}\big(r, x(r), x'(r), x\big(r - \tau(r)\big)\big) dr \right] ds
$$
$$
= \int_0^t \sqrt{1 + x'^2(s)} \varphi_q \left[\int_\eta^s \big(\varphi_p\big(x'/\sqrt{1 + x'^2(r)}\big) \big)' dr \right] ds
$$
$$
= \int_0^t x'(s) \, ds
$$
$$
= x(t) - x(0)
$$
$$
= \big[(I - P)x \big](t).
$$

Step 4. $\forall x \in \bar{\Omega}$, we get

$$
\mathcal{M}\big[Px + R(x, \lambda) \big](t)
$$
$$
= \left(\varphi_p \left(\left[x(0) + \int_0^t \sqrt{1 + x'^2(s)} \varphi_q \left[\int_\eta^s \lambda \big(\mathcal{F}\big(r, x(r), x'(r), x\big(r - \tau(r)\big)\big) \right. \right. \right. \right.
$$
$$
\left. \left. \left. \left. - (Q\mathcal{F})(r) \big) dr \right] ds \right]' \right) \right)'
$$
$$
= \big[(I - Q)N_\lambda x \big](t).
$$

Hence, N_λ is $\mathcal{M}$-compact in $\bar{\Omega}$.   □

Theorem 3.1 *Assume that assumptions* [A1], [A3], *and* [A5] *hold, then equation* (1.6) *has at least one T-periodic solution.*

Proof We will complete the proof in three steps.

  Step 1. Let $\Omega_1 = \{x \in \operatorname{dom} M : Mx = N_\lambda x, \lambda \in (0,1)\}$. We verify that Ω_1 is a bounded set. If $x \in \Omega_1$, then $Mx = N_\lambda x$, *i.e.*,

$$\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' = -\lambda f(x'(t)) - \lambda g(x(t - \tau_1(t))) + \lambda e_1(t). \tag{3.5}$$

Let t_0 be the maximum point of $x(t)$ on $[0, T]$, *i.e.*, $x(t_0) = \max_{t \in [0,T]} x(t)$. Then $x'(t_0) = 0$, by (3.5), we get

$$g(x(t_0 - \tau_1(t_0))) = -f(0) + e_1(t_0) \geq -|f(0)| - |e_1|_0$$

and

$$g(x(t_0 - \tau_1(t_0))) \leq |f(0)| + |e_1|_0,$$

which together with the condition [A5] shows that we have $-D \leq x(t_0 - \tau_1(t_0)) \leq D$. Since $x(t)$ is a T-periodic function, there is a constant $\xi^* \in [0, T]$ satisfying $t_0 - \tau_1(t_0) = \xi^* + kT, k \in \mathbb{Z}$, and we get

$$|x(\xi^*)| \leq D. \tag{3.6}$$

By (3.6) we have

$$|x|_0 = \max_{t \in [0,T]} \left| x(\xi^*) + \int_{\xi^*}^t x'(s)\, ds \right| \leq |x(\xi^*)| + \int_0^T |x'(s)|\, ds \leq D + \int_0^T |x'(s)|\, ds. \tag{3.7}$$

Multiplying two sides of equation (3.5) by $x'(t)$ and integrating over $[0, T]$, we get

$$\int_0^T \left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' x'(t)\, dt + \lambda \int_0^T f(x'(t))x'(t)\, dt + \lambda \int_0^T g(x(t - \tau_1(t)))x'(t)\, dt$$

$$= \lambda \int_0^T e_1(t)x'(t)\, dt. \tag{3.8}$$

Denote $\omega(t) = \varphi_p(\frac{x'(t)}{\sqrt{1+x'^2(t)}})$, then $x'(t) = \pm\sqrt{\frac{\varphi_q^2(\omega(t))}{1-\varphi_q^2(\omega(t))}}$ and

$$\int_0^T \left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' x'(t)\, dt = \pm \int_0^T \sqrt{\frac{\varphi_q^2(\omega(t))}{1-\varphi_q^2(\omega(t))}}\, d\omega(t) = 0.$$

Hence equation (3.8) reduces to

$$\int_0^T f(x'(t))x'(t)\, dt = -\int_0^T g(x(t - \tau_1(t)))x'(t)\, dt + \int_0^T e_1(t)x'(t)\, dt. \tag{3.9}$$

From condition [A1], we have

$$\left| \int_0^T f(x'(t))x'(t)\,dt \right| = \int_0^T \left| f(x'(t))x'(t) \right| dt \geq \sigma \int_0^T \left| x'(t) \right|^p dt,$$

which together with [A3], Lemma 2.2, (3.9), and the Hölder inequality implies

$$\sigma \int_0^T \left| x'(t) \right|^p dt \leq \int_0^T \left| g(x(t - \tau_1(t)))x'(t) \right| dt + \int_0^T \left| e_1(t)x'(t) \right| dt$$

$$\leq \int_0^T \left| g(x(t - \tau_1(t)))x'(t) \right| dt + \|e_1\|_q \left(\int_0^T \left| x'(t) \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq \int_0^T \left| \left[g(x(t - \tau_1(t))) - g(x(t)) \right] x'(t) \right| dt$$

$$+ \int_0^T \left| g(x(t))x'(t) \right| dt + \|e_1\|_q \left(\int_0^T \left| x'(t) \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq \int_0^T l \left| \left[x(t - \tau_1(t)) - x(t) \right] x'(t) \right| dt + \|e_1\|_q \left(\int_0^T \left| x'(t) \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq l \left(\int_0^T \left| x(t - \tau_1(t)) - x(t) \right|^q \right)^{1/q} \left(\int_0^T \left| x'(t) \right|^p dt \right)^{1/p}$$

$$+ \|e_1\|_q \left(\int_0^T \left| x'(t) \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq l 2^{1/q} |\tau_1|_0 \left(\int_0^T \left| x'(t) \right|^q \right)^{1/q} \left(\int_0^T \left| x'(t) \right|^p dt \right)^{1/p}$$

$$+ \|e_1\|_q \left(\int_0^T \left| x'(t) \right|^p dt \right)^{\frac{1}{p}}. \tag{3.10}$$

By $p, q > 1$, and (3.10), there must be a constant $M_1 > 0$ such that

$$\int_0^T \left| x'(t) \right|^p dt \leq M_1. \tag{3.11}$$

Thus by the Hölder inequality and (3.7), we have

$$|x|_0 \leq D + T^{1/q} \left(\int_0^T \left| x'(t) \right|^p dt \right)^{1/p} \leq D + T^{1/q} M_1^{1/p} := M_2.$$

By (3.11), we have $|x'|_0 \leq T^{-\frac{1}{p}} M_1^{\frac{1}{p}} := M_3$. Thus,

$$\|x\| < \max\{M_2, M_3\} + 1 := \widetilde{M}.$$

Step 2. Let $\Omega_2 = \{x \in \text{Ker}\,\mathcal{M} : QNx = 0\}$. We prove that Ω_2 is a bounded set. $\forall x \in \Omega_2$, then $x = a_0, a_0 \in \mathbb{R}$, and we have

$$\int_0^T \left(g(a_0) + f(0) - e_1(t) \right) dt = 0.$$

Based on the integral mean value theorem, there exists a constant $t_1 \in [0, T]$ such that

$$g(a_0) + f(0) - e_1(t_1) = 0,$$

by [A5], we have $|a_0| \le D$. So Ω_2 is a bounded set.

Step 3. Let $\Omega = \{x \in X : \|x\| < \widetilde{M}\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. $\forall (x, \lambda) \in \partial\Omega \times (0, 1)$, from the above proof, $\mathcal{M}x \ne N_\lambda x$ is satisfied. Clearly, condition (H_2) of Lemma 2.1 is satisfied. Now, we show that condition (H_3) of Lemma 2.1 holds. Take the homotopy

$$H(x, \mu) = -\mu x + (1 - \mu)JQNx, \quad x \in \bar{\Omega} \cap \operatorname{Ker} \mathcal{M}, \mu \in [0, 1],$$

where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $Ja = a, a \in \mathbb{R}$. $\forall x \in \partial\Omega \cap \operatorname{Ker} M$, we have $x = a_1, |a_1| = \widetilde{M} > D$, then

$$H(x, \mu) = -a_1\mu + (1 - \mu)\frac{1}{T} \int_0^T \left(-g(a_1) - f(0) + e_1(t)\right) dt,$$

$$a_1 H(x, \mu) = -a_1^2\mu + (1 - \mu)\frac{1}{T} \int_0^T a_1\left(-g(a_1) - f(0) + e_1(t)\right) dt.$$

By assumption [A5], we have $H(x, \mu) < 0$. Using degree theory, we have

$$
\begin{aligned}
\deg\{JQN, \Omega \cap \operatorname{Ker} \mathcal{M}, 0\} &= \deg\{H(\cdot, 0), \Omega \cap \operatorname{Ker} \mathcal{M}, 0\} \\
&= \deg\{H(\cdot, 1), \Omega \cap \operatorname{Ker} \mathcal{M}, 0\} \\
&= \deg\{-I, \Omega \cap \operatorname{Ker} \mathcal{M}, 0\} \ne 0.
\end{aligned}
$$

Using Lemma 2.1, we reach the conclusion. □

Remark 3.1 Assumptions [A1] and [A2] ensure that integrals $\int_0^T f(u(t))u(t)\,dt$ and $\int_0^T f(u(t))\,dt$ have no sign change, respectively.

Remark 3.2 Assumption [A5] is crucial for obtaining prior bounds of periodic solutions and verifying the second condition of Lemma 2.1.

Theorem 3.2 *Suppose that assumptions* [A2], [A4], *and* [A5] *hold, then equation* (1.6) *has at least one T-periodic solution if*
(1) $m = p, \sigma > \max_{t \in [0, T]} |\frac{1}{1 - \tau_1'(\gamma_1(t))}| r$, *or*
(2) $m < p$.

Proof Similar to the proof of Theorem 3.1, integrating the two sides of (3.5) over $[0, T]$, we have

$$\int_0^T f(x'(t))\,dt + \int_0^T g(x(t - \tau_1(t)))\,dt = \int_0^T e_1(t)\,dt.$$

In view of [A2], we have

$$\left| \int_0^T f(x'(t))\,dt \right| = \int_0^T |f(x'(t))| \ge \sigma \int_0^T |x'(t)|^p\,dt$$

and so

$$\sigma \int_0^T \left|x'(t)\right|^p dt \le \int_0^T \left|g\bigl(x\bigl(t - \tau_1(t)\bigr)\bigr)\right| dt + \int_0^T \left|e_1(t)\right| dt. \tag{3.12}$$

From assumption [A4], $\forall \varepsilon > 0$, we see that there is a constant $\rho > D > 0$ such that

$$\left|g(x)\right| \le (r + \varepsilon)|x|^m, \quad \text{whenever } |x| > \rho. \tag{3.13}$$

Let

$$E_1 = \bigl\{ t \in [0, T] : \left|x\bigl(t - \tau_1(t)\bigr)\right| > \rho \bigr\}, \qquad E_2 = \bigl\{ t \in [0, T] : \left|x\bigl(t - \tau_1(t)\bigr)\right| \le \rho \bigr\}.$$

By (3.12) and (3.13), we have

$$\begin{aligned}
\sigma \int_0^T \left|x'(t)\right|^p dt &\le \int_{E_1} \left|g\bigl(x\bigl(t - \tau_1(t)\bigr)\bigr)\right| dt + \int_{E_2} \left|g\bigl(x\bigl(t - \tau_1(t)\bigr)\bigr)\right| dt + T|e_1|_0 \\
&\le \int_0^T (r + \varepsilon)\left|x\bigl(t - \tau_1(t)\bigr)\right|^m dt + Tg_\rho + T|e_1|_0 \\
&\le \max_{t \in [0,T]} \left|\frac{1}{1 - \tau_1'(\gamma_1(t))}\right| (r + \varepsilon) \int_0^T |x(t)|^m dt + Tg_\rho + T|e_1|_0,
\end{aligned} \tag{3.14}$$

where $g_\rho = \max_{|v| \le \rho} |g(v)|$.

Case 1. If $p = m$ and $\sigma > \max_{t \in [0,T]} |\frac{1}{1 - \tau_1'(\gamma_1(t))}| r$. Choose $\varepsilon < \sigma - \max_{t \in [0,T]} |\frac{1}{1 - \tau_1'(\gamma_1(t))}| r$. So by (3.14), $\int_0^T |x'(t)|^p dt$ is bounded.

Case 2. If $m < p$, by the Hölder inequality and (3.14), we get

$$\begin{aligned}
\sigma \int_0^T \left|x'(t)\right|^p dt &\le \max_{t \in [0,T]} \left|\frac{1}{1 - \tau_1'(\gamma_1(t))}\right| (r + \varepsilon) T^{p - m/p} \left(\int_0^T |x(t)|^p dt \right)^{m/p} \\
&\quad + Tg_\rho + T|e_1|_0.
\end{aligned} \tag{3.15}$$

From $m/p < 1$ and (3.15), we see that $\int_0^T |x'(t)|^p dt$ is bounded. Hence we know that in two cases there exists a constant $M > 0$ such that

$$\int_0^T \left|x'(t)\right|^p dt < M,$$

which together with (3.7) yields

$$|x|_0 \le D + T^{1/q} \left(\int_0^T \left|x'(t)\right|^p dt \right)^{1/p} \le D + T^{1/q} M^{1/p}.$$

The rest of the proof of the theorem is identical to that of Theorem 3.1. □

Remark 3.3 The results of Theorem 3.2 show that the existence of a periodic solution is related to the delays.

Now, we give an existence theorem for equation (1.7).

Theorem 3.3 *Suppose that assumptions* [A6] *and* [A7] *hold, then equation* (1.7) *has at least one T-periodic solution x.*

Proof Similar to the proof of Theorem 3.1, let

$$\mathcal{M} : \operatorname{dom} \mathcal{M} \cap X \to Z, \quad (\mathcal{M}x)(t) = \left(\varphi_p \left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}} \right) \right)', \quad t \in \mathbb{R}, \tag{3.16}$$

$$N_\lambda : X \to Z, \quad (N_\lambda x)(t) = -\lambda g\big(x\big(t - \tau_2(t)\big)\big) + \lambda e_2(t), \quad t \in \mathbb{R}, \lambda \in [0,1]. \tag{3.17}$$

Let $\Omega_1 = \{x \in \operatorname{dom} \mathcal{M} : \mathcal{M}x = N_\lambda x, \lambda \in (0,1)\}$, where $\mathcal{M}$ and N_λ are defined by (3.16) and (3.17), respectively. We show that Ω_1 is a bounded set. If $x \in \Omega_1$, then $\mathcal{M}x = N_\lambda x$, i.e.,

$$\left(\varphi_p \left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}} \right) \right)' = -\lambda g\big(x\big(t - \tau_2(t)\big)\big) + \lambda e_2(t). \tag{3.18}$$

Similar to (3.6) and (3.7), by assumption [A6] there exists a constant $\eta^* \in [0, T]$ such that

$$\big|x\big(\eta^*\big)\big| \le D_1$$

and

$$|x|_0 = \max_{t \in [0,T]} \left| x\big(\eta^*\big) + \int_{\eta^*}^t x'(s)\, ds \right| \le \big|x\big(\eta^*\big)\big| + \int_0^T \big|x'(s)\big|\, ds \le D_1 + \int_0^T \big|x'(s)\big|\, ds. \tag{3.19}$$

On the other hand, multiplying the two sides of (3.18) $[0, T]$, we get

$$\int_0^T g\big(x\big(t - \tau_2(t)\big)\big)\, dt = \int_0^T e_2(t)\, dt$$

and

$$\int_0^T \frac{g(x(t))}{1 - \tau_2'(\gamma_2(t))}\, dt = \int_0^T e_2(t)\, dt.$$

It follows from [A7] that

$$\left| \int_0^T g\big(x(t)\big)\, dt \right| = \int_0^T \big|g\big(x(t)\big)\big| \ge \sigma \int_0^T \big|x(t)\big|^m\, dt$$

and so

$$\min_{s \in [0,T]} \left| \frac{1}{1 - \tau_2'(\gamma_2(s))} \right| \sigma \int_0^T \big|x'(t)\big|^m\, dt \le \int_0^T \big|e_2(t)\big|\, dt. \tag{3.20}$$

It follows from (3.19), (3.20), and the Hölder inequality that there exists a positive constant C_1 such that

$$\int_0^T \big|x'(t)\big|^m\, dt \le C_1$$

and

$$|x|_0 \leq D_1 + T^{1/n} C_1^{1/m} := C_2,$$

where $\frac{1}{m} + \frac{1}{n} = 1$. The rest of the proof of the theorem is identical to that of Theorem 3.1.

□

Remark 3.4 From the proof of the above theorems, we find that the prior bounds estimation to the periodic solutions is key for obtaining existence results. Here, we use a new method for obtaining the prior bounds, which is different from the methods of [26] and [16].

4 Uniqueness results

In this section, we will study the uniqueness of periodic solutions to equations (1.6) and (1.7).

Theorem 4.1 *Suppose that $\tau_1(t) = \varepsilon_1$ (ε_1 is sufficiently small constant) and the following condition holds*:

[B1] $(u_1 - u_2)[g(u_1) - g(u_2)] < 0$ *for all $u_1, u_2 \in \mathbb{R}, u_1 \neq u_2$.*
Then equation (1.6) has at most one T-periodic solution.

Proof Denote

$$y(t) = \varphi_p\left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}}\right) + F(x(t)),$$

where $F(x(t))$ satisfies $\frac{dF(x(t))}{dt} = f(x'(t))$. Then (1.6) can be changed into the following form:

$$\begin{cases} x'(t) = \frac{\varphi_q(y(t) - F(x(t)))}{\sqrt{1 - \varphi_q^2(y(t) - F(x(t)))}}, \\ y'(t) = g(x(t - \varepsilon_1)) + e_1(t). \end{cases}$$

The rest of the proof of the theorem is identical to that of Theorem 4.1 in [17]. □

Theorem 4.2 *Suppose that $\tau_1(t) = \varepsilon_1$ (ε_1 is sufficiently small constant), assumptions [A1], [A3], [A5], and [B1] hold, then equation (1.6) has a unique T-periodic solution x.*

Theorem 4.3 *Suppose that $\tau_1(t) = \varepsilon_1$ (ε_1 is a sufficiently small constant), and assumptions [A2], [A4], [A5], and [B1] hold, then equation (1.6) has a unique T-periodic solution x, if*

(1) $m = p, \sigma > \max_{t \in [0,T]} |\frac{1}{1 - \tau_1'(\gamma_1(t))}| r$, *or*
(2) $m < p$.

We also have the following uniqueness results to (1.7).

Theorem 4.4 *Suppose that $\tau_2(t) = \varepsilon_2$ (ε_2 is sufficiently small constant) and assumption [B1] hold. Then equation (1.7) has at most one T-periodic solution x.*

Theorem 4.5 *Suppose that $\tau_2(t) = \varepsilon_2$ (ε_2 is sufficiently small constant), assumptions [A6], [A7], and [B1] hold, then equation (1.7) has a unique T-periodic solution x.*

Remark 4.1 We give an example for verifying the existence of $F(x(t))$. Let $F(x(t)) = 5x(t), f(x'(t)) = 5x'(t)$. Then we have $\frac{dF(x(t))}{dt} = 5x'(t) = f(x'(t))$.

Remark 4.2 In [16], based on the delay $\tau(t) = 0$, Wang gave uniqueness results to equation (1.6), our results allow $\tau(t)$ to be a small positive constant, and this is more significant than the work of Wang.

5 Examples

As applications, we provide some examples to illustrate the main results.

Example 5.1

$$\left(\varphi_4\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' + (x')^3 e^{\sin x'} + g\left(x\left(t - \frac{1}{2}\sin t\right)\right) = \cos t, \tag{5.1}$$

where $p = 4, f(x') = (x')^3 e^{\sin x'}, \tau(t) = \frac{1}{2}\sin t, e(t) = \cos t, T = 2\pi,$

$$g(u) = \begin{cases} -e^{(2+\cos u)} & \text{for } u \geq 0, \\ -\frac{1}{2}u & \text{for } u < 0. \end{cases}$$

Conclusion Equation (5.1) has at least one 2π-periodic solution.

Proof From $f(x')x' = (x')^4 e^{\sin x'}$ and $\sigma|x'|^4$, choose $\sigma < e^{-1}$, then assumption [A1] holds. From $f(0) + |e|_0 = 1$, choose $D = 2$, then assumption [A3] holds. From $g'(u)$ being bounded, then assumption [A5] holds. Applying Theorem 3.1, equation (5.1) has at least one 2π-periodic solution. $\square$

Example 5.2

$$\left(\varphi_4\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' + (x')^4 + g\left(x\left(t - \frac{1}{2}\sin t\right)\right) = \cos t, \tag{5.2}$$

where $p = 4, f(x') = (x')^4, \tau(t) = \frac{1}{2}\sin t, e(t) = \cos t, T = 2\pi,$

$$g(u) = \begin{cases} -ue^{\sin u} & \text{for } u \geq 0, \\ -\frac{1}{2}u & \text{for } u < 0. \end{cases}$$

Conclusion Equation (5.2) has at least one 2π-periodic solution.

Proof Let $m = p = 4, 0 < \sigma \leq 1$. We have

$$f(x') = (x')^4 \geq \sigma|x'|^4, \qquad \lim_{|x|\to+\infty} \frac{|g(x)|}{|x|^{p-1}} = 0 := r,$$

and

$$\sigma > \max_{t\in[0,T]}\left|\frac{1}{1 - \tau_1'(\gamma_1(t))}\right| r = 0.$$

Thus, all conditions of Theorem 3.2 hold. Applying Theorem 3.2, equation (5.2) has at least one 2π-periodic solution. □

Example 5.3

$$\left(\varphi_4\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)\right)' + g\left(x\left(t - \frac{1}{2}\sin t\right)\right) = \cos t, \tag{5.3}$$

where $p = 4, \tau(t) = \frac{1}{2}\sin t, e(t) = \cos t, T = 2\pi,$

$$g(u) = \begin{cases} -e^{(2+\cos u)} & \text{for } u \geq 0, \\ -\frac{1}{2}u & \text{for } u < 0. \end{cases}$$

Conclusion Equation (5.3) has at least one 2π-periodic solution.

Proof From $|e|_0 = 1$, choose $D_1 = 2$, then assumption [A6] holds. If $u < 0$, choose $0 < \sigma \leq \frac{1}{2}$ and $m = 1$, then assumption [A7] holds. If $u \geq 0$, choose σ sufficiently small and $m = 1$, then assumption [A7] also holds. Applying Theorem 3.3, equation (5.3) has at least one 2π-periodic solution. □

6 Conclusions

In this paper, we have investigated a periodic solution problem for two types of p-Laplacian prescribed mean curvature equations. Some sufficient conditions are obtained for the existence and uniqueness of periodic solutions to the above equations. Three examples are given to show the effectiveness of the results in this paper.

We mention here that the technique used here is different from the usual methods employed to solve periodic solutions cases such as the Mawhin continuation theorem and some fixed point theorems. We develop a new method (*i.e.*, a generalized Mawhin continuation theorem) for p-Laplacian equations. This method is more extensively applicable than classical methods.

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