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A nonexistence result for a nonlinear wave equation with damping on a Riemannian manifold

Qiang Ru*

*Correspondence:
ruqiang666@163.com
Department of Mathematics, China University of Mining and Technology, Xuzhou, 221116, China

Abstract

In this paper, we study the global nonexistence of solutions to a nonlinear wave equation with critical potential $V(x)$ on a Riemannian manifold, the form of which is more general than those in (Todorova and Yordanov in C. R. Acad. Sci., Sér. 1 Math. 300:557-562, 2000). The way we follow is motivated by the work of Qi S. Zhang (C. R. Acad. Sci., Sér. 1 Math. 333:109-114, 2001). We also prove the local existence and uniqueness result.

Keywords: nonexistence; wave equation

1 Introduction and main results

In this paper, we study the global nonexistence of solutions to the following nonlinear wave equation with a damping term:

$$\begin{cases} \Delta u(x,t) + W(x)|u|^p(x,t) - u_t(x,t) - u_{tt}(x,t) = 0 \quad \text{in } \mathbb{M}^n \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{M}^n, \\ u_t(x,0) = u_1(x) \quad \text{in } \mathbb{M}^n, \end{cases} \tag{1.1}$$

where $\mathbb{M}^n$ ($n \geq 3$) is a non-compact complete Riemannian manifold, Δ is the Laplace-Beltrami operator, and $\int u_0(x)\,dx, \int u_1(x)\,dx > 0$, while the constant $p > 1$.

In [1], Todorova and Yordanov proved the following result for (1.1) when $\mathbb{M}^n = \mathbb{R}^n$ and $W(x) \equiv 1$:

> Let $1 < p < 1 + \frac{2}{n}$. If we assume that $u_0(x)$, $u_1(x)$ is compactly supported and $\int u_0(x)\,dx$, $\int u_1(x)\,dx > 0$, then the global solution of (1.1) does not exist.

However, whether or not the critical case $p = 1 + \frac{2}{n}$ belongs to the blow-up case was left open. In [2], Qi S. Zhang showed $p = 1 + \frac{2}{n}$ belongs to the blow-up case.

The investigation of nonexistence and existence of global solutions to evolution equations has a long history, We refer the reader to the surveys [3–7]. There are more recent contributions to the discussion of the test function method; we refer to [8–11] for a survey of the literature on this problem.

In this paper, we study the global nonexistence of solutions to a nonlinear wave equation with critical potential $V(x)$ on a Riemannian manifold, the form of which is more general than those in [1]. The way we follow is motivated by the work of Qi S. Zhang [2]. We also prove the local existence and uniqueness result.

Throughout the paper, for a fixed $x_0 \in \mathbb{M}^n$, we make the following assumptions (see [2]):

(i) $\frac{\partial \log g^{\frac{1}{2}}}{\partial r} \leq \frac{C}{r}$, when $r = d(x, x_0)$ is smooth; here $g^{\frac{1}{2}}$ is the volume density of the manifold;

(ii) there are positive constants $\alpha > 2$ and $m > -2$, such that
  - $C^{-1}r^{\alpha} \leq |B_r(x_0)| \leq Cr^{\alpha}$, when r is large and for all $x \in \mathbb{M}^n$;
  - $W(x)$ are non-negative L^{∞}_{loc} functions. For large $r = d(x, x_0)$, $C^{-1}r^m \leq W(x) \leq Cr^m$.

Lemma 1 (see [12]) *Under assumptions* (i) *and* (ii), *there exist positive constants C and R_0, for $R \geq R_0$ and $\frac{1}{p} + \frac{1}{q} = 1$, such that*

$$\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)\, dx \leq C \ln R + CR^{-\frac{qm}{p}+\alpha}.$$

Our result is as follows.

Theorem 1.1 *Under assumptions* (i) *and* (ii), *let $p \in (1, 1 + \frac{2+m}{\alpha}]$. If we assume that $u_0(x)$, $u_1(x)$ is compactly supported and $\int u_0(x)\, dx, \int u_1(x)\, dx > 0$, then the global solution of* (1.1) *does not exist.*

Remark Clearly $\mathbb{R}^n$ satisfies assumptions (i) and (ii), so if $\mathbb{M}^n = \mathbb{R}^n$ and $W(x) \equiv 1 (m = 0)$, from the proof of Theorem 1.1, it is in accordance with (a).

Theorem 1.2 (Local existence and uniqueness) *Let $\mathbb{M}^n$ be an n-dimensional smooth compact manifold, and u_0 be a smooth hypersurface immersion of $\mathbb{M}^n$ into $\mathbb{R}^{n+1}$. Then there exists a constant $T > 0$ such that the initial value problem*

$$\begin{cases} \Delta u(x,t) + W(x)|u|^p(x,t) - u_t(x,t) - u_{tt}(x,t) = 0 \quad \text{in } \mathbb{M}^n \times (0, \infty), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{M}^n, \\ u_t(x,0) = u_1(x) \quad \text{in } \mathbb{M}^n, \end{cases} \tag{1.2}$$

has a unique smooth solution $u(x,t)$ on $\mathbb{M}^n \times [0, T)$, where $u_1(x)$ is a smooth vector-valued function on $\mathbb{M}^n$.

Theorem 1.1 is proved in Section 2; Theorem 1.2 is proved in Section 3.

2 Global nonexistence of solutions

Proof of Theorem 1.1 From now on, C is always a constant that may change from line to line. Throughout the section, we let $\varphi, \eta \in C^\infty[0,\infty)$ be two functions satisfying

$$\begin{cases} \varphi(r) \in [0,1], & \text{if } r \in [0,\infty), \\ \varphi(r) = 1, & \text{if } r \in [0,\frac{1}{2}], \\ \varphi(r) = 0, & \text{if } r \in [1,\infty]; \\ \eta(t) \in [0,1], & \text{if } t \in [0,\infty), \\ \eta(t) = 1, & \text{if } t \in [0,\frac{1}{4}], \\ \eta(t) = 0, & \text{if } t \in [1,\infty]; \\ \frac{|\nabla\varphi|^2}{\varphi} \le C, & \text{if } r \in [0,1]; \\ \frac{\eta_t^2}{\eta} \le C, & \text{if } t \in [0,1]; \\ -C \le \varphi(r)' \le 0; \quad |\varphi(r)''| \le C; \quad -C \le \eta(t)' \le 0; \quad |\eta(t)''| \le C. \end{cases} \tag{2.1}$$

For $R > 0$, we define $Q_R = B_R(x_0) \times [0,R^2]$. We also need a cut-off function

$$\psi_R = \varphi_R\big[d(x,x_0)\big]\eta_R(t), \tag{2.2}$$

where $\varphi_R(r) = \varphi(\frac{r}{R})$ and $\eta_R(t) = \eta(\frac{t}{R^2})$. Clearly,

$$\begin{aligned} &\frac{\partial\varphi_R}{\partial r} \in \left[-\frac{C}{R},0\right]; \qquad \frac{\partial^2\varphi_R}{\partial r^2} \in \left[-\frac{C}{R^2},\frac{C}{R^2}\right]; \qquad \frac{\partial\eta_R}{\partial t} \in \left[-\frac{C}{R^2},0\right]; \\ &\frac{|\nabla\varphi_R|^2}{\varphi_R} \le \frac{C}{R^2}; \qquad \frac{(\partial_t\eta_R)^2}{\eta_R} \le \frac{C}{R^4}. \end{aligned} \tag{2.3}$$

We use the method of contradiction. Suppose that $u(x,t)$ is a global positive solution of (1.1). For $R > 0$, we set

$$I_R \stackrel{\triangle}{=} \int_{Q_R} W(x)|u|^p(x,t)\psi_R^q(x,t)\,dx\,dt, \tag{2.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Since $u(x,t)$ is a solution of (1.1), we have

$$I_R = \int_{Q_R} \big[u_t(x,t) - \Delta u(x,t) + u_{tt}(x,t)\big]\psi_R^q(x,t)\,dx\,dt = J_1 + J_2, \tag{2.5}$$

where

$$J_1 \stackrel{\triangle}{=} \int_{Q_R} \big[u_t(x,t) - \Delta u(x,t)\big]\psi_R^q(x,t)\,dx\,dt, \qquad J_2 \stackrel{\triangle}{=} \int_{Q_R} u_{tt}(x,t)\psi_R^q(x,t)\,dx\,dt. \tag{2.6}$$

We will estimate J_1 and J_2 separately.

By the Stokes formula and noting that $\psi_R = 0$ on $\partial B_R(x_0)$, we have

$$\begin{aligned} J_1 &= \int_{Q_R} u_t(x,t)\psi_R^q(x,t)\,dx\,dt - \int_0^{R^2}\int_{\partial B_R(x_0)} \frac{\partial u(x,t)}{\partial n}\psi_R^q(x,t)\,dS_x\,dt \\ &\quad + \int_{Q_R} \nabla u(x,t)\nabla\psi_R^q(x,t)\,dx\,dt \\ &= \int_{Q_R} u_t(x,t)\psi_R^q(x,t)\,dx\,dt + \int_{Q_R}\nabla u(x,t)\nabla\psi_R^q(x,t)\,dx\,dt, \end{aligned} \tag{2.7}$$

which implies, via integration by parts,

$$\begin{aligned} J_1 &= \int_{B_R(x_0)} u\bigl(x,R^2\bigr)\psi_R^q\bigl(x,R^2\bigr)\,dx - \int_{B_R(x_0)} u(x,0)\psi_R^q(x,0)\,dx \\ &\quad - q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt + \int_0^{R^2}\int_{\partial B_R(x_0)} u(x,t)\frac{\partial\varphi_R^q}{\partial n}\eta_R^q(t)\,dS_x\,dt \\ &\quad - \int_{Q_R} u(x,t)\Delta\varphi_R^q(x)\eta_R^q(t)\,dx\,dt. \end{aligned} \tag{2.8}$$

We observe that $\psi_R^q(x,R^2) = 0$; $\int u_0(x)\,dx > 0$, $\frac{\partial\varphi_R^q}{\partial n} = q\varphi_R^{q-1}\varphi_R'(\frac{\partial r}{\partial n}) = 0$ on $\partial B_R(x_0)$, so we obtain

$$J_1 \le -q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt - \int_{Q_R} u(x,t)\Delta\varphi_R^q(x)\eta_R^q(t)\,dx\,dt. \tag{2.9}$$

Since $\Delta\varphi_R^q(x) = q\varphi_R^{q-1}(x)\Delta\varphi_R(x) + q(q-1)\varphi_R^{q-2}(x)|\nabla\varphi_R(x)|^2$, (2.9) yields

$$\begin{aligned} J_1 &\le -q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt - q\int_{Q_R} u(x,t)\varphi_R^{q-1}(x)\Delta\varphi_R(x)\eta_R^q(t)\,dx\,dt \\ &\quad - q(q-1)\int_{Q_R} u(x,t)\varphi_R^{q-2}(x)\bigl|\nabla\varphi_R(x)\bigr|^2\eta_R^q(t)\,dx\,dt. \end{aligned} \tag{2.10}$$

Recalling the supports of $\varphi_R(x)$ and $\eta_R(t)$, that is,

$$\begin{cases} \eta_R(t) = 1, & \eta_R'(t) = 0, \quad \text{if } t \in [0, \frac{R^2}{4}], \\ \varphi_R(x) = 1, & \Delta\varphi_R(x) = 0, \quad \text{if } r \in [0, \frac{R}{2}], \end{cases} \tag{2.11}$$

we can reduce (2.10) to

$$\begin{aligned} J_1 &\le -q\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt \\ &\quad - q\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\Delta\varphi_R(x)\eta_R^q(t)\,dx\,dt \\ &\quad - q(q-1)\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-2}(x)\bigl|\nabla\varphi_R(x)\bigr|^2\eta_R^q(t)\,dx\,dt. \end{aligned} \tag{2.12}$$

Since φ_R is radial, we have

$$\Delta\varphi_R = \varphi_R'' + \left[\frac{n-1}{r} + \frac{\partial \log g^{\frac{1}{2}}}{\partial r}\right]\varphi_R'. \tag{2.13}$$

Taking R sufficiently large, by assumption (i), that is, $\frac{\partial \log g^{\frac{1}{2}}}{\partial r} \leq \frac{C}{r}$, we obtain

$$\Delta\varphi_R \geq -\frac{C}{R^2}, \tag{2.14}$$

when $x \in B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)$. Merging (2.12), (2.14), and (2.3), we know

$$\begin{aligned} J_1 \leq{}& \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\,dx\,dt \\ &+ \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\eta_R^q(t)\,dx\,dt \\ &- q(q-1)\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-2}(x)\left|\nabla\varphi_R(x)\right|^2\eta_R^q(t)\,dx\,dt. \end{aligned} \tag{2.15}$$

By (2.3), we have

$$\varphi_R^{q-2}(x)\left|\nabla\varphi_R(x)\right|^2 = \varphi_R^{q-1}\frac{|\nabla\varphi_R(x)|^2}{\varphi_R} \geq -\frac{C}{R^2}\varphi_R^{q-1}, \tag{2.16}$$

which yields

$$\begin{aligned} J_1 \leq{}& \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\,dx\,dt \\ &+ \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\eta_R^q(t)\,dx\,dt \\ &+ \frac{Cq(q+1)}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\eta_R^q(t)\,dx\,dt \\ \leq{}& \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\,dx\,dt \\ &+ \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\eta_R^q(t)\,dx\,dt. \end{aligned} \tag{2.17}$$

Therefore, as $\varphi_R, \eta_R \leq 1$,

$$\begin{aligned} J_1 \leq{}& \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\psi_R^{q-1}(x,t)\,dx\,dt \\ &+ \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\psi_R^{q-1}(x,t)\,dx\,dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{\frac{1}{p}}(x)\left|u(x,t)\right|\psi_R^{q-1}(x,t)W^{-\frac{1}{p}}(x)\,dx\,dt \\
&\quad + \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W^{\frac{1}{p}}(x)\left|u(x,t)\right|\psi_R^{q-1}(x,t)W^{-\frac{1}{p}}(x)\,dx\,dt. \qquad (2.18)
\end{aligned}$$

By the Hölder inequality and noticing $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
J_1 &\leq \frac{Cq}{R^2}\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W(x)|u|^p(x,t)\psi_R^q(x,t)\,dx\,dt\right]^{\frac{1}{p}} \times \left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)\,dx\,dt\right]^{\frac{1}{q}} \\
&\quad + \frac{Cq}{R^2}\left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W(x)|u|^p(x,t)\psi_R^q(x,t)\,dx\,dt\right]^{\frac{1}{p}} \\
&\quad \times \left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W^{-\frac{q}{p}}(x)\,dx\,dt\right]^{\frac{1}{q}} \\
&\leq \frac{Cq}{R^2}[I_R]^{\frac{1}{p}} \times \left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)\,dx\,dt\right]^{\frac{1}{q}} \\
&\quad + \frac{Cq}{R^2}[I_R]^{\frac{1}{p}} \times \left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W^{-\frac{q}{p}}(x)\,dx\,dt\right]^{\frac{1}{q}}. \qquad (2.19)
\end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}
\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)\,dx\,dt\right]^{\frac{1}{q}} &\leq \left\{\int_{\frac{R^2}{4}}^{R^2}\left[C\ln R + CR^{-\frac{qm}{p}+\alpha}\right]dt\right\}^{\frac{1}{q}} \\
&\leq CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}. \qquad (2.20)
\end{aligned}$$

Hence,

$$\begin{aligned}
J_1 &\leq \frac{Cq}{R^2}[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] + \frac{Cq}{R^2}[I_R]^{\frac{1}{p}} \times \left[CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\
&\leq \frac{Cq}{R^2}[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\
&= C[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}-2}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}-2}\right]. \qquad (2.21)
\end{aligned}$$

Now let us estimate J_2. Using integration by parts, we obtain

$$\begin{aligned}
J_2 &= \int_{Q_R} u_{tt}(x,t)\psi_R^q(x,t)\,dx\,dt \\
&= \int_{B_R(x_0)} u_t(x,t)\psi_R^q(x,t)\Big|_0^{R^2}\,dx - q\int_{B_R(x_0)} u\varphi_R^q(x)\eta_R^{q-1}(t)\partial_t\eta_R\Big|_0^{R^2}\,dx \\
&\quad + q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\partial_t^2\eta_R(t)\,dx\,dt \\
&\quad + q(q-1)\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-2}(t)\big(\partial_t\eta_R(t)\big)^2\,dx\,dt. \qquad (2.22)
\end{aligned}$$

We observe that $\psi_R^q(x,R^2) = \eta_R(R^2) = 0$; $\int u_0(x)\,dx$, $\int u_1(x)\,dx > 0$ and (2.3), The above implies

$$\begin{aligned} J_2 &\le q\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} |u|\varphi_R^q\eta_R^{q-1}\left|\partial_t^2\eta_R\right|\,dx\,dt \\ &\quad + q(q-1)\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} |u|\varphi_R^q\eta_R^{q-1}\frac{(\partial_t\eta_R)^2}{\eta_R}\,dx\,dt. \end{aligned} \tag{2.23}$$

Again by (2.3) and the Hölder inequality, we have

$$\begin{aligned} J_2 &\le \frac{C}{R^4}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} |u|\varphi_R^q\eta_R^{q-1}\,dx\,dt \\ &\le \frac{C}{R^4}\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W(x)|u|^p(x,t)\psi_R^q(x,t)\,dx\,dt\right]^{\frac{1}{p}} \\ &\quad \times\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)\,dx\,dt\right]^{\frac{1}{q}}. \end{aligned} \tag{2.24}$$

By (2.20), (2.24) yields

$$J_2 \le \frac{C}{R^4}[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right]. \tag{2.25}$$

Combining (2.5), (2.21), and (2.25), we obtain, for large R,

$$\begin{aligned} I_R &= J_1 + J_2 \\ &\le \frac{C}{R^2}[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] + \frac{C}{R^4}[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\ &\le C[I_R]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}-2}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}-2}\right], \end{aligned} \tag{2.26}$$

which yields

$$I_R^{\frac{1}{q}} \le CR^{\frac{2}{q}-2}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}-2}. \tag{2.27}$$

If $p \in (1, 1+\frac{2+m}{\alpha})$, then $-\frac{m}{p}+\frac{2+\alpha}{q}-2 < 0$. Let $R \to \infty$, we have

$$\int_0^\infty\int_{\mathbb{M}^n} W(x)|u|^p(x,t)\,dx\,dt = 0. \tag{2.28}$$

Hence, (2.28) is a contradiction when R is large. This is because the left-hand side of (2.28) is positive and non-decreasing while $R \longrightarrow \infty$.

If $p = 1+\frac{2+m}{\alpha}$, then $-\frac{m}{p}+\frac{2+\alpha}{q}-2 = 0$. Therefore, when R is large, (2.27) becomes

$$I_R \le C\left[CR^{\frac{2}{q}-2}\ln R + C\right]^q \le C. \tag{2.29}$$

This shows

$$\int_0^\infty\int_{\mathbb{M}^n} W(x)u^p(x,t)\,dx\,dt = \lim_{R\to\infty} I_R < \infty. \tag{2.30}$$

Hence

$$\lim_{R\to\infty}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W(x)u^p(x,t)\,dx\,dt = 0 \tag{2.31}$$

and

$$\lim_{R\to\infty}\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W(x)u^p(x,t)\,dx\,dt = 0. \tag{2.32}$$

Using the last two equalities, (2.19) and (2.24) again, we obtain

$$\int_0^\infty\int_{\mathbb{M}^n} W(x)|u|^p(x,t)\,dx\,dt = \lim_{R\to\infty} I_R = 0. \tag{2.33}$$

This is a contradiction.

Thus, the proof of Theorem 1.1 is completed. □

3 Local existence and uniqueness

Proof of Theorem 1.2 Let $u(\cdot,t):\mathbb{M}^n \longrightarrow \mathbb{R}^{n+1}$ be a one-parameter family of smooth hypersurface immersions in $\mathbb{R}^{n+1}$ and $g = \{g_{ij}\}$ be the induced metric on $\mathbb{M}$ in a local coordinate system $\{x^i\}$ $(1 \le i \le n)$.

Noting

$$\Delta u = \Delta_g u = g^{ij}\nabla_i\nabla_j u = g^{ij}\left(\frac{\partial^2 u}{\partial x^i\,\partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right), \tag{3.1}$$

the wave equation (1.1) can be equivalently rewritten as

$$u_{tt}(x,t) = g^{ij}\left(\frac{\partial^2 u}{\partial x^i\,\partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right) + W(x)|u|^p(x,t) - u_t(x,t). \tag{3.2}$$

Since

$$\Gamma^k_{ij} = g^{kl}\left(\frac{\partial^2 u}{\partial x^i\,\partial x^j}, \frac{\partial u}{\partial x^l}\right), \tag{3.3}$$

it follows that

$$u_{tt}(x,t) = g^{ij}\frac{\partial^2 u}{\partial x^i\,\partial x^j} - g^{ij}g^{kl}\left(\frac{\partial^2 u}{\partial x^i\,\partial x^j}, \frac{\partial u}{\partial x^l}\right)\frac{\partial u}{\partial x^k} + W(x)|u|^p(x,t) - u_t(x,t). \tag{3.4}$$

We note that equation (3.4) is not strictly hyperbolic. Therefore, in order to consider equation (3.4), we need to follow a trick of DeTurck [13] by modifying the flow through a diffeomorphism of $\mathbb{M}^n$, under which (3.4) turns out to be strictly hyperbolic, so that we can apply the standard theory of hyperbolic equations.

Suppose $\hat{u}(x,t)$ is a solution of equation (3.2) and $\phi_t:\mathbb{M}^n \longrightarrow \mathbb{M}^n$ is a family of diffeomorphisms of $\mathbb{M}^n$. Let

$$u(x,t) = \phi_t^*\hat{u}(x,t), \tag{3.5}$$

where ϕ_t^* is the pull-back operator of ϕ_t. We now want to find the evolution equation for the metric $u(x,t)$.

Denote

$$y(x,t) = \phi_t(x) = \left\{y^1(x,t)y^2(x,t)y^3(x,t)\cdots y^n(x,t)\right\}, \tag{3.6}$$

in local coordinates, and define $y(x,t) = \phi_t(x)$ by the following initial value problem:

$$\begin{cases} \frac{\partial^2 y^\alpha}{\partial t^2} = \frac{\partial y^\alpha}{\partial x^k} g^{jl}(\Gamma^k_{jl} - \hat{\Gamma}^k_{jl}), \\ y^\alpha(x,0) = x^\alpha, \qquad y^\alpha_t(x,0) = 0, \end{cases} \tag{3.7}$$

where $\hat{\Gamma}^k_{jl}$ is the connection corresponding to the initial metric $\hat{g}_{ij}(x)$. Since

$$\Gamma^k_{jl} = \frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^l}\frac{\partial x^k}{\partial y^\gamma}\hat{\Gamma}^\gamma_{\alpha\beta} + \frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j\,\partial x^l}, \tag{3.8}$$

the initial value problem (3.7) can be rewritten as

$$\begin{cases} \frac{\partial^2 y^\alpha}{\partial t^2} = g^{jl}(\frac{\partial^2 y^\alpha}{\partial x^j\,\partial x^l} + \frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^l}\hat{\Gamma}^\alpha_{\beta\gamma} - \frac{\partial y^\alpha}{\partial x^k}\hat{\Gamma}^k_{jl}), \\ y^\alpha(x,0) = x^\alpha, \qquad y^\alpha_t(x,0) = 0. \end{cases} \tag{3.9}$$

Obviously, (3.9) is an initial value problem for a strictly hyperbolic system. On the other hand, we note that

$$\begin{aligned} \Delta_{\hat{g}}\hat{u} &= \hat{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta u = \hat{g}^{\alpha\beta}\left(\frac{\partial^2\hat{u}}{\partial y^\alpha\,\partial y^\beta} - \hat{\Gamma}^\gamma_{\alpha\beta}\frac{\partial\hat{u}}{\partial y^\gamma}\right) \\ &= g^{kl}\frac{\partial y^\alpha}{\partial x^k}\frac{\partial y^\beta}{\partial x^l}\left\{\frac{\partial}{\partial y^\alpha}\left(\frac{\partial u}{\partial x^i}\frac{\partial x^i}{\partial y^\beta}\right) - \frac{\partial u}{\partial x^i}\frac{\partial x^i}{\partial y^\gamma}\hat{\Gamma}^\gamma_{\alpha\beta}\right\} \\ &= g^{kl}\frac{\partial^2 u}{\partial x^k\,\partial x^l} + g^{kl}\frac{\partial y^\alpha}{\partial x^k}\frac{\partial y^\beta}{\partial x^l}\frac{\partial u}{\partial x^i}\frac{\partial^2 x^i}{\partial y^\alpha\,\partial y^\beta} - g^{kl}\frac{\partial u}{\partial x^i}\left(\Gamma^i_{kl} - \frac{\partial x^i}{\partial y^\gamma}\frac{\partial^2 u y^\gamma}{\partial x^k\,\partial x^l}\right) \\ &= g^{ij}\nabla_i\nabla_j u = \Delta_g u. \end{aligned} \tag{3.10}$$

We have

$$\frac{\partial u}{\partial t} = \frac{\partial\hat{u}}{\partial t} + \frac{\partial\hat{u}}{\partial y^k}\frac{\partial y^k}{\partial t}, \tag{3.11}$$

$$\begin{aligned} u_{tt} &= \frac{\partial^2\hat{u}}{\partial y^\alpha\,\partial y^\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2\hat{u}}{\partial t\,\partial y^\beta}\frac{\partial y^\beta}{\partial t} + \frac{\partial^2\hat{u}}{\partial t^2} + \frac{\partial\hat{u}}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial t^2} \\ &= \frac{\partial^2\hat{u}}{\partial y^\alpha\partial y^\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2\hat{u}}{\partial t\,\partial y^\beta}\frac{\partial y^\beta}{\partial t} + \Delta_{\hat{g}}\hat{u} + \frac{\partial u}{\partial x^k}\frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial t^2} \\ &= \Delta_g u + \frac{\partial u}{\partial x^k}g^{ij}\left(\Gamma^k_{ij} - \hat{\Gamma}^i_{kl}\right) + \frac{\partial^2\hat{u}}{\partial y^\alpha\,\partial y^\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2\hat{u}}{\partial t\,\partial y^\beta}\frac{\partial y^\beta}{\partial t} \\ &= g^{ij}\left(\frac{\partial^2 u}{\partial x^i\,\partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right) + \frac{\partial u}{\partial x^k}g^{ij}\left(\Gamma^k_{ij} - \hat{\Gamma}^i_{kl}\right) + \frac{\partial^2\hat{u}}{\partial y^\alpha\partial y^\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2\hat{u}}{\partial t\,\partial y^\beta}\frac{\partial y^\beta}{\partial t} \\ &= g^{ij}\frac{\partial^2 u}{\partial x^i\,\partial x^j} - \frac{\partial u}{\partial x^k}g^{ij}\hat{\Gamma}^i_{kl} + \frac{\partial^2\hat{u}}{\partial y^\alpha\,\partial y^\beta}\frac{\partial y^\alpha}{\partial t}\frac{\partial y^\beta}{\partial t} + 2\frac{\partial^2\hat{u}}{\partial t\,\partial y^\beta}\frac{\partial y^\beta}{\partial t}. \end{aligned} \tag{3.12}$$

By the standard theory of hyperbolic equations (see [14]), we obtain a local existence and uniqueness result. Thus, the proof of Theorem 1.2 is completed. □

Competing interests
The author declares that they have no competing interests.

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References
1. Todorova, G, Yordanov, B: Critical exponent for a nonlinear wave equation with damping. C. R. Acad. Sci., Sér. 1 Math. **300**, 557-562 (2000)
2. Zhang, QS: A blow-up result for a nonlinear wave equation with damping: the critical case. C. R. Acad. Sci., Sér. 1 Math. **333**, 109-114 (2001)
3. Levine, HA: The role of critical exponents in blow-up theorems. SIAM Rev. **32**, 262-288 (1990)
4. Zhang, QS: A new critical phenomenon for semilinear parabolic problems. J. Math. Anal. **219**, 125-139 (1998)
5. Kong, D-X, Liu, K: Wave character of metrics and hyperbolic mean curvature flow. J. Math. Phys. **48**, 1-14 (2007)
6. He, C-L, Kong, D-X, Liu, K: Hyperbolic mean curvature flow. J. Differ. Equ. **246**, 373-390 (2009)
7. Zhu, X-P: Lectures on Mean Curvature Flows. Stud. Adv. Math., vol. 32. Am. Math. Soc./International Press, Providence (2002)
8. Zhang, QS: Blow-up results for nonlinear parabolic equations on manifolds. Duke Math. J. **97**, 515-539 (1999)
9. Wakasugi, Y: Critical exponent for the semilinear wave equation with scale invariant damping. In: Ruzhansky, M, Turunen, V (eds.) Trends in Mathematics. Trends in Mathematics, pp. 375-390. Birkhäuser, Basel (2014)
10. Li, X: Critical exponent for the semilinear wave equation with critical potential. Nonlinear Differ. Equ. Appl. **20**, 1379-1391 (2013)
11. Lin, J, Nishihara, K, Zhai, J: Critical exponent for the semilinear wave equation with time-dependent damping. Discrete Contin. Dyn. Syst., Ser. A **32**, 4307-4320 (2012)
12. Zhang, QS: Blow-up results for nonlinear parabolic equations on manifolds. Duke Math. J. **97**, 515-539 (1999)
13. DeTurck, D: Some regularity theorems in Riemannian geometry. Ann. Sci. Éc. Norm. Supér. **14**, 249-260 (1981)
14. Hörmander, L: Lectures on Nonlinear Hyperbolic Differential Equations. Math. Appl., vol. 26. Springer, Berlin (1997)