Kang *Boundary Value Problems* (2016) 2016:206
DOI 10.1186/s13661-016-0707-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Asymptotic behavior of the thermoelastic suspension bridge equation with linear memory

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Abstract

This paper is concerned with a thermoelastic suspension bridge equations with memory effects. For the suspension bridge equations without memory, there are many classical results. However, the suspension bridge equations with both viscoelastic and thermal memories were not studied before. The object of the present paper is to provide a result on the global attractor to a thermoelastic suspension bridge equation with past history.

Keywords: global attractors; suspension bridge equation; viscoelasticity; memory; thermoelasticity; asymptotically compact

1 Introduction

In recent years, several authors have been concerned with the asymptotic behavior of the following suspension bridge equations:

$$u_{tt} + \Delta^2 u + ku^+ + \delta u_t + f(u) = h(x), \tag{1.1}$$

where $u(x,t)$ is an unknown function, which represents the deflection of the roadbed in the vertical plane, $k > 0$ denotes the spring constant of the ties, and $\delta > 0$ is a given constant. The force $u^+ = \max\{u, 0\}$ is the positive part of u. The suspension bridge equations are an important mathematical model in engineering. Lazer and McKenna [1] investigated the problem of nonlinear oscillation in a suspension bridge. Lately, similar models have been considered by many authors, most of them concentrating on the existence and decay estimates of solutions; see [2–4] and references therein. Ma and Zhong [5] and Zhong et al. [6] proved the existence of global attractors of weak and strong solutions for equation (1.1), respectively. Park and Kang [7] showed the existence of pullback attractor for a nonautonomous suspension bridge equation with linear damping and in [8] obtained the existence of global attractors for the suspension bridge equations with nonlinear damping. Besides, the problem of attractor of the solutions to a coupled system of suspension bridge equations has been studied by several authors [9–12]. Recently, Kang [13] proved the long-time behavior to the suspension bridge equation when the unique damping mechanism is given by the memory term. We construct some proper Lyapunov functions to show the existence of global attractors. The asymptotic behavior of a thermoelastic system has been widely investigated by many authors. In particular, the stability of a thermoelastic system with memory was proved by several authors [14–17]. To the best of our knowledge, problem (1.1) was not earlier considered in a thermoelasticity point of view. Since thermal effect is a major feature in the theory of elastic plates, we intend to investigate the dynamical behavior of a thermoelastic version of problem (1.1). This paper is concerned with long-time behavior of a solution to the following thermoelastic suspension bridge equation with linear memory:

$$\begin{aligned} &u_{tt} + \alpha\Delta^2 u - \Delta u_t + ku^+ - \int_0^\infty \mu(s)\Delta^2 u(t-s)\,ds + \beta\Delta\theta + f(u) \\ &\quad = h(x) \quad \text{in } \Omega \times \mathbb{R}^+, \end{aligned} \tag{1.2}$$

$$\theta_t - \Delta\theta - \beta\Delta u_t - \int_0^\infty \kappa(s)\Delta\theta(t-s)\,ds = 0 \quad \text{on } \Omega \times \mathbb{R}^+, \tag{1.3}$$

$$u = 0, \qquad \Delta u = 0, \qquad \theta = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.4}$$

$$\begin{aligned} &u(x,t) = u_0(x,t), \qquad u_t(x,t) = u_1(x,t), \\ &\theta(x,t) = \theta_0(x,t), \quad (x,t) \in \Omega \times (-\infty, 0], \end{aligned} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with sufficiently smooth or rectangular boundary $\partial\Omega$, and Δ denotes the Laplace operator. Here α is the flexural rigidity of the structure, and $\beta > 0$ provides connection between deflection and temperature and depends on mechanical and thermal properties of the material. The initial conditions $u_0, \theta_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ are the prescribed past histories of u and θ, respectively. It is well known that $u = u(x,t)$ represents the deflection of the roadbed in the vertical plane and $\theta = \theta(x,t)$ is the temperature difference with respect to a fixed reference temperature. Memory kernels $\mu(s)$ and $\kappa(s)$ are supposed to be smooth decreasing convex functions vanishing at infinity.

The only way to associate a process with such equations is to view the past history of u and θ as new variables of the system, which will be ruled by a supplementary equation. To formulate system (1.2)-(1.5) in a history space setting, as in [18–21], we define new variables η and ζ by

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \qquad \zeta^t(x,s) = \int_0^s \theta(x,t-y)\,dy, \quad (x,s) \in \Omega \times \mathbb{R}^+, t \geq 0.$$

Formally, they satisfy the linear equations

$$\eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t), \qquad \zeta_t^t(x,s) + \zeta_s^t(x,s) = \theta(x,t), \quad (x,s) \in \Omega \times \mathbb{R}^+, t \geq 0,$$

and

$$\eta^t(x,0) = 0, \qquad \zeta^t(x,0) = 0, \quad x \in \Omega, t \geq 0,$$

whereas

$$\eta^0(x,s) = u_0(x,0) - u_0(x,-s), \qquad \zeta^0(x,s) = \int_0^s \theta_0(x,-y)\,dy, \quad (x,s) \in \Omega \times \mathbb{R}^+.$$

Assuming that $\mu, \nu \in L^1(\mathbb{R}^+)$ and taking $\alpha = 1 + \int_0^\infty \mu(s)\,ds$ and $\nu(s) = -\kappa'(s)$, problem (1.2)-(1.5) can be transformed into the equivalent system

$$u_{tt} + \Delta^2 u - \Delta u_t + ku^+ + \int_0^\infty \mu(s)\Delta^2\eta^t(s)\,ds + \beta\Delta\theta + f(u) = h(x) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.6}$$

$$\theta_t - \Delta\theta - \beta\Delta u_t - \int_0^\infty \nu(s)\Delta\zeta^t(s)\,ds = 0 \quad \text{on } \Omega \times \mathbb{R}^+, \tag{1.7}$$

$$\eta_t^t + \eta_s^t = u_t, \quad (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.8}$$

$$\zeta_t^t + \zeta_s^t = \theta, \quad (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.9}$$

with boundary conditions

$$\begin{aligned} &u = \Delta u = 0, \qquad \theta = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ &\eta = \Delta\eta = 0, \qquad \zeta = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{1.10}$$

and initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0(x), \\ &\eta^t(x,0) = 0, \qquad \zeta^t(x,0) = 0, \qquad \eta^0(x,s) = \eta_0(x,s), \qquad \zeta^0(x,s) = \zeta_0(x,s), \end{aligned} \tag{1.11}$$

where

$$\begin{cases} u_0(x) = u_0(x,0), \qquad u_1(x) = \partial_t u_0(x,t)|_{t=0}, \qquad \theta_0(x) = \theta_0(x,0), \quad x \in \Omega, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s), \qquad \zeta_0(x,s) = \int_0^s \theta_0(x,-y)\,dy, \quad (x,s) \in \Omega \times \mathbb{R}^+. \end{cases}$$

Because h is independent of time, the initial-boundary value problem (1.6)-(1.11) is in fact an autonomous dynamical system with respect to the unknown pair $(u(t), u_t(t), \theta(t), \eta^t, \zeta^t)$. In order to settle (1.2)-(1.5) in the framework of dynamical systems, we investigate modified equations (1.6)-(1.11). Indeed, it turns out that they are the same thing; to be more precise, the modified equations are in fact a generalization of the original equations. In the past years, the asymptotic behavior of viscoelastic equations with past history has been studied by many authors (see [22–27]).

We formulate our assumptions and results with respect to these new systems. The hypotheses and the well-posedness for the system (1.6)-(1.11) are presented in Section 2. Also, we give some notation and fundamental results of infinite-dimensional dynamical systems. In Section 3, we establish our main result on the existence of a compact global attractor.

2 Preliminaries

Now we introduce the Hilbert spaces that will be used in our analysis. Let

$$V_0 = L^2(\Omega), \qquad V_1 = H_0^1(\Omega), \qquad V_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

As usual, $(\cdot,\cdot)$ denotes the L^2-inner product, and $\|\cdot\|_p$ denotes the L^p-norm. We consider the history spaces $L^2_\mu(\mathbb{R}^+; V_2)$ and $L^2_\nu(\mathbb{R}^+; V_1)$ of measurable functions η with values in V_2

or V_1, respectively, such that

$$\|\eta\|^2_{\mu,V_2} = \int_0^\infty \mu(s)\big\|\Delta\eta(s)\big\|^2\,ds < \infty$$

and

$$\|\eta\|^2_{\nu,V_1} = \int_0^\infty \nu(s)\big\|\nabla\eta(s)\big\|^2\,ds < \infty.$$

The following Cartesian product of Hilbert spaces will play the role of a phase space for the considered model:

$$\mathcal{H} = V_2 \times V_0 \times V_0 \times L^2_\mu\big(\mathbb{R}^+; V_2\big) \times L^2_\nu\big(\mathbb{R}^+; V_1\big)$$

with the norm

$$\big\|(u,v,\theta,\eta,\zeta)\big\|^2_{\mathcal{H}} = \|\Delta u\|^2 + \|v\|^2 + \|\theta\|^2 + \|\eta\|^2_{\mu,V_2} + \|\zeta\|^2_{\nu,V_1}.$$

Let λ_1 and λ be the best constants in the Poincáre inequalities

$$\lambda_1\|u\|^2 \le \|\nabla u\|^2, \qquad \lambda\|u\|^2 \le \|\Delta u\|^2, \tag{2.1}$$

respectively.

We assume that $h \in L^2(\Omega)$ and the forcing term $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(0) = 0, \qquad \big|f(u) - f(v)\big| \le k_0\big(1 + |u|^p + |v|^p\big)|u - v|, \quad u, v \in \mathbb{R}, \tag{2.2}$$

where $k_0 > 0$ and $p > 0$. This implies that $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^{2(p+1)}(\Omega)$. Besides, we assume that, for some $k_1 \ge 0$,

$$-k_1 \le F(u) \le f(u)u, \quad u \in \mathbb{R}, \tag{2.3}$$

where $F(z) = \int_0^z f(s)\,ds$.

In addition, with respect to the memory kernels $\mu(s), \nu(s) \ge 0$, we assume that

$$\mu, \nu \in C^1\big(\mathbb{R}^+\big) \cap L^1\big(\mathbb{R}^+\big), \qquad \int_0^\infty \mu(s)\,ds = \mu_0 > 0, \qquad \int_0^\infty \nu(s)\,ds = \nu_0 > 0, \tag{2.4}$$

and that there exist constants $k_2, k_3 > 0$ such that

$$\mu'(s) \le -k_2\mu(s), \qquad \nu'(s) \le -k_3\nu(s), \quad s \ge 0. \tag{2.5}$$

The well-posedness of problem (1.6)-(1.11) can be obtained by the Faedo-Galerkin method (see [4, 5, 28]). For the problem involving a memory term, we follow arguments from [20, 21].

Theorem 2.1 *Under assumptions* (2.2)-(2.5), *we have*

(i) *For every initial data* $(u_0, u_1, \theta_0, \eta_0, \zeta_0) \in \mathcal{H}$, *problem* (1.6)-(1.11) *has a weak solution*

$$(u, u_t, \theta, \eta, \zeta) \in C\big([0, T]; \mathcal{H}\big), \quad T > 0,$$

satisfying

$$\begin{aligned} &u \in L^\infty(0, T; V_2), \qquad u_t, \theta \in L^\infty(0, T; V_0), \\ &\eta \in L^\infty\big(0, T; L^2_\mu\big(\mathbb{R}^+; V_2\big)\big), \qquad \zeta \in L^\infty\big(0, T; L^2_\nu\big(\mathbb{R}^+; V_1\big)\big). \end{aligned}$$

(ii) *The weak solutions depend continuously on the initial data in* $\mathcal{H}$. *More precisely, given any two weak solutions* z_1, z_2 *of problem* (1.6)-(1.11), *we have*

$$\big\|z_1(t) - z_2(t)\big\|_{\mathcal{H}} \le e^{ct}\big\|z_1(0) - z_2(0)\big\|_{\mathcal{H}}, \quad t \in [0, T],$$

for some constant $c > 0$.

Remark 2.1 The well-posedness of problem (1.6)-(1.11) implies that the solution operator $S(t): \mathcal{H} \to \mathcal{H}$ defined by

$$S(t)(u_0, u_1, \theta_0, \eta_0, \zeta_0) = \big(u(t), u_t(t), \theta(t), \eta^t, \zeta^t\big), \quad t \ge 0, \tag{2.6}$$

satisfies the semigroup properties and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on $\mathcal{H}$. Thus, we can study (1.6)-(1.11) as a nonlinear dynamical system $(\mathcal{H}, S(t))$.

Now, we recall some fundamental results of infinite-dimensional dynamical systems (see [29–31]).

Definition 2.1 Let $S(t)$ be a C_0-semigroup defined in a Banach space X. A global attractor for $(X, S(t))$ is a bounded closed set $\mathcal{A} \subset X$ that is fully invariant and uniformly attracting, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t > 0$, and for every bounded subset $B \subset X$,

$$\lim_{t\to\infty} \operatorname{dist}_X\big(S(t)B, \mathcal{A}\big) = 0,$$

where $\operatorname{dist}_X(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} d(y, z)$ is the Hausdorff semidistance between Y and Z in X.

Definition 2.2 A dynamical system $(X, S(t))$ is dissipative if it possesses a bounded absorbing set, that is, a bounded set $\mathcal{B} \subset X$ such that, for any bounded set $B \subset X$, there exists $t_B \ge 0$ satisfying

$$S(t)B \subset \mathcal{B}, \quad t \ge t_B.$$

Definition 2.3 Let X be a Banach space, and B be a bounded subset of X. We call a function $\phi(\cdot, \cdot)$ defined on $X \times X$ a contractive function on $B \times B$ if for any sequence $\{x_n\}_{n=1}^\infty \subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty} \lim_{l\to\infty} \phi(x_{n_k}, x_{n_l}) = 0.$$

Theorem 2.2 ([30]) *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space $(X, \|\cdot\|)$ that has a bounded absorbing set B_0. Moreover, assume that, for any $\epsilon \geq 0$, there exist $T = T(B_0,\epsilon)$ and $\phi_T(\cdot,\cdot) \in C(B)$ such that*

$$\big\|S(T)x - S(T)y\big\| \leq \epsilon + \phi_T(x,y) \quad \text{for all } x, y \in B_0,$$

where $C(B)$ is a set of all contractive functions on $B \times B$, and ϕ_T depends on T. Then $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X, that is, for any bounded sequence $\{y_n\}_{n=1}^{\infty} \subset X$ and any sequence $\{t_n\}$ with $t_n \to \infty$, $\{S(t_n)y_n\}_{n=1}^{\infty}$ is precompact in X.

Theorem 2.3 ([30]) *A dissipative dynamical system $(X, S(t))$ has a compact global attractor if and only if it is asymptotically compact.*

The main result of this paper is the following:

Theorem 2.4 *Suppose that assumptions* (2.2)-(2.5) *hold. For $k, \beta > 0$ such that*

$$\frac{1}{4} - \frac{3k}{2\lambda} - \frac{\beta}{2} > 0,$$

the dynamical system $(\mathcal{H}, S(t))$ corresponding to system (1.6)-(1.11) *has a compact global attractor $\mathcal{A} \subset \mathcal{H}$.*

3 Global attractor

To show Theorem 2.4, we apply the abstract results presented in the previous section. Accordingly, we shall first prove that the dynamical system $(\mathcal{H}, S(t))$ is dissipative. By Theorem 2.3 we need to verify the asymptotic compactness.

We have the following lemma on the system energy defined by

$$\begin{aligned} E(t) ={}& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{k}{2}\big\|u^+\big\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\big\|\eta^t\big\|^2_{\mu,V_2} + \frac{1}{2}\big\|\zeta^t\big\|^2_{\nu,V_1} \\ &+ \int_\Omega \big(F(u) - hu\big)\,dx. \end{aligned}$$

Lemma 3.1 *Along the solution of* (1.6)-(1.11), *the energy E satisfies*

$$E'(t) = -\|\nabla u_t\|^2 - \|\nabla\theta\|^2 + \frac{1}{2}\int_0^\infty \mu'(s)\big\|\Delta\eta^t(s)\big\|^2\,ds + \frac{1}{2}\int_0^\infty \nu'(s)\big\|\nabla\zeta^t(s)\big\|^2\,ds. \tag{3.1}$$

Proof Multiplying equations (1.6) and (1.7) by u_t and θ, respectively, and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{k}{2}\big\|u^+\big\|^2 + \frac{1}{2}\|\theta\|^2 + \int_\Omega \big(F(u) - hu\big)\,dx\right) \\ &\quad + \|\nabla u_t\|^2 + \|\nabla\theta\|^2 + \big(\eta^t, u_t\big)_{\mu,V_2} + \big(\zeta^t, \theta\big)_{\nu,V_1} = 0. \end{aligned} \tag{3.2}$$

From (1.8) and (1.11) we have

$$\big(\eta^t, u_t\big)_{\mu,V_2} = \big(\eta^t, \eta^t_t + \eta^t_s\big)_{\mu,V_2} = \frac{1}{2}\frac{d}{dt}\big\|\eta^t\big\|^2_{\mu,V_2} - \frac{1}{2}\int_0^\infty \mu'(s)\big\|\Delta\eta^t(s)\big\|^2\,ds. \tag{3.3}$$

Similarly, by (1.9) and (1.11) we obtain

$$\left(\zeta^t,\theta\right)_{\nu,V_1} = \left(\zeta^t,\zeta_t^t+\zeta_s^t\right)_{\nu,V_1} = \frac{1}{2}\frac{d}{dt}\left\|\zeta^t\right\|_{\nu,V_1}^2 - \frac{1}{2}\int_0^\infty \nu'(s)\left\|\nabla\zeta^t(s)\right\|^2 ds. \tag{3.4}$$

Combining (3.3) and (3.4) with (3.2), we get estimate (3.1). □

To this system, we define the Lyapunov functional

$$L(t) = ME(t) + \epsilon\phi(t) + \psi(t)$$

with

$$\phi(t) = \int_\Omega u_t(t)u(t)\,dx,$$
$$\psi(t) = -\int_\Omega u_t(t)\int_0^\infty \mu(s)\eta^t(s)\,ds\,dx - \int_\Omega \theta(t)\int_0^\infty \nu(s)\zeta^t(s)\,ds\,dx,$$

where $\epsilon > 0$ and $M > 0$ are to be fixed later.

Lemma 3.2 *For $M > 0$ sufficiently large, there exist positive constants q_0, q_1, and C_2 such that*

$$q_0E(t) - C_2|\Omega| - C_2\|h\|^2 \le L(t) \le q_1E(t) + C_2|\Omega| + C_2\|h\|^2, \quad t \ge 0, \tag{3.5}$$

for any $0 < \epsilon \le 1$.

Proof The Young inequality, (2.1), and (2.3) give that

$$\int_\Omega \left(F(u) - hu\right)dx \ge -k_1|\Omega| - \frac{1}{4}\|\Delta u\|^2 - \frac{1}{\lambda}\|h\|^2. \tag{3.6}$$

Then by energy and (3.6) we have

$$\frac{1}{4}\left\|\left(u(t),u_t(t),\theta(t),\eta^t,\zeta^t\right)\right\|_{\mathcal{H}}^2 \le E(t) + k_1|\Omega| + \frac{1}{\lambda}\|h\|^2. \tag{3.7}$$

From the Young inequality, (2.1), (2.4), and (3.7) we conclude that

$$\left|\phi(t)\right| \le \frac{1}{2}\|u_t\|^2 + \frac{1}{2\lambda}\|\Delta u\|^2 \le 2\max\left\{1,\frac{1}{\lambda}\right\}\left(E(t) + k_1|\Omega| + \frac{1}{\lambda}\|h\|^2\right),$$
$$\left|\psi(t)\right| \le \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{\mu_0}{2\lambda}\left\|\eta^t\right\|_{\mu,V_2}^2 + \frac{\nu_0}{2\lambda_1}\left\|\zeta^t\right\|_{\nu,V_1}^2$$
$$\le 2\max\left\{1,\frac{\mu_0}{\lambda},\frac{\nu_0}{\lambda_1}\right\}\left(E(t) + k_1|\Omega| + \frac{1}{\lambda}\|h\|^2\right).$$

Choosing $C_1 = 2\max\{1,\frac{1}{\lambda},\frac{\mu_0}{\lambda},\frac{\nu_0}{\lambda_1}\}$, for some $C_2 > 0$, we obtain

$$\left|L(t) - ME(t)\right| \le \epsilon\left|\phi(t)\right| + \left|\psi(t)\right| \le C_2\left(E(t) + |\Omega| + \|h\|^2\right), \quad 0 < \epsilon \le 1.$$

Then, taking $M > C_2$, we get inequality (3.5) with $q_0 = M - C_2$ and $q_1 = M + C_2$. □

Lemma 3.3 *We have the inequality*

$$\phi'(t) \le -E(t) - \left(\frac{1}{4} - \frac{3k}{2\lambda} - \frac{\beta}{2}\right)\|\Delta u\|^2 + \frac{3}{2}\|u_t\|^2 + \frac{1}{\lambda_1\beta}\|\nabla u_t\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{\beta}{\lambda_1}\|\nabla\theta\|^2 + \left(\mu_0 + \frac{1}{2}\right)\|\eta^t\|^2_{\mu,V_2} + \frac{1}{2}\|\zeta^t\|^2_{\nu,V_1}. \tag{3.8}$$

Proof Using (1.6) and (2.3) and subtracting and adding $E(t)$, we obtain

$$\begin{aligned}\phi'(t) &= -\|\Delta u\|^2 - \int_\Omega \nabla u_t \nabla u\,dx - k\int_\Omega u^+ u\,dx - \int_0^\infty \mu(s)\left(\int_\Omega \Delta\eta^t(s)\Delta u\,dx\right)ds \\ &\quad + \beta\int_\Omega \nabla\theta\nabla u\,dx - \int_\Omega f(u)u\,dx + \int_\Omega hu\,dx + \|u_t\|^2 \\ &\le -E(t) - \frac{1}{2}\|\Delta u\|^2 + \frac{3}{2}\|u_t\|^2 + \frac{k}{2}\|u^+\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mu,V_2} + \frac{1}{2}\|\zeta^t\|^2_{\nu,V_1} \\ &\quad - \int_\Omega \nabla u_t\nabla u\,dx - k\int_\Omega u^+ u\,dx - \int_0^\infty \mu(s)\left(\int_\Omega \Delta\eta^t(s)\Delta u\,dx\right)ds \\ &\quad + \beta\int_\Omega \nabla\theta\nabla u\,dx. \end{aligned} \tag{3.9}$$

Note that

$$\left|-\int_0^\infty \mu(s)\left(\int_\Omega \Delta\eta^t(s)\Delta u\,dx\right)ds\right| \le \frac{1}{4}\|\Delta u\|^2 + \mu_0\|\eta^t\|^2_{\mu,V_2} \tag{3.10}$$

by assumption (2.4). From (2.1), the Young inequality, and the inequality $|u^+| \le |u|$ we see that

$$\left|-\int_\Omega \nabla u_t\nabla u\,dx\right| \le \frac{\beta}{4}\|\Delta u\|^2 + \frac{1}{\lambda_1\beta}\|\nabla u_t\|^2, \tag{3.11}$$

$$\left|\beta\int_\Omega \nabla\theta\nabla u\,dx\right| \le \frac{\beta}{4}\|\Delta u\|^2 + \frac{\beta}{\lambda_1}\|\nabla\theta\|^2, \tag{3.12}$$

$$\left|-k\int_\Omega u^+ u\,dx\right| \le \frac{k}{\lambda}\|\Delta u\|^2. \tag{3.13}$$

Substituting (3.10)-(3.13) into (3.9), we get estimate (3.8). □

Lemma 3.4 *There exist positive constants C_3 and C_4 such that*

$$\begin{aligned}\psi'(t) &\le -\frac{\mu_0}{2}\|u_t\|^2 - \frac{\nu_0}{2}\|\theta\|^2 + \left(\delta + \frac{\delta k^2}{\lambda} + \delta k_0 C_s C_E^p\right)\|\Delta u\|^2 + \left(\delta + \delta\beta^2\right)\|\nabla u_t\|^2 \\ &\quad + \left(\delta + \delta\beta^2\right)\|\nabla\theta\|^2 + C_3\|\eta^t\|^2_{\mu,V_2} + C_4\|\zeta^t\|^2_{\nu,V_1} + \frac{1}{2}\|h\|^2 \\ &\quad - \frac{\mu(0)}{2\mu_0\lambda}\int_0^\infty \mu'(s)\|\Delta\eta^t(s)\|^2\,ds - \frac{\nu(0)}{2\nu_0\lambda_1}\int_0^\infty \nu'(s)\|\nabla\zeta^t(s)\|^2\,ds, \end{aligned} \tag{3.14}$$

where C_3 depends on μ_0, λ_1, λ, and δ, and C_4 depends on ν_0 and δ.

Proof Taking the derivative of the function ψ and using equations (1.6)-(1.9), we have

$$\begin{aligned}\psi'(t) = &\int_\Omega\left(\Delta^2 u - \Delta u_t + ku^+ + \int_0^\infty \mu(s)\Delta^2\eta^t(s)\,ds + \beta\Delta\theta + f(u) - h\right)\\ &\times\int_0^\infty \mu(s)\eta^t(s)\,ds\,dx\\ &-\int_\Omega\left(\Delta\theta + \beta\Delta u_t + \int_0^\infty \nu(s)\Delta\zeta^t(s)\,ds\right)\int_0^\infty \nu(s)\zeta^t(s)\,ds\,dx\\ &-\int_\Omega u_t\int_0^\infty \mu(s)\big(u_t - \eta_s^t(s)\big)\,ds\,dx - \int_\Omega\theta\int_0^\infty \nu(s)\big(\theta - \zeta_s^t(s)\big)\,ds\,dx. \end{aligned}\tag{3.15}$$

From the Young inequality, (2.1), and (2.4) we derive that, for any $\delta > 0$,

$$\left|\int_\Omega \Delta u\int_0^\infty \mu(s)\Delta\eta^t(s)\,ds\,dx\right| \le \delta\|\Delta u\|^2 + \frac{\mu_0}{4\delta}\|\eta^t\|^2_{\mu,V_2},\tag{3.16}$$

$$\left|\int_\Omega \nabla u_t\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\,dx\right| \le \delta\|\nabla u_t\|^2 + \frac{\mu_0}{4\delta\lambda_1}\|\eta^t\|^2_{\mu,V_2},\tag{3.17}$$

$$\left|k\int_\Omega u^+\int_0^\infty \mu(s)\eta^t(s)\,ds\,dx\right| \le \frac{\delta k^2}{\lambda}\|\Delta u\|^2 + \frac{\mu_0}{4\delta\lambda}\|\eta^t\|^2_{\mu,V_2},\tag{3.18}$$

$$\left|\int_\Omega\left(\int_0^\infty \mu(s)\Delta\eta^t(s)\,ds\right)^2 dx\right| \le \mu_0\|\eta^t\|^2_{\mu,V_2},\tag{3.19}$$

$$\left|-\beta\int_\Omega \nabla\theta\int_0^\infty \mu(s)\nabla\eta^t(s)\,ds\,dx\right| \le \delta\beta^2\|\nabla\theta\|^2 + \frac{\mu_0}{4\delta\lambda_1}\|\eta^t\|^2_{\mu,V_2},\tag{3.20}$$

$$\left|-\int_\Omega h\int_0^\infty \mu(s)\eta^t(s)\,ds\,dx\right| \le \frac{1}{2}\|h\|^2 + \frac{\mu_0}{2\lambda}\|\eta^t\|^2_{\mu,V_2}.\tag{3.21}$$

Using (2.2), (3.7), and the Sobolev embedding, since $E(t)$ is decreasing, we obtain

$$\begin{aligned}\left|\int_\Omega f(u)\int_0^\infty \mu(s)\eta^t(s)\,ds\,dx\right| &\le \int_\Omega k_0\big(1+|u|^p\big)|u|\left|\int_0^\infty \mu(s)\eta^t(s)\,ds\right|dx\\ &\le k_0\big(1+\|u\|^p_{2(p+1)}\big)\|u\|_{2(p+1)}\left\|\int_0^\infty \mu(s)\eta^t(s)\,ds\right\|\\ &\le \delta k_0 C_s C_E^p\|\Delta u\|^2 + \frac{\mu_0}{4\delta\lambda}\|\eta^t\|^2_{\mu,V_2}, \end{aligned}\tag{3.22}$$

where $C_E = 2(E(0) + k_1|\Omega| + \frac{1}{\lambda}\|h\|^2)^{1/2}$. Moreover, it follows that

$$\begin{aligned}\left|\int_\Omega u_t\int_0^\infty \mu(s)\eta_s^t(s)\,ds\,dx\right| &= \left|-\int_\Omega u_t\int_0^\infty \mu'(s)\eta^t(s)\,ds\,dx\right|\\ &\le \frac{\mu_0}{2}\|u_t\|^2 - \frac{\mu(0)}{2\mu_0\lambda}\int_0^\infty \mu'(s)\big\|\Delta\eta^t(s)\big\|^2\,ds. \end{aligned}\tag{3.23}$$

Similarly, we find that, for any $\delta > 0$,

$$\left|\int_\Omega \nabla\theta\int_0^\infty \nu(s)\nabla\zeta^t(s)\,ds\,dx\right| \le \delta\|\nabla\theta\|^2 + \frac{\nu_0}{4\delta}\|\zeta^t\|^2_{\nu,V_1},\tag{3.24}$$

$$\left|\beta\int_\Omega \nabla u_t\int_0^\infty \nu(s)\nabla\zeta^t(s)\,ds\,dx\right| \le \delta\beta^2\|\nabla u_t\|^2 + \frac{\nu_0}{4\delta}\|\zeta^t\|^2_{\nu,V_1},\tag{3.25}$$

$$\left|\int_\Omega\left(\int_0^\infty \nu(s)\nabla\zeta^t(s)\,ds\right)^2 dx\right| \le \nu_0\|\zeta^t\|^2_{\nu,V_1}, \tag{3.26}$$

$$\left|\int_\Omega \theta\int_0^\infty \nu(s)\zeta^t_s(s)\,ds\,dx\right| \le \frac{\nu_0}{2}\|\theta\|^2 - \frac{\nu(0)}{2\nu_0\lambda_1}\int_0^\infty \nu'(s)\|\nabla\zeta^t(s)\|^2\,ds. \tag{3.27}$$

Inserting (3.16)-(3.27) into (3.15), we deduce that

$$\begin{aligned}\psi'(t) \le{}& -\frac{\mu_0}{2}\|u_t\|^2 - \frac{\nu_0}{2}\|\theta\|^2 + \left(\delta + \frac{\delta k^2}{\lambda} + \delta k_0 C_s C_E^p\right)\|\Delta u\|^2\\ &+ \left(\delta + \delta\beta^2\right)\|\nabla u_t\|^2 + \left(\delta+\delta\beta^2\right)\|\nabla\theta\|^2\\ &+ \left(\mu_0 + \frac{\mu_0}{4\delta} + \frac{\mu_0}{2\lambda} + \frac{\mu_0}{2\delta\lambda_1} + \frac{\mu_0}{2\delta\lambda}\right)\|\eta^t\|^2_{\mu,V_2} + \left(\nu_0 + \frac{\nu_0}{2\delta}\right)\|\zeta^t\|^2_{\nu,V_1}\\ &+ \frac{1}{2}\|h\|^2 - \frac{\mu(0)}{2\mu_0\lambda}\int_0^\infty \mu'(s)\|\Delta\eta^t(s)\|^2\,ds - \frac{\nu(0)}{2\nu_0\lambda_1}\int_0^\infty \nu'(s)\|\nabla\zeta^t(s)\|^2\,ds. \qquad \square\end{aligned}$$

Lemma 3.5 *Suppose that conditions* (2.2)-(2.5) *hold. Then the dynamical system* $(\mathcal{H}, S(t))$ *corresponding to problem* (1.6)-(1.11) *has a bounded absorbing set* $\mathcal{B}\subset\mathcal{H}$.

Proof From (2.5), (3.1), (3.8), and (3.14) we see that

$$\begin{aligned}L'(t) \le{}& -\epsilon E(t) - \left(\frac{\mu_0}{2} - \frac{3\epsilon}{2}\right)\|u_t\|^2 - \left(\frac{\nu_0}{2} - \frac{\epsilon}{2}\right)\|\theta\|^2\\ &- \left(M - \frac{\epsilon}{\lambda_1\beta} - \left(1+\beta^2\right)\delta\right)\|\nabla u_t\|^2 - \left(M - \frac{\beta\epsilon}{\lambda_1} - \left(1+\beta^2\right)\delta\right)\|\nabla\theta\|^2\\ &- \left[\left(\frac{1}{4} - \frac{3k}{2\lambda} - \frac{\beta}{2}\right)\epsilon - \left(1 + \frac{k^2}{\lambda} + k_0 C_s C_E^p\right)\delta\right]\|\Delta u\|^2\\ &+ \left(\frac{M}{2} - \frac{\epsilon}{k_2}\left(\mu_0 + \frac{1}{2}\right) - \frac{C_3}{k_2} - \frac{\mu(0)}{2\mu_0\lambda}\right)\int_0^\infty \mu'(s)\|\Delta\eta^t(s)\|^2\,ds\\ &+ \left(\frac{M}{2} - \frac{\epsilon}{2k_3} - \frac{C_4}{k_3} - \frac{\nu(0)}{2\nu_0\lambda_1}\right)\int_0^\infty \nu'(s)\|\nabla\zeta^t(s)\|^2\,ds + \frac{1}{2}\|h\|^2.\end{aligned}$$

We choose ϵ so small that

$$\frac{\mu_0}{2} - \frac{3\epsilon}{2} > 0, \qquad \frac{\nu_0}{2} - \frac{\epsilon}{2} > 0.$$

For k, β such that $\frac{1}{4} - \frac{3k}{2\lambda} - \frac{\beta}{2} > 0$ and fixed ϵ, we take $\delta > 0$ small enough such that

$$\left(\frac{1}{4} - \frac{3k}{2\lambda} - \frac{\beta}{2}\right)\epsilon - \left(1 + \frac{k^2}{\lambda} + k_0 C_s C_E^p\right)\delta > 0.$$

Finally, we choose $M > 0$ large enough such that

$$\begin{aligned}M > \max\biggl\{&\frac{\epsilon}{\lambda_1\beta} + \left(1+\beta^2\right)\delta, \frac{\beta\epsilon}{\lambda_1} + \left(1+\beta^2\right)\delta,\\ &\frac{\epsilon}{k_2}(2\mu_0+1) + \frac{2C_3}{k_2} + \frac{\mu(0)}{\mu_0\lambda}, \frac{\epsilon}{k_3} + \frac{2C_4}{k_3} + \frac{\nu(0)}{\nu_0\lambda_1}\biggr\}.\end{aligned}$$

Then we deduce that

$$L'(t) \le -\epsilon E(t) + \frac{1}{2}\|h\|^2.$$

From (3.5) we get

$$L'(t) \le -\frac{\epsilon}{q_1}L(t) + \frac{\epsilon C_2}{q_1}\left(|\Omega| + \|h\|^2\right) + \frac{1}{2}\|h\|^2,$$

which implies that

$$\begin{aligned} L(t) &\le L(0)e^{-\frac{\epsilon}{q_1}t} + \left(\frac{\epsilon C_2}{q_1} + \frac{1}{2}\right)\left(|\Omega| + \|h\|^2\right)\int_0^t e^{-\frac{\epsilon}{q_1}(t-s)}\,ds \\ &= \left[L(0) - \left(C_2 + \frac{q_1}{2\epsilon}\right)\left(|\Omega| + \|h\|^2\right)\right]e^{-\frac{\epsilon}{q_1}t} + \left(C_2 + \frac{q_1}{2\epsilon}\right)\left(|\Omega| + \|h\|^2\right). \end{aligned}$$

Using (3.5) again, we have

$$E(t) \le \frac{q_1}{q_0}E(0)e^{-\frac{\epsilon}{q_1}t} + \left(\frac{2C_2}{q_0} + \frac{q_1}{2\epsilon q_0}\right)\left(|\Omega| + \|h\|^2\right), \quad t \ge 0.$$

Consequently, (3.7) infers that

$$\left\|\left(u(t), u_t(t), \theta(t), \eta^t, \zeta^t\right)\right\|_{\mathcal{H}}^2 \le CE(0)e^{-\frac{\epsilon}{q_1}t} + C\left(|\Omega| + \|h\|^2\right), \tag{3.28}$$

where $C = 4\max\{\frac{q_1}{q_0}, \frac{2C_2}{q_0} + \frac{q_1}{2\epsilon q_0} + \frac{1}{\lambda} + k_1\}$ is a positive constant. Thus, taking the closed ball $\mathcal{B} = \bar{B}(0,R)$ with $R = \sqrt{2C(|\Omega| + \|h\|^2)}$, we conclude from (3.28) that $\mathcal{B}$ is a bounded absorbing set of $(\mathcal{H}, S(t))$. □

Lemma 3.6 *Under the hypotheses of Theorem* 2.4, *given a bounded set* $B \subset \mathcal{H}$, *let* $z_1 = (u, u_t, \theta, \eta, \zeta)$ *and* $z_2 = (\tilde{u}, \tilde{u}_t, \tilde{\theta}, \tilde{\eta}, \tilde{\zeta})$ *be two weak solutions of system* (1.6)-(1.11) *with corresponding initial conditions* $z_1(0) = (u_0, u_1, \theta_0, \eta_0, \zeta_0)$ *and* $z_2(0) = (\tilde{u}_0, \tilde{u}_1, \tilde{\theta}_0, \tilde{\eta}_0, \tilde{\zeta}_0) \in B$. *Then there exist positive constants* γ, $\tilde{C}_0$, *and* $\tilde{C}_1$ *depending on* B *such that*

$$\begin{aligned} \left\|z_1(t) - z_2(t)\right\|_{\mathcal{H}}^2 &\le \tilde{C}_0 e^{-\gamma t}\left\|z_1(0) - z_2(0)\right\|_{\mathcal{H}}^2 \\ &\quad + \tilde{C}_1\int_0^t e^{-\gamma(t-s)}\left\|u(s) - \tilde{u}(s)\right\|_{2(p+1)}^2\,ds, \quad t \ge 0. \end{aligned} \tag{3.29}$$

Proof We set $w = u - \tilde{u}$, $\vartheta = \theta - \tilde{\theta}$, $\xi = \eta - \tilde{\eta}$, and $\tau = \zeta - \tilde{\zeta}$. Then $(w, w_t, \vartheta, \xi, \tau)$ is a weak solution of

$$w_{tt} + \Delta^2 w - \Delta w_t + ku^+ - k\tilde{u}^+ + \int_0^\infty \mu(s)\Delta^2\xi^t(s)\,ds + \beta\Delta\vartheta + f(u) - f(\tilde{u}) = 0, \tag{3.30}$$

$$\vartheta_t - \Delta\vartheta - \beta\Delta w_t - \int_0^\infty \nu(s)\Delta\tau^t(s)\,ds = 0, \tag{3.31}$$

$$\xi_t = -\xi_s + w_t, \tag{3.32}$$

$$\tau_t = -\tau_s + \vartheta, \tag{3.33}$$

with initial conditions

$$\begin{aligned}
&w(0) = u_0 - \tilde{u}_0, \qquad w_t(0) = u_1 - \tilde{u}_1, \qquad \vartheta(0) = \theta_0 - \tilde{\theta}_0, \\
&\xi^0 = \eta_0 - \tilde{\eta}_0, \qquad \tau^0 = \zeta_0 - \tilde{\zeta}_0.
\end{aligned}$$

Now we consider the energy functional

$$\begin{aligned}
G(t) = {} & \frac{1}{2}\|\Delta w(t)\|^2 + \frac{1}{2}\|w_t(t)\|^2 \\
& + \frac{1}{2}\|\vartheta(t)\|^2 + \frac{1}{2}\|\xi^t\|^2_{\mu,V_2} + \frac{1}{2}\|\tau^t\|^2_{\nu,V_1}, \quad t \geq 0.
\end{aligned} \tag{3.34}$$

Step 1. There exists a constant $C_5 > 0$ such that

$$\begin{aligned}
G'(t) \leq {} & -\|\nabla w_t\|^2 - \|\nabla\vartheta\|^2 + \frac{\delta_0}{2}\|w_t\|^2 + C_5\|w\|^2_{2(p+1)} \\
& + \frac{1}{2}\int_0^\infty \mu'(s)\|\Delta\xi^t(s)\|^2\,ds + \frac{1}{2}\int_0^\infty \nu'(s)\|\nabla\tau^t(s)\|^2\,ds,
\end{aligned} \tag{3.35}$$

where C_5 depends on k, k_0, δ_0, c_0, and C_B.

To show this, we multiply (3.30) by w_t and (3.31) by ϑ, respectively. Integrating and using (3.32) and (3.33), we obtain

$$\begin{aligned}
G'(t) = {} & -\|\nabla w_t\|^2 - \|\nabla\vartheta\|^2 - k\int_\Omega (u^+ - \tilde{u}^+)w_t\,dx - \int_\Omega \bigl(f(u) - f(\tilde{u})\bigr)w_t\,dx \\
& + \frac{1}{2}\int_0^\infty \mu'(s)\|\Delta\xi^t(s)\|^2\,ds + \frac{1}{2}\int_0^\infty \nu'(s)\|\nabla\tau^t(s)\|^2\,ds.
\end{aligned} \tag{3.36}$$

By the Young inequality we get

$$\begin{aligned}
\left|-k\int_\Omega (u^+ - \tilde{u}^+)w_t\,dx\right| & \leq \frac{k^2}{\delta_0}\|u^+ - \tilde{u}^+\|^2 + \frac{\delta_0}{4}\|w_t\|^2 \\
& \leq \frac{k^2 c_0}{\delta_0}\|w\|^2_{2(p+1)} + \frac{\delta_0}{4}\|w_t\|^2,
\end{aligned} \tag{3.37}$$

where we have used the facts that $|u^+ - \tilde{u}^+| \leq |u - \tilde{u}|$ and that $c_0 > 0$ is an embedding constant for $L^{2(p+1)}(\Omega) \hookrightarrow L^2(\Omega)$. In addition, from (2.2) and (3.28), by the generalized Hölder inequality with $\frac{p}{2(p+1)} + \frac{1}{2(p+1)} + \frac{1}{2} = 1$ and the Young inequality we have

$$\begin{aligned}
\left|-\int_\Omega \bigl(f(u) - f(\tilde{u})\bigr)w_t\,dx\right| & \leq k_0\int_\Omega \bigl(1 + |u|^p + |\tilde{u}|^p\bigr)|w||w_t|\,dx \\
& \leq k_0\bigl(|\Omega|^{\frac{p}{2(p+1)}} + \|u\|^p_{2(p+1)} + \|\tilde{u}\|^p_{2(p+1)}\bigr)\|w\|_{2(p+1)}\|w_t\| \\
& \leq \frac{k_0^2 C_B}{\delta_0}\|w\|^2_{2(p+1)} + \frac{\delta_0}{4}\|w_t\|^2,
\end{aligned} \tag{3.38}$$

where C_B is a constant depending on B. Combining (3.37) and (3.38) with (3.36) we see that (3.35) holds.

Step 2. Let us define the functional

$$\Phi(t)=\int_\Omega w_t(t)w(t)\,dx.$$

Then there exists a constant $C_6>0$ such that

$$\begin{aligned}\Phi'(t)\le{}&-G(t)-\left(\frac{1}{4}-\frac{k}{\lambda}-\frac{\beta}{2}\right)\|\Delta w\|^2+\frac{3}{2}\|w_t\|^2+\frac{1}{\lambda_1\beta}\|\nabla w_t\|^2+\frac{1}{2}\|\vartheta\|^2\\&+\frac{\beta}{\lambda_1}\|\nabla\vartheta\|^2+\left(2\mu_0+\frac{1}{2}\right)\|\xi^t\|^2_{\mu,V_2}+\frac{1}{2}\|\tau^t\|^2_{\nu,V_1}+C_6\|w\|^2_{2(p+1)},\end{aligned}\tag{3.39}$$

where C_6 depends on k_0, λ, and C_B.

Indeed, differentiating the function Φ, using (3.30), and adding and subtracting $G(t)$, we obtain

$$\begin{aligned}\Phi'(t)={}&-G(t)-\frac{1}{2}\|\Delta w\|^2+\frac{3}{2}\|w_t\|^2+\frac{1}{2}\|\vartheta\|^2+\frac{1}{2}\|\xi^t\|^2_{\mu,V_2}+\frac{1}{2}\|\tau^t\|^2_{\nu,V_1}\\&-k\int_\Omega\left(u^+-\tilde{u}^+\right)w\,dx+\int_\Omega\Delta w_t w\,dx-\int_\Omega\int_0^\infty\mu(s)\Delta\xi^t(s)\,ds\Delta w\,dx\\&-\beta\int_\Omega\Delta\vartheta w\,dx-\int_\Omega\big(f(u)-f(\tilde{u})\big)w\,dx.\end{aligned}\tag{3.40}$$

By a similar procedure used in Step 1, from (2.1), (2.2), (2.4), and the Young inequality we derive the following estimates:

$$\left|-k\int_\Omega\left(u^+-\tilde{u}^+\right)w\,dx\right|\le k\|w\|^2\le\frac{k}{\lambda}\|\Delta w\|^2,\tag{3.41}$$

$$\left|-\int_\Omega\int_0^\infty\mu(s)\Delta\xi^t(s)\,ds\Delta w\,dx\right|\le\frac{1}{8}\|\Delta w\|^2+2\mu_0\|\xi^t\|^2_{\mu,V_2},\tag{3.42}$$

$$\left|-\int_\Omega\big(f(u)-f(\tilde{u})\big)w\,dx\right|\le\frac{1}{8}\|\Delta w\|^2+\frac{2k_0^2C_B}{\lambda}\|w\|^2_{2(p+1)},\tag{3.43}$$

$$\left|-\int_\Omega\nabla w_t\nabla w\,dx\right|\le\frac{\beta}{4}\|\Delta w\|^2+\frac{1}{\lambda_1\beta}\|\nabla w_t\|^2,\tag{3.44}$$

$$\left|\beta\int_\Omega\nabla\vartheta\nabla w\,dx\right|\le\frac{\beta}{4}\|\Delta w\|^2+\frac{\beta}{\lambda_1}\|\nabla\vartheta\|^2.\tag{3.45}$$

Substituting (3.41)-(3.45) into (3.40), we get

$$\begin{aligned}\Phi'(t)\le{}&-G(t)-\left(\frac{1}{4}-\frac{k}{\lambda}-\frac{\beta}{2}\right)\|\Delta w\|^2+\frac{3}{2}\|w_t\|^2+\frac{1}{\lambda_1\beta}\|\nabla w_t\|^2+\frac{1}{2}\|\vartheta\|^2\\&+\frac{\beta}{\lambda_1}\|\nabla\vartheta\|^2+\left(2\mu_0+\frac{1}{2}\right)\|\xi^t\|^2_{\mu,V_2}+\frac{1}{2}\|\tau^t\|^2_{\nu,V_1}+\frac{2k_0^2C_B}{\lambda}\|w\|^2_{2(p+1)}.\end{aligned}\tag{3.46}$$

Step 3. Let us define the functional

$$\Psi(t)=-\int_\Omega w_t(t)\int_0^\infty\mu(s)\xi^t(s)\,ds\,dx-\int_\Omega\vartheta(t)\int_0^\infty\nu(s)\tau^t(s)\,ds\,dx.$$

Then there exist constants C_7, C_8, and $C_9 > 0$ such that

$$\begin{aligned}\Psi'(t) \le {}& -\frac{\mu_0}{2}\|w_t\|^2 - \frac{\nu_0}{2}\|\vartheta\|^2 + \left(\delta_1 + \frac{\delta_1 k^2}{\lambda}\right)\|\Delta w\|^2 + \left(\delta_1 + \delta_1\beta^2\right)\|\nabla w_t\|^2 \\ & + \left(\delta_1 + \delta_1\beta^2\right)\|\nabla\vartheta\|^2 + C_7\|w\|^2_{2(p+1)} + C_8\left\|\xi^t\right\|^2_{\mu,V_2} + C_9\left\|\tau^t\right\|^2_{\nu,V_1} \\ & - \frac{\mu(0)}{2\mu_0\lambda}\int_0^\infty \mu'(s)\left\|\Delta\xi^t(s)\right\|^2 ds - \frac{\nu(0)}{2\nu_0\lambda_1}\int_0^\infty \nu'(s)\left\|\nabla\tau^t(s)\right\|^2 ds, \end{aligned} \tag{3.47}$$

where C_7, C_8, and C_9 depend on δ_1, μ_0, ν_0, λ_1, λ, and C_B. To prove this, we observe that, by (3.30) and (3.31),

$$\begin{aligned}\Psi'(t) = {}& \int_\Omega\left(\Delta^2 w - \Delta w_t + ku^+ - k\tilde{u}^+ + \int_0^\infty \mu(s)\Delta^2\xi^t(s)\,ds + \beta\Delta\vartheta\right) \\ & \times \int_0^\infty \mu(s)\xi^t(s)\,ds\,dx \\ & + \int_\Omega\big(f(u) - f(\tilde{u})\big)\int_0^\infty \mu(s)\xi^t(s)\,ds\,dx - \int_\Omega w_t\int_0^\infty \mu(s)\xi^t_t(s)\,ds\,dx \\ & - \int_\Omega\left(\Delta\vartheta + \beta\Delta w_t + \int_0^\infty \nu(s)\Delta\tau^t(s)\,ds\right)\int_0^\infty \nu(s)\tau^t(s)\,ds\,dx \\ & - \int_\Omega\vartheta\int_0^\infty \nu(s)\tau^t_t(s)\,ds\,dx. \end{aligned}$$

Integrating with respect to s and using (2.1), (2.4), (3.32), (3.33), and the Young inequality, we find that

$$\begin{aligned}-\int_\Omega w_t\int_0^\infty \mu(s)\xi^t_t(s)\,ds\,dx &= -\mu_0\|w_t\|^2 - \int_\Omega w_t\int_0^\infty \mu'(s)\xi^t(s)\,ds\,dx \\ &\le -\frac{\mu_0}{2}\|w_t\|^2 - \frac{\mu(0)}{2\mu_0\lambda}\int_0^\infty \mu'(s)\left\|\Delta\xi^t(s)\right\|^2 ds \end{aligned}$$

and

$$-\int_\Omega\vartheta\int_0^\infty \nu(s)\tau^t_t(s)\,ds\,dx \le -\frac{\nu_0}{2}\|\vartheta\|^2 - \frac{\nu(0)}{2\nu_0\lambda_1}\int_0^\infty \nu'(s)\left\|\nabla\tau^t(s)\right\|^2 ds.$$

In addition, from (2.1), (2.4), and the Young inequality we have the following estimates for any $\delta_1 > 0$:

$$\begin{aligned}&\left|\int_\Omega \Delta w\int_0^\infty \mu(s)\Delta\xi^t(s)\,ds\,dx\right| \le \delta_1\|\Delta w\|^2 + \frac{\mu_0}{4\delta_1}\left\|\xi^t\right\|^2_{\mu,V_2}, \\ &\left|\int_\Omega \nabla w_t\int_0^\infty \mu(s)\nabla\xi^t(s)\,ds\,dx\right| \le \delta_1\|\nabla w_t\|^2 + \frac{\mu_0}{4\delta_1\lambda_1}\left\|\xi^t\right\|^2_{\mu,V_2}, \\ &\left|k\int_\Omega\left(u^+ - \tilde{u}^+\right)\int_0^\infty \mu(s)\xi^t(s)\,ds\,dx\right| \le \frac{\delta_1 k^2}{\lambda}\|\Delta w\|^2 + \frac{\mu_0}{4\delta_1\lambda}\left\|\xi^t\right\|^2_{\mu,V_2}, \\ &\left|\int_\Omega\left(\int_0^\infty \mu(s)\Delta\xi^t(s)\,ds\right)^2 dx\right| \le \mu_0\left\|\xi^t\right\|^2_{\mu,V_2}, \end{aligned}$$

$$\left|-\beta\int_\Omega \nabla\vartheta\int_0^\infty \mu(s)\nabla\xi^t(s)\,ds\,dx\right| \le \delta_1\beta^2\|\nabla\vartheta\|^2 + \frac{\mu_0}{4\delta_1\lambda_1}\left\|\xi^t\right\|^2_{\mu,V_2},$$
$$\left|\int_\Omega \bigl(f(u)-f(\tilde u)\bigr)\int_0^\infty \mu(s)\xi^t(s)\,ds\,dx\right| \le \delta_1 k_0^2 C_B\|w\|^2_{2(p+1)} + \frac{\mu_0}{4\delta_1\lambda}\left\|\xi^t\right\|^2_{\mu,V_2}.$$

Moreover, we obtain that, for any $\delta_1 > 0$,

$$\left|\int_\Omega \nabla\vartheta\int_0^\infty \nu(s)\nabla\tau^t(s)\,ds\,dx\right| \le \delta_1\|\nabla\vartheta\|^2 + \frac{\nu_0}{4\delta_1}\left\|\tau^t\right\|^2_{\nu,V_1},$$
$$\left|\beta\int_\Omega \nabla w_t\int_0^\infty \nu(s)\nabla\tau^t(s)\,ds\,dx\right| \le \delta_1\beta^2\|\nabla w_t\|^2 + \frac{\nu_0}{4\delta_1}\left\|\tau^t\right\|^2_{\nu,V_1},$$
$$\left|\int_\Omega \left(\int_0^\infty \nu(s)\nabla\tau^t(s)\,ds\right)^2 dx\right| \le \nu_0\left\|\tau^t\right\|^2_{\nu,V_1}.$$

Therefore, we conclude that

$$\begin{aligned}\Psi'(t) \le{}& -\frac{\mu_0}{2}\|w_t\|^2 - \frac{\nu_0}{2}\|\vartheta\|^2 + \left(\delta_1 + \frac{\delta_1 k^2}{\lambda}\right)\|\Delta w\|^2 + \left(\delta_1 + \delta_1\beta^2\right)\|\nabla w_t\|^2\\ &+ \left(\delta_1 + \delta_1\beta^2\right)\|\nabla\vartheta\|^2 + \left(\mu_0 + \frac{\mu_0}{4\delta_1} + \frac{\mu_0}{2\delta_1\lambda_1} + \frac{\mu_0}{2\delta_1\lambda}\right)\left\|\xi^t\right\|^2_{\mu,V_2}\\ &+ \left(\nu_0 + \frac{\nu_0}{2\delta_1}\right)\left\|\tau^t\right\|^2_{\nu,V_1} + \delta_1 k_0^2 C_B\|w\|^2_{2(p+1)}\\ &- \frac{\mu(0)}{2\mu_0\lambda}\int_0^\infty \mu'(s)\left\|\Delta\xi^t(s)\right\|^2 ds - \frac{\nu(0)}{2\nu_0\lambda_1}\int_0^\infty \nu'(s)\left\|\nabla\tau^t(s)\right\|^2 ds.\end{aligned}$$

Step 4. We consider the functional

$$\mathcal{G}(t) = NG(t) + \varepsilon\Phi + \Psi,$$

where $\varepsilon \in (0,1)$ and $N > 0$ are to be fixed later. Then there exists a constant $n_0 > 0$ such that, for $N > n_0$,

$$n_1 G(t) \le \mathcal{G}(t) \le n_2 G(t), \quad t \ge 0, \tag{3.48}$$

where $n_1 = N - n_0$ and $n_2 = N + n_0$. Indeed, it is easy to see that

$$\left|\Phi(t)\right| \le \frac{1}{2}\|w_t\|^2 + \frac{1}{2\lambda}\|\Delta w\|^2,$$
$$\left|\Psi(t)\right| \le \frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|\vartheta\|^2 + \frac{\mu_0}{2\lambda}\left\|\xi^t\right\|^2_{\mu,V_2} + \frac{\nu_0}{2\lambda_1}\left\|\tau^t\right\|^2_{\nu,V_1}.$$

Therefore, choosing n_0 large enough, we get

$$\left|\mathcal{G}(t) - NG(t)\right| \le \varepsilon\left|\Phi(t)\right| + \left|\Psi(t)\right| \le n_0 G(t),$$

and hence (3.48) holds.

Step 5. From (2.5), (3.35), (3.39), and (3.47) we have

$$\begin{aligned}\mathcal{G}'(t) \le {} & -\varepsilon G(t) - \left(\frac{\mu_0}{2} - \frac{3\varepsilon}{2} - \frac{\delta_0 N}{2}\right)\|w_t\|^2 - \left(\frac{\nu_0}{2} - \frac{\varepsilon}{2}\right)\|\vartheta\|^2 \\ & - \left(N - \frac{\varepsilon}{\lambda_1\beta} - \left(1+\beta^2\right)\delta_1\right)\|\nabla w_t\|^2 - \left(N - \frac{\beta\varepsilon}{\lambda_1} - \left(1+\beta^2\right)\delta_1\right)\|\nabla\vartheta\|^2 \\ & - \left[\left(\frac{1}{4} - \frac{k}{\lambda} - \frac{\beta}{2}\right)\varepsilon - \left(1 + \frac{k^2}{\lambda}\right)\delta_1\right]\|\Delta w\|^2 + (C_5 N + C_6\varepsilon + C_7)\|w\|_{2(p+1)}^2 \\ & + \left(\frac{N}{2} - \left(2\mu_0 + \frac{1}{2}\right)\frac{\varepsilon}{k_2} - \frac{C_8}{k_2} - \frac{\mu(0)}{2\mu_0\lambda}\right)\int_0^\infty \mu'(s)\left\|\Delta\xi^t(s)\right\|^2 ds \\ & + \left(\frac{N}{2} - \frac{\varepsilon}{2k_3} - \frac{C_9}{k_3} - \frac{\nu(0)}{2\nu_0\lambda_1}\right)\int_0^\infty \nu'(s)\left\|\nabla\tau^t(s)\right\|^2 ds.\end{aligned}$$

We first take $\varepsilon > 0$ so small that

$$\frac{\mu_0}{2} - \frac{3\varepsilon}{2} > 0, \qquad \frac{\nu_0}{2} - \frac{\varepsilon}{2} > 0.$$

For fixed ε, we choose $\delta_1 > 0$ so small that

$$\left(\frac{1}{4} - \frac{k}{\lambda} - \frac{\beta}{2}\right)\varepsilon - \left(1 + \frac{k^2}{\lambda}\right)\delta_1 > 0.$$

Next, for fixed δ_1 and ϵ, we take N so large that

$$\begin{aligned}N > \max\biggl\{ & \frac{\varepsilon}{\lambda_1\beta} + \left(1+\beta^2\right)\delta_1, \frac{\beta\varepsilon}{\lambda_1} + \left(1+\beta^2\right)\delta_1, \\ & (4\mu_0 + 1)\frac{\varepsilon}{k_2} + \frac{2C_8}{k_2} + \frac{\mu(0)}{\mu_0\lambda}, \frac{\varepsilon}{k_3} + \frac{2C_9}{k_3} + \frac{\nu(0)}{\nu_0\lambda_1}\biggr\}.\end{aligned}$$

Finally, choosing $\delta_0 > 0$ small enough, we get that there exist constants $\varepsilon_0, C_{10} > 0$ such that

$$\mathcal{G}'(t) \le -\varepsilon_0 G(t) + C_{10}\|w\|_{2(p+1)}^2, \quad t \ge 0. \tag{3.49}$$

Combining (3.48) with (3.49), we obtain

$$\mathcal{G}'(t) \le -\frac{\varepsilon_0}{n_2}\mathcal{G}(t) + C_{10}\|w\|_{2(p+1)}^2,$$

and so

$$\mathcal{G}(t) \le \mathcal{G}(0)e^{-\frac{\varepsilon_0}{n_2}t} + C_{10}\int_0^t e^{-\frac{\varepsilon_0}{n_2}(t-s)}\left\|w(s)\right\|_{2(p+1)}^2 ds, \quad t \ge 0.$$

Using (3.48) again, we see that

$$G(t) \le \frac{n_2}{n_1}G(0)e^{-\frac{\varepsilon_0}{n_2}t} + \frac{C_{10}}{n_1}\int_0^t e^{-\frac{\varepsilon_0}{n_2}(t-s)}\left\|w(s)\right\|_{2(p+1)}^2 ds, \quad t \ge 0.$$

Since $G(t) = \|z_1(t) - z_2(t)\|_{\mathcal{H}}^2$, we get (3.29) with $\tilde{C}_0 = \frac{n_2}{n_1}$, $\gamma = \frac{\varepsilon_0}{n_2}$, and $\tilde{C}_1 = \frac{C_{10}}{n_1}$. □

Using the ideas presented in [25, 26], we easily get the following lemma.

Lemma 3.7 *Under the assumptions of Theorem* 2.4, *the dynamical system* $(\mathcal{H}(t), S(t))$ *corresponding to problem* (1.6)-(1.11) *is asymptotically smooth.*

Proof Let B be a bounded subset of $\mathcal{H}$ positively invariant with respect to $S(t)$. Denote by C_B several positive constants that depend on B but not on t. For $z_1^0, z_2^0 \in B$, $S(t)z_1^0 = (u(t), u_t(t), \theta(t), \eta^t, \zeta^t)$ and $S(t)z_2^0 = (\tilde{u}(t), \tilde{u}_t(t), \tilde{\theta}(t), \tilde{\eta}^t, \tilde{\zeta}^t)$ are the solutions of (1.6)-(1.9). Then, given $\epsilon > 0$, by inequality (3.29) we can choose $T > 0$ such that

$$\|S(T)z_1^0 - S(T)z_2^0\|_{\mathcal{H}} \le \epsilon + C_B\left(\int_0^T \|u(s) - \tilde{u}(s)\|_{2(p+1)}^2\, ds\right)^{\frac{1}{2}}, \tag{3.50}$$

where $C_B > 0$ is a constant depending only on the size of B. The condition $p > 0$ implies that $2 < 2(p+1) < \infty$. Taking $\alpha_0 = \frac{p}{2(p+1)}$ and applying the Gagliardo-Nirenberg interpolation inequality, we have

$$\|u(t) - \tilde{u}(t)\|_{2(p+1)} \le C\|\Delta(u(t) - \tilde{u}(t))\|^{\alpha_0}\|u(t) - \tilde{u}(t)\|^{1-\alpha_0} \le C_B\|u(t) - \tilde{u}(t)\|^{1-\alpha_0}.$$

Since $\|u(t)\|$ and $\|\tilde{u}(t)\|$ are uniformly bounded, there exists a constant $C_B > 0$ such that

$$\|u(t) - \tilde{u}(t)\|_{2(p+1)}^2 \le C_B\|u(t) - \tilde{u}(t)\|^{2(1-\alpha_0)}. \tag{3.51}$$

Therefore, from (3.50) and (3.51) we obtain

$$\|S(T)z_1^0 - S(T)z_2^0\|_{\mathcal{H}} \le \epsilon + \Phi_T(z_1^0, z_2^0)$$

with

$$\Phi_T(z_1^0, z_2^0) = C_B\left(\int_0^T \|u(s) - \tilde{u}(s)\|^{2(1-\alpha_0)}\, ds\right)^{\frac{1}{2}}.$$

Thus, by Theorem 2.2 it remains to prove that ϕ_T is a contractive function on $B \times B$. Indeed, given a sequence $(z_n^0) = (u_n^0, u_n^1, \theta_n^0, \eta_n^0, \zeta_n^0) \in B$, let us write $S(t)(z_n^0) = (u_n(t), u_{n,t}(t), \theta_n(t), \eta_n^t, \zeta_n^t)$. Because B is positively invariant by $S(t)$, $t \ge 0$, it follows that the sequence $(u_n(t), u_{n,t}(t), \theta_n(t), \eta_n^t, \zeta_n^t)$ is uniformly bounded in $\mathcal{H}$. On the other hand,

$$(u_n, u_{n,t}) \text{ is bounded in } C([0,T], V_2 \times V_0), \quad T > 0.$$

By the compact embedding $V_2 \subset V_0$ the Aubin lemma implies that there exists a subsequence (u_{n_k}) that converges strongly in $C([0,T], V_0)$. Hence, we see that

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T \|u_{n_k}(s) - u_{n_l}(s)\|^{2(1-\alpha_0)}\, ds = 0.$$

This completes the proof of Lemma 3.7. □

Proof of Theorem 2.4 From Lemmas 3.5 and 3.7 we conclude that $(\mathcal{H}, S(t))$ is a dissipative dynamical system, which is asymptotically smooth. Therefore, by Theorem 2.3 it has compact global attractor in $\mathcal{H}$. □

Competing interests
The author declares that she has no competing interests.

Acknowledgements
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2014R1A1A1003440).

Received: 19 July 2016 Accepted: 8 November 2016 Published online: 24 November 2016

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