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Uniqueness results for inverse Sturm-Liouville problems with partial information given on the potential and spectral data

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Abstract
We consider the inverse spectral problem for a Sturm-Liouville problem on the unit interval [0, 1]. We obtain some uniqueness results, which imply that the potential q can be completely determined even if only partial information is given on q together with partial information on the spectral data, consisting of the spectrum and normalizing constants. Moveover, we also investigate the problem of missing both eigenvalues and normalizing constants in the situation where the potential q is C^{2k-1} near a suitable point.

1 Introduction

In this paper, we consider the Sturm-Liouville operator $L := L(q, h_0, h_1)$ in the Liouville form

$$Lu = -u'' + qu \tag{1.1}$$

on the unit interval $[0, 1]$ associated with the boundary conditions

$$u'(0) - h_0 u(0) = 0, \tag{1.2}$$

$$u'(1) - h_1 u(1) = 0. \tag{1.3}$$

We assume that the potential $q \in L^1[0, 1]$ is real-valued and $h_0 \in \mathbb{R} \cup \{\infty\}$, $h_1 \in \mathbb{R}$, where $h_0 = \infty$ singles out the Dirichlet boundary condition $u(0) = 0$. Denote the spectrum of L by $\sigma(L)$, which consists of simple real eigenvalues

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

Given a complex value z, define $v(x, z)$ as a solution of equation $Lv = zv$. Let $z = \lambda_n$, and $v_n(x) := v(x, \lambda_n)$ be the eigenfunction of the operator L associated with the eigenvalue λ_n.

Then there are two sorts of norming constants κ_n and α_n corresponding to λ_n:

$$\kappa_n = \frac{v_n(0)}{v_n(1)} \quad \text{and} \quad \alpha_n = \frac{\int_0^1 v_n^2(x)\,dx}{|v_n(1)|^2}. \tag{1.4}$$

In order to distinguish κ_n and α_n, in general, κ_n is called the *ratio*, and α_n is called the *normalizing constant*. Indeed, based on the relation (see [1], p.18)

$$\kappa_n \alpha_n = \dot{\omega}(\lambda_n) \tag{1.5}$$

between κ_n and α_n, where $\omega(\cdot)$ is the characteristic function of L defined further by (2.5), and $\dot{\omega}(\lambda) = d\omega(\lambda)/d\lambda$, the pair of sequences $\Gamma_1 := \{\lambda_n, \alpha_n; n \in \mathbb{N}_0\}$ is equivalent to the pair of sequences $\Gamma_2 := \{\lambda_n, \kappa_n; n \in \mathbb{N}_0\}$. It is also known [1–3] that knowing the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}_0}$ and the normalizing constants $\{\alpha_n\}_{n\in\mathbb{N}_0}$ is equivalent to knowing the singular measure defined by the spectral function for problem (1.1)-(1.3); moreover, the normalizing constants $\{\alpha_n\}_{n\in\mathbb{N}_0}$ can be constructed from the two sequences of eigenvalues, $\Gamma_3 := \{\lambda_n, \tilde{\lambda}_n\}_{n\in\mathbb{N}_0}$, where $\{\tilde{\lambda}_n\}_{n\in\mathbb{N}_0}$ is the spectrum of another operator $L(q, \tilde{h}_0, h_1)$ with $h_0 \neq \tilde{h}_0$.

The uniqueness problem of determining the potential q in terms of one of the above-mentioned three sets of spectral data Γ_j ($j = 1, 2, 3$) is well known (see, e.g., [2, 3]). A comprehensive review for the inverse problem in these cases is presented by McLaughlin [4].

This paper is related immediately to a earlier paper [5] by the second author and Xu in that it provides some uniqueness results, which imply that the potential q and h_1 can completely be determined even if only partial information is given on q together with partial information on the spectral data, consisting of either one full spectrum and a subset of ratios κ_n or a subset of pairs of eigenvalues and the corresponding ratios κ_n. In the present paper, we consider the same uniqueness problem under the same circumstances but with the ratios $\{\kappa_n\}$ replaced by normalizing constants $\{\alpha_n\}$. In other words, we mainly investigate the uniqueness problem when only partial information on q, on the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, and on the normalizing constants $\{\alpha_n\}_{n\in\mathbb{N}}$ is available.

Our original motivation for the above works is theorems of Hochstadt-Lieberman [6] and Gesztesy-Simon [7]. Specifically, in 1978, Hochstadt and Lieberman [6] proved that the whole spectrum uniquely determines q when it is already known on $[0, 1/2]$. In 2000, Gesztesy and Simon [7] gave several important generalizations of the Hochstadt-Lieberman theorem to the case where the $L^1[0, 1]$ potential q is known on a larger interval $[0, a]$ with $a \in [1/2, 1)$ and the set of common eigenvalues is sufficiently large. Another result in [7] is obtained under the assumption that the potential q belongs to C^{2k} for some $k \in \mathbb{N}_0$ near 1/2 so that C^{2k}-smoothness can replace the knowledge of some $k + 1$ eigenvalues, that is, $k + 1$ eigenvalues may be missing. These results have been generalized and improved in a variety of ways; see [8–14]. Our aim here is to realize that, for the question of uniqueness for the Sturm-Liouville problem, normalizing constants play an equal role with eigenvalues. In other words, the number of normalizing constants is, in a sense, equivalent to the number of eigenvalues.

Here is one of the main results of this paper.

Theorem 1.1 *Let L be defined by* (1.1) *with boundary conditions* (1.2)-(1.3), *and $h_0 \in \mathbb{R} \cup \{\infty\}$ and $h_1 \in \mathbb{R}$. Suppose that, for some $k \in \mathbb{N}_0$ and $\varepsilon > 0$, q is $C^{2k-1}[0, \varepsilon)$ when $h_0 \in \mathbb{R}$ or q*

is $C^{2k}[0,\varepsilon)$ *when* $h_0 = \infty$. *Let* $\Lambda_e = \{i_1, i_2, \ldots, i_l\} \subset \mathbb{N}_0$ *and* $\Lambda_n = \{j_1, j_2, \ldots, j_m\} \subset \mathbb{N}_0$ *satisfy*

$$\Lambda_e \subseteq \Lambda_n, \quad l + m = k + 1. \tag{1.6}$$

Then h_0, $q^{(n)}(0)$ *for* $n = 0, 1, \ldots, (2k-1)$ *when* $h_0 \in \mathbb{R}$ *or for* $n = 0, 1, \ldots, 2k$ *when* $h_0 = \infty$, *all the eigenvalues* $\{\lambda_n\}_{n\in\mathbb{N}}$ *except for* $n \in \Lambda_e$, *and all the normalizing constants* $\{\alpha_n\}_{n\in\mathbb{N}}$ *except for* $n \in \Lambda_n$ *uniquely determine* h_1 *and* q *on* $[0,1]$.

Remark 1.2 Under the assumption that $\sigma(L)$ is known, from (1.6) it follows that the result of Theorem 1.1 remains valid if the condition of unknown normalizing constants $\{\alpha_{j_i}\}_{i=1}^{k+1}$ is replaced with the condition of unknown ratios $\{\kappa_{j_i}\}_{i=1}^{k+1}$. This is the same as [5], Thm. 1.1. However, Theorem 1.1 here shows that both some eigenvalues and normalizing constants may be missing.

For any $\alpha = \{\alpha_j\}_{j\in\mathbb{N}_0} \subset \mathbb{C}$ with $|\alpha_0| \le |\alpha_1| \le |\alpha_2| \le \cdots$, set

$$n_\alpha(t) = \#\bigl\{j \in \mathbb{N}_0 : |\alpha_j| \le t\bigr\} \quad \text{for } t \ge 0. \tag{1.7}$$

The following theorem treats the case where partial information is given on the set of the spectral data $\Gamma_1 = \{\lambda_m, \alpha_n : m, n \in \mathbb{N}_0\}$ when q is known a priori on $[0,a]$ with $a \in [0,1)$.

Theorem 1.3 *Let L be defined by* (1.1) *with boundary conditions* (1.2)-(1.3), *and* $h_0 \in \mathbb{R} \cup \{\infty\}$ *and* $h_1 \in \mathbb{R}$. *Let*

$$S_n \subseteq S_e \subseteq \sigma(L) \quad \textit{and} \quad \Pi_n = \{\alpha_j : \lambda_j \in S_n\}.$$

Then q *on* $[0,a]$ *for some* $a \in [0,1)$, h_0, *and two subsets* S_e *and* Π_n *satisfying*

$$n_{S_n}(t) + n_{S_e}(t) \ge 2(1-a)n_{\sigma(L)}(t) + (a-1) \quad \textit{if } h_0 \in \mathbb{R} \tag{1.8}$$

or

$$n_{S_n}(t) + n_{S_e}(t) \ge 2(1-a)n_{\sigma(L)}(t) \quad \textit{if } h_0 = \infty \tag{1.9}$$

for all sufficiently large $t \in \mathbb{R}$ *uniquely determine* h_1 *and* q *on* $[0,1]$.

Remark 1.4 As in the case $a = 0$, we have an extension of the same type as in Theorem 1.1. Explicitly, if $h_0 \in \mathbb{R}$ and q is assumed to be C^{2k-1} near $x = a$, then, instead of condition (1.8), we only need the condition

$$n_{S_n}(t) + n_{S_e}(t) \ge 2(1-a)n_{\sigma(L)}(t) + a - (k+2). \tag{1.10}$$

Remark 1.5 Comparing the result of Theorem 1.3 with that of [7], Thm. 1.4, we can see that the lack of a certain number of normalizing constants can be reduced to the situation of lack the same number of eigenvalues.

This result is related to another paper by the authors [15], where we consider an analog of Theorem 1.3 for finite tridiagonal (Jacobi) matrices. Moreover, as a particular case of Theorem 1.3, we have the following corollary, which is parallel with [5], Thm. 4.2, where the problem of the partial information on the subset of the pair of sequences $\Gamma_2 := \{\lambda_n, \kappa_n; n \in \mathbb{N}_0\}$ is concerned.

Corollary 1.6 *Under the assumptions of Theorem* 1.3, *let* $S_n = S_e$. *Then* q *on* $[0,a]$ *for some* $a \in [0,1)$, h_0, *and two subsets* Π_n *and* S_e *satisfying the condition*

$$n_{S_e}(t) \geq (1-a)n_{\sigma(L)}(t) + (a-1)/2 \quad \text{if } h_0 \in \mathbb{R} \tag{1.11}$$

or

$$n_{S_e}(t) \geq (1-a)n_{\sigma(L)}(t) \quad \text{if } h_0 = \infty \tag{1.12}$$

for all sufficiently large $t \in \mathbb{R}$ *uniquely determine* h_1 *and* q *on* $[0,1]$.

All the results obtained concern mainly with a spectrum with $h_0 \in \mathbb{R} \cup \{\infty\}$ being fixed. Furthermore, in Section 4, we generalize these results to more general circumstances associated with the spectral data of different operators $L(q, h_{0,n}, h_1)$, where $h_{0,n}$ are allowed to belong to different values.

Moreover, note that the case of Dirichlet boundary condition at $x = 1$ demands a separate treatment. Nevertheless, we expect that the method of the paper can be applied in this case.

The results presented in this paper are based on the uniqueness theorem of the Weyl m-function developed by Marchenko [16] and introduced to deal with inverse problems with partial information by Gesztesy, Simon, and del Rio [7, 17–19]. Our proof in the paper is based on two multiple zeros of the Wronskian of two Sturm-Liouville problems. At this point, we note that our proof here is different from that of the results in [5] for dealing with the unique determination problem of q and h_1 in terms of eigenvalues and ratios, where the known ratios are transformed to known eigenvalues by a particular solution of the equation $Lu = \lambda u$ such that the Weyl m-function technique can be used.

The paper is organized as follows. In the next section, we recall the uniqueness theorem of Marchenko [16] and give a proof of Theorem 1.1. The proof of Theorem 1.3 is presented in Section 3. In Section 4, we extend Theorem 1.3 to a more general case, associated with different boundary conditions at the endpoint $x = 0$, and further establish some new uniqueness results.

2 Preliminaries and proof of Theorem 1.1

In this section, we first recall the uniqueness theorem of Marchenko and formulate some asymptotic expansions of m-functions and solutions of Eq. (2.1), which will be used later to prove our principal results.

Throughout this paper, by the statement "q on $[0,a]$, eigenvalues λ_n, and normalizing constants α_n determine uniquely q and h_1" we mean that there are no two distinct potentials q_1 and q_2 on $[0,1]$ with the two properties: (i) $q_1 = q_2$ a.e. on $[0,a]$, and (ii) λ_n and α_n are common eigenvalues and normalizing constants for q_1 and q_2.

Unless explicitly stated otherwise, h_0 will be known, and all potentials q, q_1, and q_2 will be real valued and in $L^1[0,1]$ for the rest of this paper.

For a real-valued potential $q \in L^1[0,1]$, consider the initial-value problem

$$-u'' + qu = zu \tag{2.1}$$

on $[0,1]$ with initial conditions

$$u_-(0) = 1, \qquad u_-'(0) = h_0, \tag{2.2}$$

$$u_+(1) = 1, \qquad u_+'(1) = h_1. \tag{2.3}$$

Let $u_- := u_-(x,z)$ and $u_+ := u_+(x,z)$ be the solutions of problem (2.1)-(2.2) and problem (2.1) and (2.3), respectively. If $z = \lambda_n \in \sigma(L)$, where the operator L is defined by problem (1.1)-(1.3), then both $u_-(x,\lambda_n) =: u_{-,n}$ and $u_+(x,\lambda_n) =: u_{+,n}$ are eigenfunctions of the operator L corresponding to the eigenvalue λ_n, and

$$u_{+,n} = \kappa_n u_{-,n}, \tag{2.4}$$

where $\kappa_n = u_{+,n}(0) = u_{-,n}^{-1}(1)$ is the ratio corresponding to the eigenvalue λ_n; hence, $\kappa_n \neq 0, \infty$. Denote

$$\omega(z) = \bigl[u_+(x,z), u_-(x,z)\bigr] := \begin{vmatrix} u_+(x,z) & u_-(x,z) \\ u_+'(x,z) & u_-'(x,z) \end{vmatrix}, \tag{2.5}$$

where $[u_+(x,z), u_-(x,z)]$ is the Wronskian of $u_+(x,z)$ and $u_-(x,z)$. By Green's formula for the Wronskian, $[u_+, u_-]$ does not depend on x. The function $\omega(z)$ is called the characteristic function of the operator L. It is easy to see from (2.4) that

$$\alpha_n := \int_0^1 \bigl|u_-(x,\lambda_n)\bigr|^2\,dx = \frac{1}{\kappa_n^2}\int_0^1 \bigl|u_+(x,\lambda_n)\bigr|^2\,dx, \tag{2.6}$$

where α_n is the normalizing constant corresponding to λ_n. The following lemma, which is proved in [19], p.8, presents the relation among λ_n, α_n, and κ_n.

Lemma 2.1 *We have the relation*

$$\kappa_n \alpha_n = -\dot{\omega}(\lambda_n) \tag{2.7}$$

for all $n \in \mathbb{N}_0$, *where* $\dot{\omega}(z) = d\omega/dz$, *and* κ_n *are defined by* (1.4).

We next formulate the main uniqueness theorem in the literature, proved by Marchenko [16]. For the solution $u_+(x,z)$ of Eq. (2.1), the Weyl m_+-function is defined by

$$m_+(a,z) = \frac{u_+'(a,z)}{u_+(a,z)} \tag{2.8}$$

for $a \in [0,1)$. Marchenko's [16] fundamental uniqueness theorem of inverse spectral theory then reads as follows.

Theorem 2.2 *$m_+(a,z)$ uniquely determines h_1 and q (a.e.) on $[a,1]$.*

Consider a problem with boundary condition (1.3) at $x = 1$. We need to know the high-energy asymptotic behavior of the m_+-function with $x \in [0,1)$. It is known [20] that, under the general hypothesis $q \in L^1[0,1]$,

$$m_+(a,z) = i\sqrt{z} + o(1) \tag{2.9}$$

uniformly in $a \in [0,1-\delta]$ for $\delta > 0$ as $|z| \to \infty$ in any sector $\varepsilon < \operatorname{Arg}(z) < \pi - \varepsilon$ for $\varepsilon > 0$, where $\sqrt{z}$ is the square root branch with $\operatorname{Im}(\sqrt{z}) \geq 0$. It is also known [20] that if q is C^n near $a \in [0,1)$ for some $n \in \mathbb{N}_0$, then $m_+(a,z)$ have asymptotic expansions of the form

$$m_+(a,z) = i\sqrt{z} + \sum_{l=1}^{n+1} c_l(a)\frac{1}{z^{(l+1)/2}} + o\left(\frac{1}{z^{(n+1)/2}}\right) \tag{2.10}$$

as $|z| \to \infty$ in any sector $\varepsilon < \operatorname{Arg}(z) < \pi - \varepsilon$ for $\varepsilon > 0$. Here $c_l(a)$ are the universal functions of $q(a), q'(a), \ldots, q^{(l-2)}(a)$ and can be computed recursively as follows:

$$c_0(a) = 1, \qquad c_1(a) = 0, \qquad c_2(a) = -\frac{1}{2}q(a),$$
$$c_j(a) = \frac{i}{2}c'_{j-1}(a) - \frac{1}{2}\sum_{l=1}^{j-1} c_l(a)c_{j-l}(a), \quad j \geq 3.$$

Let q be given on $[0,a]$ with some $a \in [0,1)$. Let q_1 and q_2 be two candidates for q extended to $[0,1]$. Let $u_{1,+}(x,z)$ and $u_{2,+}(x,z)$ be solutions of Eq. (2.1) corresponding to q_1 and q_2, respectively, where $u_{j,+}(x,z)$ satisfies the initial conditions

$$u_{j,+}(1,z) = 1, \qquad u'_{j,+}(1,z) = h_j, \quad j = 1,2, \tag{2.11}$$

with $h_1, h_2 \in \mathbb{R}$. It is well known [1, 3] that for each $x \in [0,1]$, $u_{j,+}(x,z)$ and $u'_{j,+}(x,z)$ are entire functions of z and satisfy the asymptotic expansions

$$u_{j,+}(x,z) = \cos\bigl(\sqrt{z}(1-x)\bigr) + O\bigl(e^{\operatorname{Im}(\sqrt{z})(1-x)}\sqrt{z}\bigr), \tag{2.12}$$
$$u'_{j,+}(x,z) = \sqrt{z}\sin\bigl(\sqrt{z}(1-x)\bigr) + O\bigl(e^{\operatorname{Im}(\sqrt{z})(1-x)}\bigr) \tag{2.13}$$

as $|z| \to \infty$ for all $x \in [0,1]$. For $j = 1,2$, let

$$\omega_j(z) = \begin{cases} u'_{j,+}(0,z) - h_0 u_{j,+}(0,z) & \text{if } h_0 \in \mathbb{R}, \\ u_{j,+}(0,z) & \text{if } h_0 = \infty, \end{cases} \tag{2.14}$$

which are the *characteristic* functions of the operators $L(q_j, h_0, h_j) =: L_j$. Then $\sigma(L_j) = \{\lambda_{j,n}\}_{n=0}^{\infty}$ are precisely the zeros of $\omega_j(z)$.

Since the zeros of $u_{j,+}(a,\cdot)$ and $u'_{j,+}(a,\cdot)$ are all real and uniformly bounded below, by adding (if necessary) a sufficiently large constant to q_1 and q_2, we may assume that all zeros of $u_{j,+}(a,\cdot)$, $u'_{j,+}(a,\cdot)$, and $\omega_j(\cdot)$ are in $[1,\infty)$. In this case, all these six functions are m-type (see [7], p.2781). Therefore, $u_{j,+}(a,\cdot)$, $u'_{j,+}(a,\cdot)$, and $\omega_j(\cdot)$ are bounded by $C_1\exp(C_2|z|^{1/2})$ for some constants $C_1, C_2 > 0$ and are of the form (see [7])

$$c\prod_{n=0}^{\infty}\left(1-\frac{z}{x_n}\right)$$

for suitable $\{x_n\}_{n=0}^{\infty} \subset [1,\infty)$.

Let

$$U_+(a,z) = \left[u_{1,+}(a,z), u_{2,+}(a,z)\right] := \begin{vmatrix} u_{1,+}(a,z) & u_{2,+}(a,z) \\ u'_{1,+}(a,z) & u'_{2,+}(a,z) \end{vmatrix} \tag{2.15}$$

for $a \in [0,1]$. Then we have the following lemma, which plays a key role in this paper.

Lemma 2.3 *Assume that $q_1 = q_2$ a.e. on $[0,a]$ for some $a \in [0,1)$. If $\lambda_{1,n} = \lambda_{2,n}$ for some $n \in \mathbb{N}_0$, then $U_+(a,\lambda_{1,n}) = 0$; if, in addition, $\alpha_{1,n} = \alpha_{2,n}$, then*

$$\dot{U}_+(a,\lambda_{1,n}) = 0, \tag{2.16}$$

that is, in this case, $\lambda_{1,n}$ is a two-multiple root of the equation $U_+(a,z) = 0$.

Proof By the assumption of $q_1 = q_2$ a.e. on $[0,a]$, if $h_0 \in \mathbb{R}$, then it is easy to see that

$$\begin{aligned} U_+(a,z) &= U_+(0,z) + \int_0^a \frac{\partial}{\partial t}\left[u_{1,+}(t,z), u_{2,+}(t,z)\right] dt \\ &= U_+(0,z) - \int_0^a (q_1 - q_2)(t)(u_{1,+}u_{2,+})(t,z)\, dt \\ &= U_+(0,z) \\ &= \begin{vmatrix} u_{1,+}(0,z) & u_{2,+}(0,z) \\ \omega_1(z) & \omega_2(z) \end{vmatrix}. \end{aligned} \tag{2.17}$$

It follows from (2.4) and the last identity that if $\lambda_{1,n} = \lambda_{2,n}$, then $\omega_j(\lambda_{1,n}) = 0$ for $j = 1, 2$, and therefore $U_+(a,\lambda_{1,n}) = 0$. Furthermore, since

$$\dot{U}_+(0,z) = \begin{vmatrix} \dot{u}_{1,+}(0,z) & \dot{u}_{2,+}(0,z) \\ \omega_1(z) & \omega_2(z) \end{vmatrix} + \begin{vmatrix} u_{1,+}(0,z) & u_{2,+}(0,z) \\ \dot{\omega}_1(z) & \dot{\omega}_2(z) \end{vmatrix},$$

substituting $z = \lambda_{1,n}$ into this formula, we have

$$\dot{U}_+(0,\lambda_{1,n}) = u_{1,+}(0,\lambda_{1,n})\dot{\omega}_2(\lambda_{1,n}) - u_{2,+}(0,\lambda_{1,n})\dot{\omega}_1(\lambda_{1,n}).$$

Note that $\kappa_{j,n}\alpha_{j,n} = -\dot{\omega}_j(\lambda_{j,n})$ and $\kappa_{j,n} = u_{j,+}(0,\lambda_{j,n})$ for $j = 1, 2$ by Lemma 2.1 and (2.4). It is easy to check that $\dot{U}_+(0,\lambda_{1,n}) = (\alpha_{2,n} - \alpha_{1,n})\kappa_{1,n}\kappa_{2,n} = 0$ when, in addition, $\alpha_{1,n} = \alpha_{2,n}$. Thus, from (2.17) we have that $\dot{U}_+(a,\lambda_{1,n}) = 0$. Moreover, the same approach can be used to deal with the case $h_0 = \infty$. This completes the proof. □

According to the preliminaries, we now prove Theorem 1.1.

Proof of Theorem 1.1 We prove the theorem when $h_0 \in \mathbb{R}$. The case $h_0 = \infty$ is similar. Let $\{\lambda_{j,n}, \alpha_{j,n}\}_{n\in\mathbb{N}_0}$ be the spectral data corresponding to the operators $L(q_j, h_0, h_j)$ for $j = 1, 2$. Without loss of generality, we assume that

$$\lambda_{1,n} = \lambda_{2,n} \quad \text{for all } n \geq l \quad \text{and} \quad \alpha_{1,n} = \alpha_{2,n} \quad \text{for all } n \geq m.$$

Let us consider the function $H(z)$ defined by

$$H(z) = \frac{U_+(0,z)}{\omega_1(z)^2} \prod_{t=0}^{m-1} (z - \lambda_{1,t})^2 \prod_{s=m}^{l-1} (z - \lambda_{1,s}). \tag{2.18}$$

By Lemma 2.3 the cross ratio $U_+(0,z) \prod_{t=1}^{m-1} (z - \lambda_{1,t})^2 \prod_{s=1}^{l-1} (z - \lambda_{1,s})$ with two-multiple zeros vanishes at each point where $\omega_1(z)^2$ vanishes, and $\omega_1(z)^2$ necessarily has two-multiple zeros since $L(q_1, h_0, h_1)$ has a simple spectrum. Thus, H is an entire function. In addition, from $\inf_{\theta\in[0,2\pi]} |\omega_j((\pi(n+\frac{1}{2}))^2 e^{i\theta})| \geq \pi n + O(1)$, for sufficiently large n (see [7], p.2771) and the fact that the functions $u_{j,+}(0,z)$ are m-type we conclude that $H(z)$ satisfies

$$\left|H(z)\right| \leq C_1 e^{C_2|z|^{1/2}}.$$

As a matter of fact, it follows from (2.19) that the last inequality holds whenever $|z| = (\pi(n + 1/2))^2$ for n sufficiently large; it then extends to all z by the maximum modulus principle. Furthermore, since $q_1^{(j)}(0) = q_2^{(j)}(0)$ for $j = 0, 1, \ldots, 2k - 1$, by (2.10) and (2.12) we infer that

$$\begin{aligned}
&\left|m_{1,+}(0, iy) - m_{2,+}(0, iy)\right| = o\left(|y|^{-k}\right), \\
&\left|u_{j,+}(0, iy)\right| = \frac{1}{2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} \left(1 + o(1)\right), \\
&\left|\omega_j(iy)\right| = \frac{1}{2} |y|^{1/2} e^{\exp \operatorname{Im}(\sqrt{i})|y|^{1/2}} \left(1 + o(1)\right)
\end{aligned}$$

as y (real) $\to \infty$ for $j = 1, 2$. This, together with (2.18), shows that

$$\begin{aligned}
\left|H(iy)\right| &\leq \left|\frac{(u_{1,+}u_{2,+})(0, iy)(m_{1,+}(0, iy) - m_{2,+}(0, iy))}{\omega_1(iy)^2}\right| \\
&\quad \times \left|\prod_{t=0}^{m-1} (iy - \lambda_{1,t})^2 \prod_{s=m}^{l-1} (iy - \lambda_{1,s})\right| \\
&= \frac{e^{\operatorname{Im}(\sqrt{i})2|y|^{1/2}} (1 + o(1)) o(|y|^{-k})}{|y| e^{\operatorname{Im}(\sqrt{i})2|y|^{1/2}} (1 + o(1))} O\left(|y|^{(k+1)}\right) \\
&= o(1) \quad \left(y \text{ (real)} \to \infty\right).
\end{aligned} \tag{2.19}$$

It turns out that $|H(iy)| \to 0$ as y (real) $\to \infty$. By [7], Prop. B.6 we obtain $H \equiv 0$. We can multiply H by

$$\frac{\omega_1(z)^2}{\prod_{t=0}^{m-1} (z - \lambda_{1,t}) \prod_{s=0}^{l-1} (z - \lambda_{1,s})},$$

which has two-multiple zeros and poles, to conclude that $U_+(z) = 0$ for all $z \in \mathbb{C}$. This, together with (2.8) and (2.15), yields $m_{1,+}(0,z) = m_{2,+}(0,z)$. By Theorem 2.2, $q_1 = q_2$ a.e. on $[0,1]$ and $h_1 = h_2$. The proof is therefore complete. □

3 Proof of Theorem 1.3

Our goal in this section is to prove Theorem 1.3. We first establish a lemma, which will be used later to prove the theorem and its generalizations (see Section 4 for details).

Given a sequence $S := \{x_n\}_{n=0}^{\infty}$ of positive reals such that $1 \le x_0 \le x_1 \le \cdots$ and

$$\sum_{n=0}^{\infty} \frac{1}{x_n^{\rho}} < \infty \quad \text{for all } \rho > \rho_0, \tag{3.1}$$

where $\rho_0 \in (0,1)$ is fixed, define the function G_S by

$$G_S(z) = \prod_{x_n \in S} \left(1 - \frac{z}{x_n}\right). \tag{3.2}$$

It is known [21], Sects. II.48 and II.49 that $G_S(z)$ is an entire function with

$$\left|G_S(z)\right| \le C_1 e^{C_2 |z|^{\rho}} \quad \text{for all } \rho > \rho_0, \tag{3.3}$$

where both C_1 and C_2 are positive constants. It should be noted that (3.1) holds if and only if $n_S(t) \le C_0 |t|^{\rho}$ for all $\rho > \rho_0$, where C_0 is a positive constant. Conversely, if G_S is an entire function satisfying (3.3) with all its zeros $\{x_n\}_{n=0}^{\infty}$ in $[1,\infty)$, then its zeros satisfy (3.1), and G_S has the canonical product expansion (3.2) (see [21] for details). In contrast to [7], Def. B.3, the function $G_S(z)$ is also said to be m-type in a generalized sense. It should be emphasized here that the above argument may involve the case that $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$.

Lemma 3.1 *Let $\sigma(L) =: \{\lambda_j\}_{j=0}^{\infty}$ be the spectrum of problem* (1.1)-(1.3), *let $S := \{x_n\}_{n=0}^{\infty}$ with $1 \le x_0 \le x_1 \le \cdots$ satisfy* (3.1), *and G_S be defined by* (3.2). *Assume that*

$$n_S(t) \ge A n_{\sigma(L)}(t) + B \tag{3.4}$$

for all sufficiently large $t \in \mathbb{R}$, where both A and B are real constants with $A > 0$. Then

$$\left|G_S(iy)\right| \ge C_1 |y|^{(B+A/2)} e^{\operatorname{Im}(\sqrt{i}) A |y|^{1/2}} \quad \text{if } h_0 \in \mathbb{R} \tag{3.5}$$

or

$$\left|G_S(iy)\right| \ge C_1 |y|^{B} e^{\operatorname{Im}(\sqrt{i}) A |y|^{1/2}} \quad \text{if } h_0 = \infty. \tag{3.6}$$

Proof Define

$$G_{\sigma(L)}(z) = \prod_{\lambda_j \in \sigma(L)} \left(1 - \frac{z}{\lambda_j}\right).$$

By the definition (3.2) of G_S, we have, by integration by parts,

$$\begin{aligned}
\ln\bigl|G_S(iy)\bigr| &= \sum_{x_j\in S}\frac{1}{2}\ln\biggl(1+\frac{y^2}{x_j^2}\biggr)\\
&= \frac{1}{2}\int_0^\infty \ln\biggl(1+\frac{y^2}{t^2}\biggr)\,dn_S(t)\\
&= \int_0^\infty \frac{y^2}{t^3+ty^2}n_S(t)\,dt \quad \bigl(\text{since } n_S(0)=0\bigr)\\
&= \int_1^\infty \frac{y^2}{t^3+ty^2}n_S(t)\,dt \quad \bigl(n_S(t)=0 \text{ if } t\in[0,1)\bigr).
\end{aligned} \tag{3.7}$$

Furthermore, by hypothesis (3.4) on S of the lemma there are constants $t_0 \geq 1$ and $C \geq 0$ such that

$$n_S(t) \geq \begin{cases} An_{\sigma(L)}(t)+B & \text{if } t>t_0,\\ An_{\sigma(L)}(t)-C & \text{if } t\leq t_0.\end{cases}$$

Hence, by (3.7) and (3.4), noting the relation

$$\frac{y^2}{t^3+ty^2} = -\frac{d}{dt}\biggl(\frac{1}{2}\ln\biggl(1+\frac{y^2}{t^2}\biggr)\biggr),$$

we deduce that

$$\begin{aligned}
\ln\bigl|G_S(iy)\bigr| &= \int_1^{t_0}\frac{y^2}{t^3+ty^2}n_S(t)\,dt + \int_{t_0}^\infty \frac{y^2}{t^3+ty^2}n_S(t)\,dt\\
&\geq A\int_1^\infty \frac{y^2}{t^3+ty^2}n_{\sigma(L)}(t)\,dt + B\int_1^\infty \frac{y^2}{t^3+ty^2}\,dt + C_0\\
&= A\ln\bigl|G_{\sigma(L)}(iy)\bigr| + \frac{B}{2}\ln\bigl(1+y^2\bigr) + C_0,
\end{aligned} \tag{3.8}$$

where $C_0 = -|B-C|\ln(t_0)$.

Because $\sigma(L)$ is the full set of the eigenvalues of the self-adjoint operator L on $[0,1]$, we get that, asymptotically,

$$\bigl|G_{\sigma(L)}(iy)\bigr| = \frac{1}{2}|y|^{1/2}e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}}\bigl(1+o(1)\bigr) \quad \text{if } h_0\in\mathbb{R}$$

and

$$\bigl|G_{\sigma(L)}(iy)\bigr| = \frac{1}{2}e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}}\bigl(1+o(1)\bigr) \quad \text{if } h_0=\infty$$

as y (real) $\to\infty$. It thus turns out from (3.8) that there exists a positive constant C_1 such that (3.5)-(3.6) hold. The proof is complete. □

We now are in position to prove Theorem 1.3.

Proof of Theorem 1.3 Let $\{\lambda_{j,n},\alpha_{j,n}\}_{n\in\mathbb{N}_0}$ be the spectral data corresponding to the operators $L(q_j,h_0,h_j)$ for $j=1,2$. We only prove the theorem when $h_0\in\mathbb{R}$. The same approach can be used to deal with the case $h_0 = \infty$. By the hypothesis on S_n and S_e define

$$G_{S_e}(z) = \prod_{\lambda_{1,n}\in S_e}\left(1 - \frac{z}{\lambda_{1,n}}\right), \qquad G_{S_n}(z) = \prod_{\lambda_{1,n}\in S_n}\left(1 - \frac{z}{\lambda_{1,n}}\right).$$

Since $\lambda_{1,n} = n^2\pi^2 + O(1)$ (see [1], p.10), it follows that the functions G_{S_e} and G_{S_n} are m-type, and therefore

$$(G_{S_e}G_{S_n})(z) = \prod_{\lambda_{1,n}\in S_e}\left(1 - \frac{z}{\lambda_{1,n}}\right)^{k(n)} \tag{3.9}$$

is also m-type, where $k(n) = 1$ when $\lambda_{1,n} \in S_e\backslash S_n$ and $k(n) = 2$ when $\lambda_{1,n} \in S_n$. Let us consider the function $H(z)$ defined by

$$H(z) = \frac{U_+(a,z)}{(G_{S_e}G_{S_n})(z)}, \tag{3.10}$$

where $U_+(a,z)$ is defined by (2.15). Then by the hypothesis of $S_n \subseteq S_e$ we have from Lemma 2.3 that

$$U_+(a,\lambda_{1,j}) = 0 \quad \text{if } \lambda_{1,j} \in S_e, \qquad U_+(a,\lambda_{1,j}) = \dot{U}_+(a,\lambda_{1,j}) = 0 \quad \text{if } \lambda_{1,j} \in S_n$$

since $q_1 = q_2$ on $[0,a]$. This implies that $H(z)$ is an entire function. Recall that

$$\left|u_{j,+}(a,iy)\right| = \frac{1}{2}e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}\left(1 + o(1)\right), \quad j = 1,2, \tag{3.11}$$

and $m_+(a,iy) = i\sqrt{iy} + o(1)$ as y (real) $\to \infty$. Thus, by Lemma 3.1, (3.10), and (1.8) we have $A = 2(1-a)$, $B = a-1$, and

$$\begin{aligned}
\left|H(iy)\right| &\leq \left|\frac{u_{1,+}(a,iy)u_{2,+}(a,iy)(m_{1,+}(a,iy) - m_{2,+}(a,iy))}{G_{S_e}(z)G_{S_n}(z)}\right| \\
&\leq \frac{e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}}(1 + o(1))}{e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}}}o(1) \\
&= o(1).
\end{aligned} \tag{3.12}$$

This yields $H(z) = 0$, and therefore $U_+(z) = 0$ for all $z \in \mathbb{C}$ by the argument of the proof of Theorem 1.1. Thus, $m_{1,+}(a,z) = m_{2,+}(a,z)$. By Theorem 2.2, $q_1 = q_2$ a.e. on $[0,1]$ and $h_1 = h_2$. The proof is complete. □

4 Uniqueness results for a more general case

In this section, we extend Theorem 1.3 by that the spectral data $\{\lambda_m, \alpha_m\}_{m\in\mathbb{N}_0}$ can be selected in terms of different Sturm-Liouville operators $L(q,h_0,h_1) =: L(h_0)$ with h_0 being different numbers in the boundary condition (1.2).

It is well known (see [1]) that Borg proved the famous two-spectra theorem that the spectra for two boundary conditions of a regular Sturm-Liouville operator uniquely determine the potential q. Later refinements (see, e.g., [12, 13, 19]) show that the knowing eigenvalues associated with a number of different boundary conditions can also determine the potential uniquely. In particular, McLaughlin and Rundell [22] used fixed jth eigenvalues $\lambda_j(q, h_{0,l}, h_1)$ with $l \in \mathbb{N}_0$ for a countable number of different boundary conditions at $x = 0$ to establish the uniqueness of q. Moreover, Horváth considered the same uniqueness problem when the known eigenvalues are taken from finite different spectra, which are corresponding to a finite number of boundary conditions at $x = 0$.

In our uniqueness results to be given further, the known eigenvalues and normalizing constants are of problem (1.1)-(1.3) where a countable number of different boundary conditions at $x = 0$ may be involved. These results not only generalize the results of [13, 19, 22] but also give some new uniqueness results for the inverse Sturm-Liouville problems through normalizing constants instead of eigenvalues. It is essential that, roughly speaking, for the unique determination problem of the potentials q and h_1, the number of normalizing constants is, in a sense, equivalent to the number of eigenvalues.

Given a sequence $\{h_{0,l}\}_{l=0}^{\infty} \subset \mathbb{R} \cup \{\infty\}$, we consider the operator $L(q, h_{0,l}, h_1) =: L(h_{0,l})$ for each $h_{0,l}$ and denote by $\sigma(L(h_{0,l})) =: \{\lambda_m(h_{0,l})\}_{m=0}^{\infty}$ the spectrum of $L(h_{0,l})$. Throughout this section, we always assume that the eigenvalue sequence

$$\left\{\lambda_{m_l}(h_{0,l})\right\}_{l=0}^{\infty} =: \left\{\lambda(h_{0,l})\right\}_{l=0}^{\infty} \quad \text{(for simplicity)} \tag{4.1}$$

is increasing and satisfies the condition

$$\sum_{l=0}^{\infty} \frac{1}{|\lambda(h_{0,l})|^{\rho}} < \infty \quad \text{for all } \rho > \rho_0, \tag{4.2}$$

where $\rho_0 \in (0,1)$ is fixed. In this case, we denote by $\alpha(h_{0,l})$ the normalizing constant corresponding to the eigenvalue $\lambda(h_{0,l})$ and $\alpha_m(h_{0,l})$ corresponding to the $(m+1)$th eigenvalue $\lambda_m(h_{0,l})$ of the operator $L(h_{0,l})$.

We mention some properties of these eigenvalues, which we need further. For their proofs, we refer to [23, 24].

Lemma 4.1

(i) *If* $h_{0,l_1} \neq h_{0,l_2}$, *then* $\lambda(h_{0,l_1}) \neq \lambda(h_{0,l_2})$, *where* $\lambda(h_{0,l_j})$ *are any eigenvalues of* $L(h_{0,l_j})$ *for* $j = 1, 2$.

(ii) *Let* $m \in \mathbb{N}_0$. *Then* $\lambda_m(h_0)$ *is strictly decreasing in* $h_0 \in \mathbb{R}$ *for any fixed* q *and* h_1. *Furthermore, for* $m \geq 1$, *we have*

$$\lim_{h_0 \to \infty} \lambda_m(h_0) = \lambda_{m-1}(\infty), \qquad \lim_{h_0 \to -\infty} \lambda_m(h_0) = \lambda_m(\infty),$$

where $\{\lambda_m(\infty)\}_{m=0}^{\infty} = \sigma(L(\infty))$ *with* $h_0 = \infty$.

Here is our main result of this section.

Theorem 4.2 *Let* $L(q, h_{0,l}, h_1)$ *be defined by* (1.1) *associated with boundary conditions* (1.2)-(1.3) *with* h_0 *being replaced by* $h_{0,l}$, *where* $h_{0,l} \in \mathbb{R} \cup \{\infty\}$ *for* $l \in \mathbb{N}_0$ *and* $h_1 \in \mathbb{R}$. *Let* $S_n \subseteq S_e \subseteq \bigcup_{l=0}^{\infty} \sigma(L(h_{0,l}))$ *and*

$$S_e = \left\{\lambda(h_{0,l})\right\}_{l=0}^{\infty}, \qquad \Pi_n = \left\{\alpha_l(h_{0,l}) : \lambda(h_{0,l}) \in S_n\right\},$$

where $\{\lambda(h_{0,l})\}_{l=0}^{\infty}$ *are increasing and satisfy* (4.2).

Then q on $[0,a]$ for some $a \in [0,1)$, $\{h_{0,l}\}_{l=0}^{\infty}$, and two sets Π_n and S_e such that, for some fixed $h_0 \in \mathbb{R}$,

$$n_{S_n \cup S_e}(t) \geq 2(1-a) n_{\sigma(L(h_0))}(t) - (1-a) \tag{4.3}$$

for all sufficiently large $t \in \mathbb{R}$ uniquely determine h_1 and q on $[0,1]$.

Proof Let $\{\lambda_j(h_{0,l}), \alpha_j(h_{0,l})\}$ for all $l \in \mathbb{N}_0$ be the pairs of the eigenvalues and its corresponding normalizing constants of the operators $L(q_j, h_{0,l}, h_j)$ for $j = 1,2$. Since $S_{e,j} = \{\lambda_j(h_{0,l})\}_{l=0}^{\infty}$ is an increasing sequence for each $j = 1,2$, it follows that it is bounded below. By adding (if necessary) a sufficiently large constant to q_1 and q_2, we assume that all $\lambda_j(h_{0,l})$ are in $[1,\infty)$. In this case, let us define

$$G_{S_e}(z) = \prod_{\lambda_1(h_{0,l}) \in S_e} \left(1 - \frac{z}{\lambda_1(h_{0,l})}\right), \qquad G_{S_n}(z) = \prod_{\lambda_1(h_{0,l}) \in S_n} \left(1 - \frac{z}{\lambda_1(h_{0,l})}\right).$$

Consider the function H defined by

$$H(z) = \frac{U_+(a,z)}{(G_{S_e} G_{S_n})(z)}, \tag{4.4}$$

where $U_+(a,z)$ is defined by (2.15). Then by the hypothesis on S_n and S_e, we have from Lemma 2.3 that

$$\begin{aligned} &2U_+\big(a, \lambda(h_n)\big) = 0 \quad \text{if } \lambda(h_n) \in S_e, \\ &U_+\big(a, \lambda(h_n)\big) = \dot{U}_+\big(a, \lambda(h_n)\big) = 0 \quad \text{if } \lambda(h_n) \in S_n \end{aligned}$$

since $q_1 = q_2$ on $[0,a]$. This implies that $H(z)$ is an entire function. Recall that

$$\big|u_{j,+}(a,iy)\big| = \frac{1}{2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} \big(1 + o(1)\big), \quad j = 1,2.$$

Thus, by Lemma 3.1, (2.9), and (4.3) we have that $A = 2(1-a)$, $B = a - 1$, and

$$\begin{aligned} \big|H(iy)\big| &\leq \left| \frac{u_{1,+}(a,iy) u_{2,+}(a,iy)(m_{1,+}(a,iy) - m_{2,+}(a,iy))}{G_{S_e}(iy) G_{S_n}(iy)} \right| \\ &\leq \frac{e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}}(1 + o(1))}{e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}}} o(1) = o(1). \end{aligned} \tag{4.5}$$

This yields $H(z) = 0$, and therefore $m_{1,+}(a,z) = m_{2,+}(a,z)$ for all $z \in \mathbb{C}$ by the argument of the proof of Theorem 1.1. By Theorem 2.2, $q_1 = q_2$ a.e. on $[0,1]$ and $h_1 = h_2$. The proof is complete. □

Remark 4.3 By the previous argument, if $\sigma(L(h_0))$ is replaced by $\sigma_0 := \{m^2\}_{m=0}^{\infty}$ in (4.3), then the result of Theorem 4.2 also holds. In fact, note that

$$\left| z \prod_{m=1}^{\infty} \left(1 - \frac{z}{m^2}\right) \right|_{z=iy} = \big|\sqrt{iy} \sin(\sqrt{iy})\big| = \frac{1}{2} |y|^{1/2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}}.$$

By Lemma 3.1 we infer $|G_{S_e}(iy) G_{S_n}(iy)| \geq e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}}$. This implies that (4.5) holds.

As a particular case of Theorem 4.2, we have the following corollary, which concerns the uniqueness problem of q and h_1 in terms of eigenvalues and normalizing constants associated with a (countable) number of different boundary conditions.

Corollary 4.4 *Under the assumptions of Theorem* 4.2, *if* $a = 0$,

$$\lambda(h_{0,l}) = \lambda_l(h_{0,l}), \quad \textit{and} \quad \inf\{h_{0,l}\}_{l=0}^{\infty} > -\infty, \tag{4.6}$$

then $\{h_{0,l}\}_{l=0}^{\infty}$, $\{\lambda_l(h_{0,l})\}_{l=0}^{\infty}$ *and* $\{\alpha_l(h_{0,l})\}_{l=0}^{\infty}$ *except for one uniquely determine* h_1 *and* q *on* $[0,1]$.

Proof In Theorem 4.2, taking $a = 0$ and $h_0 := \inf\{h_{0,l}\}_{l=0}^{\infty}$, by Lemma 4.1 we have

$$\lambda_l(h_{0,l}) \le \lambda_l(h_0) \tag{4.7}$$

for all $l \in \mathbb{N}_0$. In this case, letting $S_e = \{\lambda_l(h_{0,l})\}_{l=0}^{\infty}$ and $S_n = \{\lambda_l(h_{0,l})\}_{l=1}^{\infty}$ (without loss of generality), it is easy to verify from Lemma 4.1 that $\{\lambda_l(h_{0,l})\}$ are distinct and

$$\sum_{l=0}^{\infty} \frac{1}{\lambda_l(h_{0,l})^{\rho}} \le \frac{1}{\lambda_0(h_{0,0})^{\rho}} + \sum_{l=0}^{\infty} \frac{1}{\lambda_l(\infty)^{\rho}} < \infty \quad \text{for all } \rho > \frac{1}{2}$$

and from (4.2) that $n_{S_n \cup S_e}(t) \ge 2n_{\sigma(L(h_0))}(t) - 1$ for all $t > \lambda_0(h_{0,0})$. Thus, by Theorem 4.2 we easily obtain the result of Corollary 4.4. □

As another particular case of Theorem 4.2, we also have the following corollary, which concerns our uniqueness problem in terms of eigenvalues only, associated with a countable number of different boundary conditions.

Corollary 4.5 *Under the assumptions of Theorem* 4.2, *given* $\{h_{0,l}, h'_{0,l}\}_{l=0}^{\infty}$ *satisfying*

$$h_{0,l} \ne h'_{0,l} \quad \textit{for all } l \in \mathbb{N}_0, \qquad \inf\{h_{0,l}, h'_{0,l}\}_{n=0}^{\infty} > -\infty, \tag{4.8}$$

if $a = 0$, *then* $\{h_{0,l}, h'_{0,l}\}_{l=0}^{\infty}$ *and the eigenvalues* $\{\lambda_l(h_{0,l}), \lambda_l(h'_{0,l})\}_{l=0}^{\infty}$ *except for one, uniquely determine* h_1 *and* q *on* $[0,1]$.

Proof Without loss of generality, we assume $h_{0,l} < h'_{0,l}$ for all $l \in \mathbb{N}_0$. In Theorem 4.2, taking $a = 0$ and $h_0 := \inf\{h_{0,l}, h'_{0,l}\}_{l=0}^{\infty}$, letting $S_n = \varnothing$ and

$$S_e = \{\lambda_0(h_{0,0}), \lambda_1(h'_{0,1}), \lambda_1(h_{0,1}), \lambda_2(h'_{0,2}), \dots\}$$

with $\lambda_0(h'_{0,0})$ being missing, by Lemma 4.1 we see that the sequence S_e is strictly increasing and satisfies

$$\sum_{l=0}^{\infty} \left(\frac{1}{\lambda_l(h_{0,l})^{\rho}} + \frac{1}{\lambda_l(h'_{0,l})^{\rho}} \right) \le \frac{1}{\lambda_0(h_{0,0})^{\rho}} + 2\sum_{l=0}^{\infty} \frac{1}{\lambda_l(\infty)^{\rho}} < \infty$$

for all $\rho > 1/2$. Moreover, it easy to check that $n_{S_e}(t) \ge 2n_{\sigma(L(h_0))}(t) - 1$ for all $t > \lambda_0(h'_{0,0})$. By Theorem 4.2 we complete the proof. □

The corollary is a generalization of the Borg's two-spectra theorem. In fact, if $h_{0,l} = h_0$ and $h'_{0,l} = h'_0$ for all $l \in \mathbb{N}_0$, and $\{\lambda_l(h_0)\}_{l=0}^{\infty}$ and $\lambda_l(h'_0)$, except any one of them, are known, then q on $[0,1]$ and h_1 are determined uniquely. Furthermore, this can be also viewed as a generalization of two-thirds spectra theorem by del Rio, Gesztesy, and Simon [19], Cor. 3.3. However, Corollary 4.5 here shows that the known eigenvalues are allowed to belong to a countable number of different spectra.

Finally, we give a generalization of half-inverse theorem of Hochstadt and Lieberman, which is involved in a countable number of different boundary conditions.

Corollary 4.6 *Under the assumptions of Theorem* 4.2, *if* $a = 1/2$, *then* $\{h_{0,l}\}_{l=0}^{\infty}$ *and the eigenvalues* $\{\lambda_l(h_{0,l})\}_{l=0}^{\infty}$ *uniquely determine* h_1 *and* q *on* $[0,1]$.

Proof By Lemma 4.1 the sequence $S_e = \{\lambda_l(h_{0,l})\}_{l=0}^{\infty}$ is strictly increasing and satisfies

$$\lambda_{l-1}(\infty) < \lambda_l(h_{0,l}) < \lambda_l(\infty)$$

for all $l \in \mathbb{N}$. In this case, it is easy to ensure that (4.1) holds for $\lambda(h_{0,l}) = \lambda_l(h_{0,l})$ and $n_{S_e}(t) \geq n_{\sigma(L(\infty))}(t)$ for all $t > 0$; hence, by Theorem 4.2, h_1 and q on $[0,1]$ are uniquely determined. □

It should be noted that if all the $h_{0,l} = h_0$ where $h_0 \in \mathbb{R}$ or $h_0 = \infty$, then h_1 and q on $[0,1]$ are uniquely determined. This is the half-inverse theorem of Hochstadt and Lieberman.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this article. All the authors read and approved the final manuscript.

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