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Cauchy problem for the Laplace equation in a radially symmetric hollow cylinder

Yun-Jie Ma[1*] and Chu-Li Fu[2]

*Correspondence:
mayunjie001@sina.com
[1]School of Mathematics and Informational Science, Yantai University, Yantai, 264005, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, an axisymmetric Cauchy problem for the Laplace equation in an unbounded hollow cylinder is considered. The Cauchy data are given on the inside surface of the cylinder, and the solution on the whole domain is sought. We propose a Fourier method with *a priori* and *a posteriori* parameter choice rules to solve this ill-posed problem. It is shown that the approximate solutions are stably convergent to the exact ones with explicit error estimates. A further comparison in the numerical aspects demonstrates the effectiveness and accuracy of the presented methods.

MSC: 35R25; 35R30

Keywords: Cauchy problem for the Laplace equation; hollow cylinder; ill-posed problem; regularization; error estimates

1 Introduction

The Cauchy problem for the Laplace equation is an old yet persistent problem arising in many practical applications, and the general form is

$$\begin{cases} \triangle u = 0, \quad x \in \Omega, \\ u|_{\Gamma} = g, \\ \frac{\partial u}{\partial n}|_{\Gamma} = h, \end{cases} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$, $\Gamma \subset \partial\Omega$ is part of the boundary, g and h are given functions in $L^2(\Gamma)$, and the solution u is sought in the whole domain Ω. This problem arises in many practical contexts, for example, in the problem of electrical prospecting, u denotes the potential of the electrostatic field artificially created in the interior of the Earth. We have $\triangle u = 0, x \in \Omega$, where $u|_{\Gamma}$ is the magnitude of the potential u and $\frac{\partial u}{\partial n}|_{\Gamma}$ is the intensity of the potential, both measured at the accessible surface Γ of the Earth. In general, this problem is ill-posed since for some Cauchy data g and h there is not a solution, and even if there exists a solution it does not always depend continuously on the data. Therefore, several regularization methods have been presented to solve it such as the quasi-reversibility method [1, 2], the boundary element method [3, 4], the Fourier regularization method [5, 6], the central difference regularization method [7], the mollification method [8], *etc.* However, most of the results are in two dimensions. For the high dimensional case, both theoretical analysis and numerical computation are very difficult. In [9], the authors transfer high dimensional Cauchy problem for Laplace equation into moment problem, and then construct a series of polynomial functions to approximate solutions of the moment problem. In [10], a quasi-boundary-value method together with left-preconditioned generalized minimum residual method are proposed to deal with an ill-posed Cauchy problem for a 3D elliptic partial differential equation with variable coefficients.

In this paper, we consider the problem of an extension of the field potential specified on the inside surface of a hollow cylinder into space, and it is reduced to the axisymmetric Cauchy problem for the Laplace equation. This problem is involved in practical calculations of various electron optic systems. The hollow cylinder case is interesting since the hole leaves spaces for the measurement devices or devices that generate electric or magnetic fields. For example, electric fields with rotational symmetry are usually generated by electrodes in the shape of cylinders, cups and diaphragms. In recent years, Lu *et al.* [11] applied an analytical approach to study the transient heat conduction in a composite hollow cylinder. Cheng *et al.* [12] studied the inverse heat conduction problem in a hollow spherically symmetric domain. Marin and Marinescu [13, 14] investigated the existence, uniqueness and the asymptotic partition of total energy for the solutions of the initial boundary value problem within the context of the thermoelasticity of initially stressed bodies, and further considered micropolar thermoelastic body occupying a prismatic cylinder [15]. Şeremet and Şeremet [16] presented new steady-state Green's functions for displacements and thermal stresses for plane problem within a rectangular region, and the proposed technique could be extended to many 3D problems. More detailed descriptions of the model of hollow cylinders can be found in [17].

In this paper, suppose the considered 3D hollow cylinder domain is regular, and the internal and external radii are denoted r_0 and R, respectively. Under the assumption that the inside surface of the cylinder is composed of insulating materials, it makes most sense to use cylindrical coordinates to transfer problem (1.1) into the following problem:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, & r_0 < r < R, z \in \mathbb{R}, \\ u(r_0, z) = g(z), & z \in \mathbb{R}, \\ u_r(r_0, z) = 0, & z \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $r = \sqrt{x^2 + y^2}$, and $g(z)$ is a known potential distribution along the inside surface.

In practice, the data $g(z)$ is often obtained by the instrument installed inside the hollow cylinder, and there exist unavoidable errors. We assume that instead of exact data $g(z) \in L^2(\mathbb{R})$, only a noisy data $g^\delta(z) \in L^2(\mathbb{R})$ with

$$\left\| g(z) - g^\delta(z) \right\| \le \delta \tag{1.3}$$

is available. $\delta > 0$ represents the 'noise level' and $\| \cdot \|$ denotes the L^2-norm.

We further assume that

$$u(r, \cdot) \in L^2(\mathbb{R}), \quad \text{for each } r \in (r_0, R), \tag{1.4}$$

and the following *a priori* bound holds:

$$\left\| u(R, \cdot) \right\| \le E, \tag{1.5}$$

where E is a positive constant.

This paper is organized as follows. In Section 2, we present the expression of the solution and analyze the ill-posedness of problem (1.2). The *a priori* and *a posteriori* parameter choice rules which yield error estimates of Hölder type are suggested in Section 3. In Section 4, some numerical examples are given to illustrate the validity of the theoretical results. Finally, Section 5 ends this paper with a short conclusion.

2 Expression of the solution

For $f(z) \in L^1(\mathbb{R})$, $\hat{f}(\xi)$ denotes its Fourier transform, which is defined by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) e^{-i\xi z}\, dz.$$

Thus by using Fourier transform, the problem (1.2) is transformed into the following initial problem (2.1) of ODE in frequency domain:

$$\begin{cases} \hat{u}_{rr} + \frac{1}{r}\hat{u}_r - \xi^2 \hat{u} = 0, & r_0 < r < R, \xi \in \mathbb{R}, \\ \hat{u}(r_0, \xi) = \hat{g}(\xi), & \xi \in \mathbb{R}, \\ \hat{u}_r(r_0, \xi) = 0, & \xi \in \mathbb{R}. \end{cases} \tag{2.1}$$

Lemma 2.1 *The solution of problem* (2.1) *is given by*

$$\hat{u}(r, \xi) = r_0 |\xi| \Phi(r, \xi) \hat{g}(\xi), \quad r \in [r_0, R], \xi \in \mathbb{R}, \tag{2.2}$$

where

$$\Phi(r, \xi) = I_0\bigl(r|\xi|\bigr) K_1\bigl(r_0|\xi|\bigr) + K_0\bigl(r|\xi|\bigr) I_1\bigl(r_0|\xi|\bigr), \tag{2.3}$$

and $I_0(\cdot)$, $I_1(\cdot)$, $K_0(\cdot)$, $K_1(\cdot)$ *denote the modified Bessel function.*

Proof From [18], we know that the modified Bessel equation has two linearly independent solutions I_0 and K_0, then the general solution of equation in problem (2.1) is

$$\hat{u}(r, \xi) = C_1(\xi) I_0\bigl(r|\xi|\bigr) + C_2(\xi) K_0\bigl(r|\xi|\bigr), \quad r \in [r_0, R], \xi \in \mathbb{R}.$$

Combining the boundary conditions in (2.1) with the properties $I_0'(x) = I_1(x)$, $K_0'(x) = -K_1(x)$, and $I_0(x)K_1(x) + I_1(x)K_0(x) = \frac{1}{x}$ for $x > 0$, we get equation (2.2) of the solution to problem (2.1). □

Note that if $r_0 = 0$, it is a special case to problem (1.2) and the corresponding solution is

$$u(r, z) = \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}} e^{iz\xi} I_0(r\xi) \hat{g}^\delta(\xi)\, d\xi.$$

There is only one modified Bessel function I_0 in this expression, and this case has been discussed in [19].

In order to get a better understanding of the property of solution (2.2), it is necessary to list some important properties of function $\Phi(r, \xi)$. The following lemma establishes the relationship between $\Phi(r, \xi)$ and some basic elementary functions.

Lemma 2.2 *For $\xi \neq 0$, there exist positive constants C_1 and C_2 such that the following inequalities hold*:

$$C_1 \leq \frac{|\xi|\Phi(r,\xi)}{e^{(r-r_0)|\xi|}} \leq C_2. \tag{2.4}$$

Proof *Case* 1: $|\xi| \geq 1$.

According to [18], the 'asymptotic expansions for large arguments' of modified Bessel functions I_0, I_1, K_0, and K_1 are as follows:

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi}}\frac{e^x}{\sqrt{x}}, \qquad K_\nu(x) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}, \quad \nu = 0,1, \tag{2.5}$$

and then, combining with the continuous property of Bessel functions, we know that there exist four pairs of positive constants c_ν, c'_ν, d_ν, and d'_ν ($\nu = 0,1$), such that

$$c_\nu\frac{e^x}{\sqrt{x}} \leq I_\nu(x) \leq c'_\nu\frac{e^x}{\sqrt{x}}, \qquad d_\nu\frac{e^{-x}}{\sqrt{x}} \leq K_\nu(x) \leq d'_\nu\frac{e^{-x}}{\sqrt{x}}, \quad x \geq r_0. \tag{2.6}$$

Furthermore, on denoting $\mu_1 = c_0 d_1$ and $\mu_2 = c'_0 d'_1 + c'_1 d'_0$, then the following inequalities are straightforward calculations by using equation (2.3) and the above inequalities (2.6):

$$\frac{\mu_1 e^{(r-r_0)|\xi|}}{|\xi|} \leq \Phi(r,\xi) \leq \frac{\mu_2 e^{(r-r_0)|\xi|}}{|\xi|}, \quad \text{for } |\xi| \geq 1. \tag{2.7}$$

Case 2: $0 < |\xi| < 1$.

Based on equation (2.3), we have

$$\Phi(r,\xi) \sim \frac{1}{r_0|\xi|}, \quad \text{for } |\xi| \to 0. \tag{2.8}$$

For ease of use, an alternative form of (2.8) is as follows:

$$\Phi(r,\xi) \sim \frac{e^{(r-r_0)|\xi|}}{r_0|\xi|}, \quad \text{for } |\xi| \to 0,$$

and combining with the continuous property of $\Phi(r,\xi)$ on $[r_0, R] \times (0,1)$, we know that there exist two positive constants μ_3 and μ_4 such that the following inequalities hold:

$$\frac{\mu_3 e^{(r-r_0)|\xi|}}{|\xi|} \leq \Phi(r,\xi) \leq \frac{\mu_4 e^{(r-r_0)|\xi|}}{|\xi|}, \quad \text{for } 0 < |\xi| < 1.$$

If we take $C_1 = \min(\mu_1, \mu_3)$, and $C_2 = \max(\mu_2, \mu_4)$, then for $\xi \neq 0$ inequalities (2.4) are obtained. □

By the Parseval equality, we know

$$\left\|u(r,\cdot)\right\|^2 = \int_{\mathbb{R}} \left|r_0\xi\Phi(r,\xi)\hat{g}(\xi)\right|^2 d\xi. \tag{2.9}$$

Combining with (1.4), (2.9), and Lemma 2.2, we know that the Fourier transform $\hat{g}(\xi)$ of the exact data $g(z)$ must decay rapidly for $|\xi| \to \infty$ in order to ensure the convergence of (2.9). However, for the noisy data $g^\delta(z)$, its Fourier transform may not possess such a property, and the noisy perturbation will be multiplied by the diverging factor $|\xi\Phi(r,\xi)|$.

3 Fourier method and error estimates

Since the ill-posedness of problem (1.2) is caused by the high frequency perturbation of the noisy data, it is reasonable to stabilize the problem by eliminating high frequencies of the noisy data directly from the solution. This is the so-called Fourier method, it was put forward first by Lars Eldén *et al.* to deal with the inverse heat conduction problem [20]. Afterwards, this method has been successfully applied to deal with various inverse problems, *e.g.* the problem of a numerical pseudodifferential operator [21], the problem of numerical analytic continuation [22], the Cauchy problem for the Helmholtz equation [23], *etc.* However, for the Cauchy problem in a hollow cylinder, there are few efficient numerical methods, especially with *a posteriori* regularization parameter choice rule. In the following, we attempt to solve this problem by using Fourier method together with both *a priori* and *a posteriori* parameter choice rules.

According to [20], we eliminate all high frequencies from the solution and consider (2.2) only for $|\xi| < \xi_{\max}$, *i.e.*, define the Fourier regularization solution of problem (1.2) as

$$v^\delta_{\xi_{\max}}(r,z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{iz\xi} r_0|\xi|\Phi(r,\xi)\hat{g}^\delta(\xi)\chi_{\xi_{\max}}\,d\xi, \tag{3.1}$$

where $\xi_{\max} > 0$ is the regularization parameter to be determined, and χ is the characteristic function. In the following, we will establish error estimates between the exact solution and its regularization approximations.

3.1 *A priori* parameter choice rule

Theorem 3.1 *Assume the conditions* (1.3)-(1.5) *hold. The regularized solution of problem* (1.2) *is defined by* (3.1). *If we take the regularization parameter* $\xi^*_{\max}$ *to be*

$$\xi^*_{\max} = \frac{1}{R-r_0}\ln\frac{E}{\delta}, \tag{3.2}$$

then the following Hölder stability holds:

$$\left\|u(r,\cdot) - v^\delta_{\xi^*_{\max}}(r,\cdot)\right\| \le \left(r_0 + C_1^{-1}\right)C_2 E^{\frac{r-r_0}{R-r_0}}\delta^{\frac{R-r}{R-r_0}}, \quad r_0 < r < R. \tag{3.3}$$

Proof For the exact data $g(z)$, we define

$$v_{\xi_{\max}}(r,z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{iz\xi} r_0|\xi|\Phi(r,\xi)\hat{g}(\xi)\chi_{\xi_{\max}}\,d\xi.$$

Combining with (2.2) and (3.1), we have

$$\begin{aligned}\left\|u(r,\cdot) - v^\delta_{\xi_{\max}}(r,\cdot)\right\| &\le \left\|u(r,\cdot) - v_{\xi_{\max}}(r,\cdot)\right\| + \left\|v_{\xi_{\max}}(r,\cdot) - v^\delta_{\xi_{\max}}(r,\cdot)\right\| \\ &\le \left(\int_{|\xi|>\xi_{\max}} \left|r_0\xi\Phi(r,\xi)\hat{g}\right|^2 d\xi\right)^{\frac{1}{2}}\end{aligned}$$

$$
+ \left(\int_{|\xi| \le \xi_{\max}} \left| r_0 \xi \Phi(r,\xi) \left(\hat{g} - \hat{g}^\delta \right) \right|^2 d\xi \right)^{\frac{1}{2}}
$$
$$
:= I_1 + I_2.
$$

We will estimate I_1 and I_2 separately.

For the first term I_1, due to (1.5), we have

$$
\begin{aligned}
I_1 &= \left(\int_{|\xi| > \xi_{\max}} \left| r_0 \xi \Phi(r,\xi) \hat{g} \right|^2 d\xi \right)^{\frac{1}{2}} \le E \sup_{|\xi| > \xi_{\max}} \left[\frac{\Phi(r,\xi)}{\Phi(R,\xi)} \right] \\
&\le \frac{C_2 E}{C_1} \sup_{|\xi| > \xi_{\max}} e^{-(R-r)|\xi|} \le \frac{C_2 E}{C_1} e^{-(R-r)\xi_{\max}}.
\end{aligned}
$$

For the second term I_2, due to (1.3), we know that

$$
\begin{aligned}
I_2 &= \left(\int_{|\xi| \le \xi_{\max}} \left| r_0 \xi \Phi(r,\xi) \left(\hat{g} - \hat{g}^\delta \right) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\le \delta \sup_{|\xi| \le \xi_{\max}} r_0 |\xi| \Phi(r,\xi) \le C_2 r_0 \delta \sup_{|\xi| \le \xi_{\max}} e^{(r-r_0)|\xi|} \le C_2 r_0 \delta e^{(r-r_0)\xi_{\max}}.
\end{aligned}
$$

Combining I_1 and I_2, we have

$$
\left\| u(r,\cdot) - v^\delta_{\xi_{\max}}(r,\cdot) \right\| \le \frac{C_2 E}{C_1} e^{-(R-r)\xi_{\max}} + C_2 r_0 \delta e^{(r-r_0)\xi_{\max}}.
$$

If we replace $\xi_{\max}$ by $\xi^*_{\max}$ defined by (3.2), the final estimate is obtained as (3.3). The proof is completed. □

3.2 *A posteriori* parameter choice rule

Set

$$
\rho(\xi_{\max}) := \left\| v^\delta_{\xi_{\max}}(r_0,z) - g^\delta(z) \right\|. \tag{3.4}
$$

From equation (3.1) of the regularized solution $v^\delta_{\xi_{\max}}$, it is easy to see that

$$
\rho(\xi_{\max}) = \left\| \hat{g}^\delta \chi_{\xi_{\max}} - \hat{g}^\delta \right\| = \left(\int_{|\xi| > \xi_{\max}} \left| \hat{g}^\delta(\xi) \right|^2 d\xi \right)^{\frac{1}{2}}.
$$

Lemma 3.1 *The function* $\rho(\xi_{\max})$ *satisfies*

1. $\rho(\xi_{\max})$ *is a continuous and decreasing function on* $(0,\infty)$,
2. $\lim_{\xi_{\max}\to\infty} \rho(\xi_{\max}) = 0$,
3. $\lim_{\xi_{\max}\to 0} \rho(\xi_{\max}) = \|g^\delta\|$.

According to the discrepancy principle, we will take the solution $\xi_{\max}$ of the equation

$$
\rho(\xi_{\max}) = \tau\delta \tag{3.5}
$$

to be the regularization parameter, where $\tau > 1$ is a constant. In practice, we always have $\|g^\delta\| > \delta$, otherwise, $v^\delta_{\xi_{\max}} \equiv 0$ would be an acceptable approximation to u. Therefore, for an appropriate constant $\tau > 1$, equation (3.5) is always solvable, and if solution of (3.5) is not unique, the $\xi_{\max}$ will be understood as the minimal solution of the equation.

For the choice rule of regularization parameter, we have a range estimate for $\xi_{\max}$ given by the following lemma.

Lemma 3.2 *Assume that conditions* (1.3)-(1.5) *hold, if* $\xi_{\max}$ *is taken as the solution of equation* (3.5), *then the following inequality holds*:

$$e^{(R-r_0)\xi_{\max}} \le \frac{E}{C_1 r_0(\tau-1)\delta}. \tag{3.6}$$

Proof It is easy to observe that

$$\begin{aligned} \|\hat{g}\chi_{\xi_{\max}} - \hat{g}\| &= \left(\int_{|\xi|>\xi_{\max}} \left|\hat{g}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\ &= \left(\int_{|\xi|>\xi_{\max}} \left|r_0\xi\Phi(R,\xi)\hat{g}(\xi)\right|^2 r_0^{-2}\xi^{-2}\Phi^{-2}(R,\xi)\,d\xi\right)^{\frac{1}{2}} \\ &\le E\sup_{|\xi|>\xi_{\max}}\left[r_0|\xi|\Phi(R,\xi)\right]^{-1} \le \frac{E}{C_1 r_0} e^{-(R-r_0)\xi_{\max}}. \end{aligned} \tag{3.7}$$

In view of equation (3.5) and the triangle inequality, we also have

$$\begin{aligned} \|\hat{g}\chi_{\xi_{\max}} - \hat{g}\| &= \left\|(1-\chi_{\xi_{\max}})\hat{g}\right\| = \left\|(1-\chi_{\xi_{\max}})\left(\hat{g}-\hat{g}^\delta+\hat{g}^\delta\right)\right\| \\ &\ge \left\|(1-\chi_{\xi_{\max}})\hat{g}^\delta\right\| - \left\|(1-\chi_{\xi_{\max}})\left(\hat{g}-\hat{g}^\delta\right)\right\| \\ &\ge (\tau-1)\delta. \end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), we have

$$\frac{E}{C_1 r_0} e^{-(R-r_0)\xi_{\max}} \ge (\tau-1)\delta,$$

and therefore (3.6) holds. □

Lemma 3.3 *Assume that conditions* (1.3)-(1.5) *hold, then we have*

$$\begin{aligned} &\left\|v_{\xi_{\max}}^\delta(r,\cdot) - u(r,\cdot)\right\| \le C\left\|v_{\xi_{\max}}^\delta(r_0,\cdot) - u(r_0,\cdot)\right\|^{\frac{R-r}{R-r_0}}\left\|v_{\xi_{\max}}^\delta(R,\cdot) - u(R,\cdot)\right\|^{\frac{r-r_0}{R-r_0}}, \\ &r_0 < r < R, \end{aligned} \tag{3.9}$$

where C is a constant independent on δ *and E.*

Proof Let the index $\alpha = \frac{R-r}{R-r_0}$, it can be deduced from Lemma 2.2 that there exists a positive constant C such that

$$\left(r_0|\xi|\Phi(R,\xi)\right)^\alpha \frac{\Phi(r,\xi)}{\Phi(R,\xi)} \le C, \quad \text{for } |\xi| > 0.$$

Then we have the following estimate:

$$\begin{aligned}
&\left\| v^{\delta}_{\xi_{\max}}(r,\cdot) - u(r,\cdot) \right\| \\
&\quad = \left\| r_0 \xi \Phi(r,\xi) \hat{g}^{\delta} \chi_{\xi_{\max}} - r_0 \xi \Phi(r,\xi) \hat{g} \right\| \\
&\quad = \left[\int_{\xi \in \mathbb{R}} \left(r_0 |\xi| \Phi(r,\xi) \right)^2 \left(\hat{g}^{\delta} \chi_{\xi_{\max}} - \hat{g} \right)^2 d\xi \right]^{\frac{1}{2}} \\
&\quad = \left[\int_{\xi \in \mathbb{R}} \left(\hat{g}^{\delta} \chi_{\xi_{\max}} - \hat{g} \right)^{2\alpha} \left(r_0 |\xi| \Phi(R,\xi) \left(\hat{g}^{\delta} \chi_{\xi_{\max}} - \hat{g} \right) \right)^{2-2\alpha} \right. \\
&\qquad \left. \times \left(r_0 |\xi| \Phi(R,\xi) \right)^{2\alpha} \frac{\Phi^2(r,\xi)}{\Phi^2(R,\xi)} \, d\xi \right]^{\frac{1}{2}} \\
&\quad \le C \left\| v^{\delta}_{\xi_{\max}}(r_0,\cdot) - u(r_0,\cdot) \right\|^{\frac{R-r}{R-r_0}} \left\| v^{\delta}_{\xi_{\max}}(R,\cdot) - u(R,\cdot) \right\|^{\frac{r-r_0}{R-r_0}}.
\end{aligned}$$

Thus the proof is completed. □

Lemma 3.4 *Assume that conditions* (1.3)-(1.5) *hold. If* $\xi^*_{\max}$ *is taken as the solution of equation* (3.5), *then we have*

$$\left\| u(r_0,\cdot) - v^{\delta}_{\xi^*_{\max}}(r_0,\cdot) \right\| \le (1+\tau)\delta, \tag{3.10}$$

$$\left\| u(R,\cdot) - v^{\delta}_{\xi^*_{\max}}(R,\cdot) \right\| \le C^* E, \tag{3.11}$$

where C^* *is a constant independent on* δ *and* E.

Proof From noise level (1.3) and the choice rule for $\xi^*_{\max}$, the first inequality is easy to obtain,

$$\left\| u(r_0,\cdot) - v^{\delta}_{\xi^*_{\max}}(r_0,\cdot) \right\| \le \left\| g - g^{\delta} \right\| + \left\| g^{\delta} - v^{\delta}_{\xi^*_{\max}}(r_0,\cdot) \right\| \le (1+\tau)\delta.$$

For the second inequality, we have

$$\begin{aligned}
&\left\| u(R,\cdot) - v^{\delta}_{\xi^*_{\max}}(R,\cdot) \right\| \\
&\quad \le \left\| u(R,\cdot) - v_{\xi^*_{\max}}(R,\cdot) \right\| + \left\| v_{\xi^*_{\max}}(R,\cdot) - v^{\delta}_{\xi^*_{\max}}(R,\cdot) \right\| \\
&\quad \le E + \left[\int_{|\xi| \le \xi^*_{\max}} \left(r_0 \xi \Phi(R,\xi) \left(\hat{g} - \hat{g}^{\delta} \right) \right)^2 d\xi \right]^{\frac{1}{2}} \\
&\quad \le E + \delta \sup_{|\xi| \le \xi^*_{\max}} r_0 |\xi| \Phi(R,\xi) \\
&\quad \le E + C_2 r_0 \delta e^{(R-r_0)\xi^*_{\max}}.
\end{aligned}$$

Combining with Lemma 3.2, we have

$$\begin{aligned}
\left\| u(R,\cdot) - v^{\delta}_{\xi^*_{\max}}(R,\cdot) \right\| &\le E + \frac{C_2 E}{C_1(\tau - 1)} \\
&= C^* E,
\end{aligned}$$

where $C^* = 1 + \frac{C_2}{C_1(\tau-1)}$. The proof of this lemma is completed. □

Theorem 3.2 *Assume that conditions* (1.3)-(1.5) *hold. Further suppose that* $\delta < \|g^\delta\|$. *Choose* $\tau > 1$ *such that* $0 < \tau\delta < \|g^\delta\|$. *If* $\xi^*_{\max}$ *is taken as the solution of equation* (3.5), *then the following error estimates are satisfied*:

$$\left\| v^\delta_{\xi^*_{\max}}(r,\cdot) - u(r,\cdot) \right\| \le C\left(C^* E\right)^{\frac{r-r_0}{R-r_0}} \left[(1+\tau)\delta\right]^{\frac{R-r}{R-r_0}}, \quad r_0 < r < R, \tag{3.12}$$

where C *and* C^* *are the same as in Lemmas* 3.3 *and* 3.4, *respectively.*

The proposition of Theorem 3.2 follows immediately from Lemmas 3.3 and 3.4.

4 Numerical experiment

In this section, we present the numerical implementation of the Fourier method with *a priori* and *a posteriori* parameter choice rule, respectively. The fast Fourier transform and the inverse fast Fourier transform are used to compute the approximate solutions, and all computations are done in Matlab 7.0. In the computation, we always take $R = 1$, $r_0 = 0.1$, and we consider the problem in domain $\{0.1 < r < 1, -10 < z < 10\}$. The numbers of grids on the (r,z) domain are denoted M and K. In practical applications, the data $g(z)$ is obtained by measurement and there are inevitable errors. Thus in our experiment, we will consider the noisy data g^δ created by

$$g^\delta = g\left(1 + \varepsilon\, \mathrm{rand}\left(\mathrm{size}(g)\right)\right),$$

where $g = (g(z_1), g(z_2), \dots, g(z_K))$, $z_k = -10 + \frac{20(k-1)}{K-1}$, $k = 1, 2, \dots, K$, and the noise level

$$\delta = \left\| g - g^\delta \right\|_{L^2} = \sqrt{\frac{20}{K} \sum_{k=1}^{K} \left(g(z_k) - g^\delta(z_k)\right)^2}.$$

The function 'rand(·)' generates arrays of uniformly distributed random numbers.

For the *a priori* parameter choice rule, the regularization parameter depends on both the *a priori* bound and the noise level. The *a priori* bound is computed according to

$$E = \sqrt{\frac{20}{K} \sum_{k=1}^{K} \left|f(z_k)\right|^2}.$$

To test the accuracy of the computed approximations, we use the relative root mean square error (RES), which is defined as

$$\mathrm{RES}\left(v^\delta_N(r,\cdot)\right) = \frac{\sqrt{\sum_{k=1}^{K} (u(r,z_k) - v^\delta_N(r,z_k))^2}}{\sqrt{\sum_{k=1}^{K} u^2(r,z_k)}}. \tag{4.1}$$

We will consider two examples where there are no exact solutions. The data function $g(z)$ is obtained by solving the following direct problem:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, & r_0 < r < R, z \in \mathbb{R}, \\ u_r(r_0, z) = 0, & z \in \mathbb{R}, \\ u(R, z) = f(z), & z \in \mathbb{R}, \end{cases} \tag{4.2}$$

Table 1 Relative errors RES with $r = 0.2$, $\varepsilon = 0.001$, $M = 31$ for Example 1

K	**99**	**109**	**139**	**169**	**199**
RES (*a priori*)	0.0038	0.0036	0.0036	0.0034	0.0035
RES (*a posteriori*)	0.0037	0.0035	0.0036	0.0033	0.0038

Table 2 Relative errors RES with $r = 0.2$, $\varepsilon = 0.001$, $K = 109$ for Example 1

M	**21**	**31**	**41**	**51**	**61**
RES (*a priori*)	0.0036	0.0036	0.0036	0.0036	0.0035
RES (*a posteriori*)	0.0035	0.0035	0.0034	0.0034	0.0034

Table 3 Comparison of relative errors for Example 1

r	**$\varepsilon = 0.005$**			**$\varepsilon = 0.05$**		
	0.2	**0.6**	**0.9**	**0.2**	**0.6**	**0.9**
RES (*a priori*)	0.0129	0.0588	0.1759	0.1191	0.1359	0.2093
RES (*a posteriori*)	0.0117	0.0538	0.1362	0.1147	0.1228	0.1721

where $f(z)$ is selected to be a function with some interesting features. The well-posed problem (4.2) could then be solved by the standard five-point difference scheme and the corresponding data function $g(z)$ can be computed.

Example 1 $f(z) = e^{-z^2}$, $z \in \mathbb{R}$.

Tables 1 and 2 show the effect of increasing M and K on accuracy. From these tables, we find that M and K have small influence on the results when they become large. That is to say, the degree of ill-posedness of numerical problems does not increase with the refinement of the mesh used. Thus we shall always take $M = 31$, $K = 109$ in the numerical experiment.

Table 3 lists the relative root mean square error (RES) between the exact solution and its approximations with different perturbations in the data. The approximations are obtained by the Fourier method with both the *a priori* and the *a posteriori* parameter choice rule. For the *a priori* Fourier method, the regularization parameter is selected according to Theorem 3.1. For the *a posteriori* Fourier method, we choose $\xi^*_{\max}$ as the solution of equation (3.5), where τ is some constant greater than unity, which can be taken heuristically to be 1.1. Note that the regularization parameter chosen by the *a posteriori* rule is only dependent on noise level δ, and computing accuracy is improved. This table shows that the smaller r, the better the computed results, which is consistent with the theoretical result in Theorems 3.1 and 3.2. The suggested method is still stable for higher noise levels on the data, and the smaller ε, the more accurate the approximations.

Figure 1 is the comparison of exact solution and its approximations at different values of the radius $r = 0.2, 0.6, 0.9$. In Figure 1(a1)-(c1), the perturbation is $\varepsilon = 0.005$. In Figure 1(a2)-(c2), the perturbation is $\varepsilon = 0.05$.

Figure 2 gives the corresponding comparison of exact solution and its approximations in terms of $u(\cdot, z)$ distributions at different constant $z = -10, 0, 2, 5$ for $\varepsilon = 0.05$.

From Figures 1 and 2, and Table 3, we see that there is almost no difference for the *a posteriori* and *a priori* Fourier method with exact *a priori* bound E when r is relatively small. However, with the increase of r, the numerical effect of the *a posteriori* Fourier

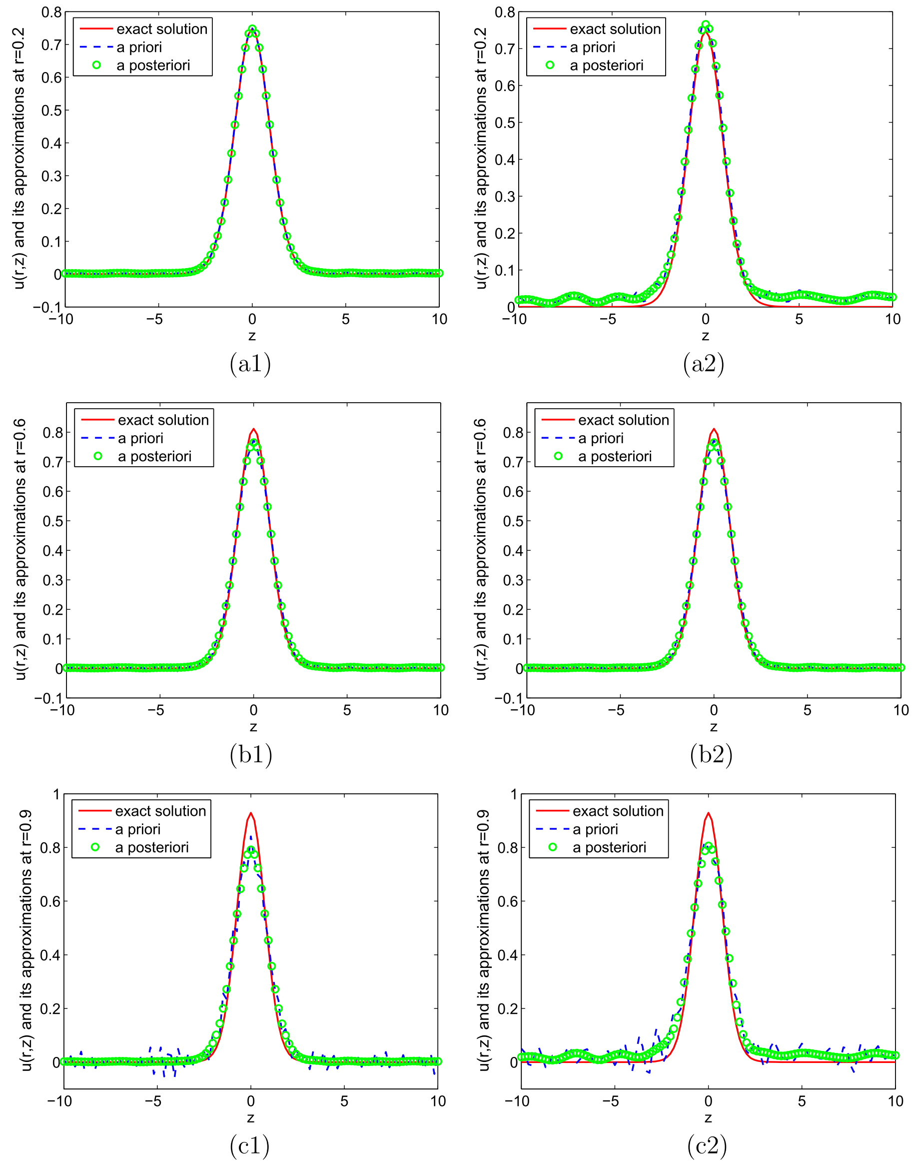

Figure 1 Comparisons of exact solution and its approximations at $r = 0.2, 0.6, 0.9$ for Example 1. The results for $\varepsilon = 0.005$ are **(a1)**-**(c1)** and the results for $\varepsilon = 0.05$ are **(a2)**-**(c2)**.

method is better than the *a priori* one. It is generally known that *a priori* bound E has a great influence on the accuracy of regularized solutions computed by *a priori* method, and a wrong *a priori* bound may lead to bad regularized solutions. This is just the weakness of the *a priori* parameter choice rule. The following example will also confirm this matter.

Example 2 $f(z) = \sin \frac{\pi z}{10}$, $z \in \mathbb{R}$.

Table 4 lists the RES between the exact solution and its approximations with different perturbations in the data for Example 2.

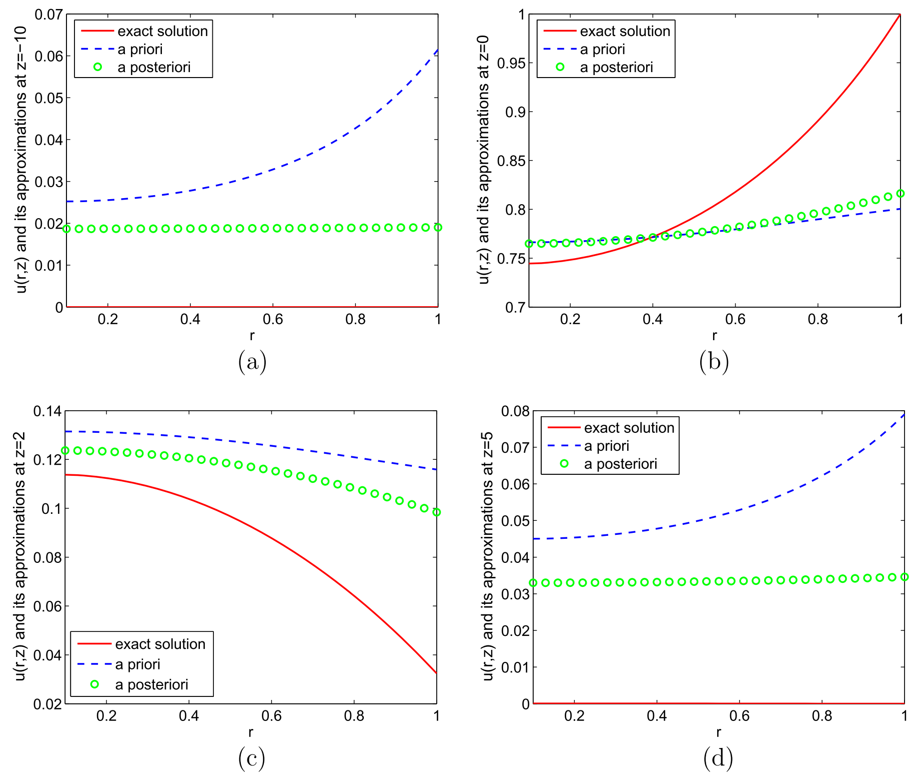

Figure 2 Comparisons of exact solution and its approximations at $z = -10, 0, 2, 5$ with $\varepsilon = 0.05$ for Example 1.

Table 4 Comparison of relative errors for Example 2

r	$\varepsilon = 0.005$			$\varepsilon = 0.05$		
	0.2	**0.6**	**0.9**	**0.2**	**0.6**	**0.9**
RES (*a priori*)	0.0041	0.0147	0.0871	0.0376	0.0468	0.0949
RES (*a posteriori*)	0.0040	0.0096	0.0350	0.0363	0.0363	0.0377

Figure 3 is a comparison of the exact solution and its approximations at different values of the radius $r = 0.2, 0.6, 0.9$ for Example 2. In Figure 3(a1)-(c1), the perturbation is $\varepsilon = 0.005$. In Figure 3(a2)-(c2), the perturbation is $\varepsilon = 0.05$.

The explanation for Table 4 and Figure 3 is similar to that for Example 1, but it is worth noting that numerical results for the *a posteriori* Fourier method are more accurate at $r = 0.9$. The reason for this phenomenon is mainly the that regularization parameter selected by *a posteriori* choice rule depends only on the noise level δ, and is not related to other factors. Table 5 gives a discussion as regards the impact of bound E on the relative error of the regularized approximation for Example 2, and we also see that a wrong constant E may lead to bad regularized solutions.

For linear ill-posed problems defined on a 'strip' or 'cylinder' domain, the Fourier method is the most simple and a very effective regularization method. We repeated the computations of Examples 1 and 2 using the modified Tikhonov regularization method. The advantage of this method is that explicit error estimate for some specific problems

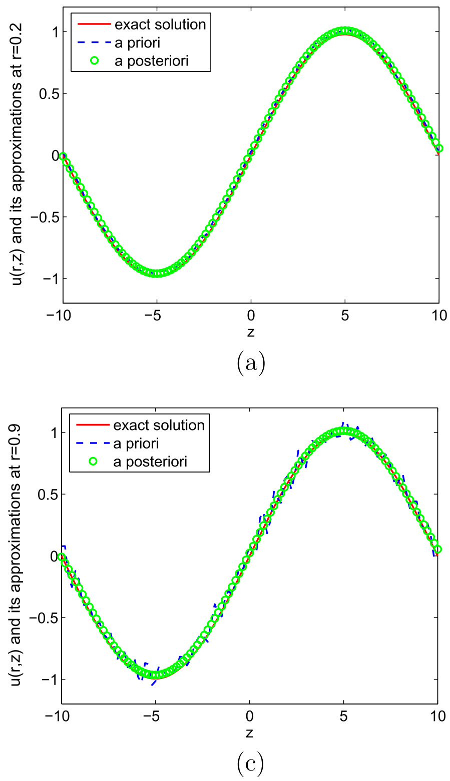

Figure 3 Comparisons of exact solution and its approximations at $r = 0.2, 0.6, 0.9$ with $\varepsilon = 0.05$ for Example 2.

Table 5 The impact of *a priori* bound *E* on the relative errors for Example 2 at $r = 0.9$ with $\varepsilon = 10^{-2}$

E	1	3.1463	5	7	9
RES	0.0344	0.0782	0.1265	0.1605	0.1605

could be obtained. The expression of the modified Tikhonov regularized solution and the corresponding error estimate are listed in the appendix.

Figure 4 shows the comparison between the *a posteriori* Fourier method and the modified Tikhonov method on $r = 0.9$ with different perturbations. From this figure, it is easy to see that the Fourier method is much stable and better for larger r and ε.

5 Conclusion

In this paper, we have applied the Fourier method together with *a priori* parameter choice rule and *a posteriori* parameter choice rule to solve the Cauchy problem for the Laplace equation in a hollow cylinder domain. The Hölder type error estimates between the exact solution and its approximation are obtained. As for any *a priori* regularization method, the choice of the regularization parameter usually depends on both the *a priori* bound and the noise level. In general, the *a priori* bound cannot be known exactly in practice, and working with a wrong *a priori* bound may lead to bad regularization solution. The advantage of the *a posteriori* method is that one does not need to know the smoothness and the *a priori* bound of the unknown solution. The numerical results also show that the

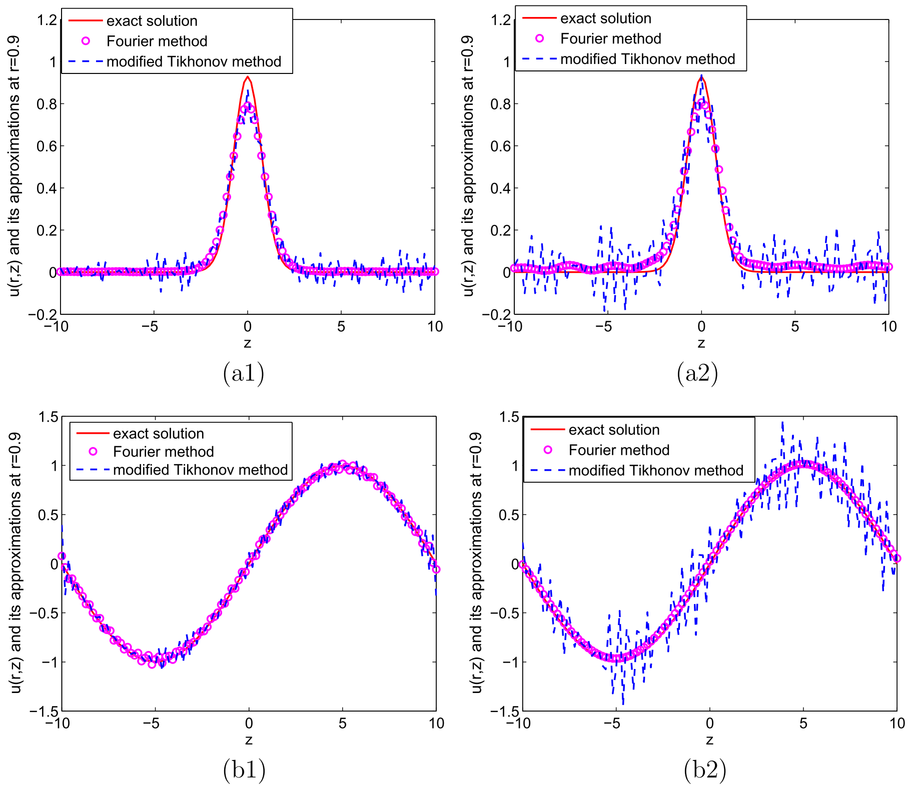

Figure 4 Comparisons of exact solution and its approximations by using different methods at $r = 0.9$. The results for Example 1 are **(a1)** and **(a2)** with $\varepsilon = 0.005$ and $\varepsilon = 0.05$, respectively. The results for Example 2 are **(b1)** and **(b2)** with $\varepsilon = 0.005$ and $\varepsilon = 0.05$, respectively.

Fourier method with *a posteriori* parameter choice rule is much stable than the one with *a priori* parameter choice rule for larger r and ε. However for the *a posteriori* method, some important information as regards the solution is concealed and hidden for the discrepancy principle, such that the theoretical analysis of the convergence rate is rather difficult obtain for some problems. The related theory is particularly worthy of further development.

Appendix

In order to compare the results with Fourier method, we repeat the computations of Examples 1 and 2 using the modified Tikhonov regularization method. Define the approximation as follows:

$$v_\alpha^\delta(r,z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{i\xi z} r_0|\xi|\Phi(r,\xi)}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \hat{g}^\delta(\xi)\,d\xi. \tag{A.1}$$

Before giving the explicit error estimate, we present first the following lemma which is crucial for the error estimate.

Lemma ([24]) *Let $0 < x < l$, then we have*

$$\sup_{s\ge 0} \frac{e^{(l-x)s}}{1+\omega^2 e^{2ls}} \le \omega^{\frac{x-l}{l}}, \qquad \sup_{s\ge 0} \frac{e^{(2l-x)s}}{1+\omega^2 e^{2ls}} \le \omega^{\frac{x-2l}{l}}. \tag{A.2}$$

Theorem *Assume that conditions* (1.3)-(1.5) *hold. If the regularization parameter* α *is selected to be*

$$\alpha = \frac{\delta}{E}, \tag{A.3}$$

then the Hölder type error estimate holds,

$$\left\| u(r,z) - v_\alpha^\delta(r,z) \right\| \le C' E^{\frac{r-r_0}{R-r_0}} \delta^{\frac{R-r}{R-r_0}}, \tag{A.4}$$

where C' *is a constant independent on* δ *and* E.

Proof For fixed $r \in (r_0, R)$,

$$\begin{aligned}
\left\| u(r,z) - v_\alpha^\delta(r,z) \right\| &= \left\| \hat{u}(r,\xi) - v_\alpha^\delta(r,\xi) \right\| = \left\| r_0|\xi|\Phi(r,\xi)\hat{g} - \frac{r_0|\xi|\Phi(r,\xi)}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2}\hat{g}^\delta \right\| \\
&\le \left\| \frac{r_0|\xi|\Phi(r,\xi)(\hat{g}^\delta - \hat{g})}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \right\| + \alpha^2 \left\| \frac{(r_0|\xi|)^3\Phi(r,\xi)\Phi^2(R,\xi)\hat{g}}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \right\| \\
&= I_1 + \alpha^2 I_2.
\end{aligned}$$

Now we estimate I_1 and I_2 separately.

$$\begin{aligned}
I_1 &= \left\| \frac{r_0|\xi|\Phi(r,\xi)(\hat{g}^\delta - \hat{g})}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \right\| \le \delta \sup \frac{r_0|\xi|\Phi(r,\xi)}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \\
&\le \delta \sup \frac{C_2 r_0 e^{(r-r_0)|\xi|}}{1+\alpha^2 C_1^2 r_0^2 e^{2(R-r_0)|\xi|}} \le \delta C_2 r_0 (C_1 r_0 \alpha)^{-\frac{r-r_0}{R-r_0}}, \\
I_2 &= \left\| \frac{(r_0|\xi|)^3\Phi(r,\xi)\Phi^2(R,\xi)\hat{g}}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \right\| = \left\| \frac{(r_0|\xi|)^2\Phi(r,\xi)\Phi(R,\xi)[r_0|\xi|\Phi(R,\xi)\hat{g}]}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \right\| \\
&\le E \sup \frac{(r_0|\xi|)^2\Phi(r,\xi)\Phi(R,\xi)}{1+\alpha^2[r_0|\xi|\Phi(R,\xi)]^2} \le E(C_2 r_0)^2 \sup \frac{e^{(R+r-2r_0)|\xi|}}{1+(C_1\alpha)^2 e^{2(R-r_0)|\xi|}} \\
&\le E(C_2 r_0)^2 (C_1\alpha)^{\frac{2r_0-R-r}{R-r_0}}.
\end{aligned}$$

From the estimate of I_1 and I_2, we have

$$\left\| u(r,z) - v_\alpha^\delta(r,z) \right\| \le \delta C_2 r_0 (C_1 r_0 \alpha)^{-\frac{r-r_0}{R-r_0}} + E(C_2 r_0)^2 (C_1\alpha)^{\frac{2r_0-R-r}{R-r_0}}.$$

If we select $\alpha = \frac{\delta}{E}$, then we have

$$\left\| u(r,z) - v_\alpha^\delta(r,z) \right\| \le C' E^{\frac{r-r_0}{R-r_0}} \delta^{\frac{R-r}{R-r_0}},$$

where $C' = C_1^{-\frac{r-r_0}{R-r_0}} C_2 r_0^{\frac{R-r}{R-r_0}} + C_1^{\frac{2r_0-R-r}{R-r_0}} C_2^2 r_0^2$. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Informational Science, Yantai University, Yantai, 264005, P.R. China. [2]School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, P.R. China.

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