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The existence of stationary star solutions for compressible magnetohydrodynamic flows

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Abstract
In this paper, we are concerned with the compressible Euler-Poisson system coupled to a magnetic field in the three-dimensional space. Based on a variational method and the exact expression of the Green's function for an elliptic equation in spherical coordinates, we prove the existence of stationary star solutions.

Keywords: Euler-Poisson system; magnetic field; variational method; stationary star solutions

1 Introduction

The Euler-Poisson system of compressible fluids coupled to a magnetic field is given by

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ (\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p(\rho) = -\rho \nabla \Psi + \mu_0 (\nabla \times H) \times H, \\ H_t = \nabla \times (v \times H), \\ \Delta \Psi = 4\pi G \rho, \\ \operatorname{div} H = 0, \end{cases} \tag{1.1}$$

where ρ is the density, $v = (v_1, v_2, v_3)$ is the velocity field, $H = (H_1, H_2, H_3)$ is the magnetic field, p is the pressure function, Ψ is the Newtonian potential, G is the gravitational constant, and μ_0 is the permeability of vacuum. We consider the polytropic gases for which the equation of state is given by

$$p = p(\rho) = \rho^{\alpha}, \tag{1.2}$$

where $\alpha > 1$ is the adiabatic exponent. The gravitational potential Ψ is given by

$$\Psi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy = -\rho * \frac{1}{|x|}, \tag{1.3}$$

where $*$ denotes convolution.

Extensive works have been done on the existence of stationary solutions for system (1.1) in [1–9] and references therein. Recently, some important progress has been made for system (1.1) without magnetic field. For the pressure α-law in (1.2) with the adiabatic exponent $\alpha > \frac{3}{2}$, the global existence of weak solutions was obtained in [10] when the spatial dimension is three in the framework of Lions and Feireisl for the compressible Navier-Stokes equations [11, 12]. Furthermore, Cai and Tan [2] have proved the existence and uniqueness of stationary solutions of the three-dimensional compressible Navier-Stokes-Poisson equations basing on the weighted L^2 method and the contraction mapping principle.

For a nonrotating gaseous star, it is important to investigate the spherically symmetric motion since the stable equilibrium configuration is spherically symmetric. Moreover, the spherically symmetric solution minimizes the energy among all possible configurations [7], which are called Lane-Emden solutions. More importantly, Luo, Xin, and Zeng [13] were concerned with three-dimensional spherically symmetric solutions of the compressible Navier-Stokes-Poisson equations with free boundary condition. They also proved the nonlinear asymptotic stability of the Lane-Emden solutions for spherically symmetric motions of viscous gaseous stars if the adiabatic constant α lies in the stability range $(\frac{4}{3}, 2)$.

For the rotating nonmagnetic stationary solutions, Auchmuty and Beals [1] gave a priori bound for the maximum of the density ρ. When the rotation is a fixed axis with constant angular velocity, Chanillo and Li [3] obtained a priori bound for the support of the relative equilibrium form of a homogeneous, gravitating, and compressible mass of fluid. Coupling to the magnetic field, Federbush, Luo, and Smoller [4] first proved the existence of axisymmetric stationary solutions of system (1.1). They utilized a variational method and proved the existence of a stationary solution expressed by density, which is a minimizer of the associated energy functional. To prove the main result, an elliptic equation is derived for the magnetic potential in cylindrical coordinates in $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$. They looked for solutions of the following form:

$$\begin{cases} \rho(x) = \rho(r,z), \qquad \Psi(x) = \Psi(r,z), \\ H(x) = H_r(r,z)e_r + H_\theta(r,z)e_\theta + H_z(r,z)e_z, \end{cases} \tag{1.4}$$

where $e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)^T$, $e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0)^T$, $e_z = (0,0,1)^T$.

It is well known that almost every Newtonian gaseous star has crystal body. Auchmuty and Beals [1] have obtained a nonrotating nonmagnetic spherically symmetric solution of some nonlinear integro-differential equations in $\mathbb{R}^3$, which are of interest in astrophysics. Also, they formulated each problem as a variational problem and looked for a solution among an appropriate class of nonnegative functions ρ. In this paper, for the nonrotating magnetic case, we suppose that the stationary solutions should have the following form:

$$\begin{cases} \rho(x) = \rho(r,\theta), \\ H(x) = H(r,\theta) = (H_r(r,\theta), H_\theta(r,\theta), 0), \\ \Psi(x) = \Psi(r,\theta), \\ r^2 = x_1^2 + x_2^2 + x_3^2, \qquad \theta = \arccos\frac{x_3}{r}. \end{cases} \tag{1.5}$$

Motivated by the paper [4], we show the existence of stationary star solutions (1.5) of system (1.1) in spherical coordinates by using the variational methods.

The paper is organized as follows. In Section 2, we obtain an expression of stationary equations of system (1.1) in spherical coordinates by using the formula in [14]. Owing to the methods in [7, 8, 15], we see that the stationary solution can be expressed by the density that is a minimizer of the corresponding the energy functional. Thus, we only need to derive the minimizer of the corresponding energy functional in Section 3.

2 Euler-Poisson system coupling with a magnetic field

In this paper, we are interested in the stationary solutions of (1.1), which represent an important class of equilibrium configurations. The stationary solutions ($\nu = 0$) satisfy the following system:

$$\begin{cases} \nabla p(\rho) = -\rho\nabla\Psi + \mu_0(\nabla \times H) \times H, \\ \Delta\Psi = 4\pi G\rho, \\ \operatorname{div} H = 0, \end{cases} \tag{2.1}$$

where ρ is the density, $H = (H_1, H_2, H_3)$ is the magnetic field, p is the pressure function, Ψ is the Newtonian potential, G is the gravitational constant, and μ_0 is the permeability of vacuum.

2.1 The expression of stationary equation in spherical coordinates

Now, we will give a new method to get a specific expression of (2.1) in spherical coordinates. Based on a recent paper [14], Wang and Wang gave the general definitions of curl and cross products on a 3-D Riemanion manifold (M, g_{ij}) with metric

$$ds^2 = g_{ij}\,dx^i\,dx^j \quad (i,j = 1,2,3). \tag{2.2}$$

Let $A = (A^1, A^2, A^3)$ and $B = (B^1, B^2, B^3)$ be smooth vector fields. From [14] we immediately get the following formulae for $\nabla \times A$ and $A \times B$:

$$\nabla A = \left(g^{11}\frac{\partial A^1}{\partial x^1}, g^{22}\frac{\partial A^2}{\partial x^2}, g^{33}\frac{\partial A^3}{\partial x^3}\right), \tag{2.3}$$

$$\nabla \times A = \frac{1}{\sqrt{g}}\left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}, \frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}, \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right), \tag{2.4}$$

$$\begin{aligned} A \times B = \Bigg(&\sqrt{\frac{g_{22}g_{33}}{g_{11}}}\left(A^2B^3 - A^3B^2\right), \sqrt{\frac{g_{11}g_{33}}{g_{22}}}\left(A^3B^1 - A^1B^3\right), \\ &\sqrt{\frac{g_{11}g_{22}}{g_{33}}}\left(A^1B^2 - A^2B^1\right)\Bigg), \end{aligned} \tag{2.5}$$

where $(A_1, A_2, A_3) = (g_{11}A^1, g_{22}A^2, g_{33}A^3)$.

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r^2 = x_1^2 + x_2^2 + x_3^2$, and $\theta = \arccos\frac{x_3}{r}$. In the spherical coordinates, the metric is

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2. \tag{2.6}$$

Obviously, we have

$$\begin{cases} g_{11} = 1, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2\theta, \\ g_{ij} = 0 \quad (i \neq j), \qquad g^{kk} = \frac{1}{g_{kk}} \quad (k = 1,2,3), \\ \sqrt{g} = r^2\sin\theta. \end{cases} \tag{2.7}$$

Let $H(x) = H(r,\theta,\varphi) = (H^1, H^2, H^3)$. Hence, from (2.7) we have

$$H_1 = g_{1j}H^j = H^1, \qquad H_2 = g_{2j}H^j = r^2H^2, \qquad H_3 = g_{3j}H^j = r^2\sin^2\theta H^3. \tag{2.8}$$

Using the definition of curl product (2.4), we get

$$\begin{aligned}\nabla\times H &= \frac{1}{\sqrt{g}}\left(\frac{\partial H_3}{\partial\theta}-\frac{\partial H_2}{\partial\varphi},\frac{\partial H_1}{\partial\varphi}-\frac{\partial H_3}{\partial r},\frac{\partial H_2}{\partial r}-\frac{\partial H_1}{\partial\theta}\right)\\ &= \frac{1}{r^2\sin\theta}\biggl(2r^2\sin\theta\cos\theta H^3+r^2\sin\theta\frac{\partial H^3}{\partial\theta}-r^2\frac{\partial H^2}{\partial\varphi},\\ &\quad\frac{\partial H^1}{\partial\varphi}-2r\sin^2\theta H^3-r^2\sin^2\theta\frac{\partial H^3}{\partial r},2r\sin\theta H^2+r^2\frac{\partial H^2}{\partial r}-\frac{\partial H^1}{\partial\theta}\biggr).\end{aligned}\tag{2.9}$$

It follows from the definition of cross product (2.5) and (2.9) that

$$\begin{aligned}&(\nabla\times H)\times H\\ &\quad=\biggl(H^2\frac{\partial H^1}{\partial\theta}+H^3\frac{\partial H^1}{\partial\varphi}-2r\bigl(H^2\bigr)^2-2r\sin^2\theta\bigl(H^3\bigr)^2-r^2H^2\frac{\partial H^2}{\partial r}-r^2\sin^2\theta H^3\frac{\partial H^3}{\partial r},\\ &\qquad H^1\frac{\partial H^2}{\partial r}+H^3\frac{\partial H^2}{\partial\varphi}+\frac{2}{r}H^1H^2-\frac{1}{r^2}H^1\frac{\partial H^1}{\partial\theta}-2\sin\theta\cos\theta\bigl(H^3\bigr)^2-\sin^2\theta H^3\frac{\partial H^3}{\partial\theta},\\ &\qquad H^1\frac{\partial H^3}{\partial r}+H^2\frac{\partial H^3}{\partial\theta}+\frac{2}{r}H^1H^3+\frac{2\cos\theta}{\sin\theta}H^2H^3-\frac{1}{r^2\sin^2\theta}H^1\frac{\partial H^1}{\partial\varphi}\\ &\qquad-\frac{1}{\sin^2\theta}H^2\frac{\partial H^2}{\partial\varphi}\biggr).\end{aligned}\tag{2.10}$$

Combining (2.3) and (2.7), it is easy to see that

$$\begin{aligned}&\nabla P(\rho)+\rho\nabla\Psi\\ &\quad=\left(\frac{\partial p}{\partial r}+\rho\frac{\partial\Psi}{\partial r},\frac{1}{r^2}\frac{\partial p}{\partial\theta}+\frac{\rho}{r^2}\frac{\partial\Psi}{\partial\theta},\frac{1}{r^2\sin^2\theta}\frac{\partial p}{\partial\varphi}+\frac{\rho}{r^2\sin^2\theta}\frac{\partial\Psi}{\partial\varphi}\right).\end{aligned}\tag{2.11}$$

Hence, we have, by (2.1), (2.10), and (2.11),

$$\begin{cases}\frac{\partial p}{\partial r}+\rho\frac{\partial\Psi}{\partial r}=\mu_0(H^2\frac{\partial H^1}{\partial\theta}+H^3\frac{\partial H^1}{\partial\varphi}-2r(H^2)^2-2r\sin^2\theta(H^3)^2\\ \qquad-r^2H^2\frac{\partial H^2}{\partial r}-r^2\sin^2\theta H^3\frac{\partial H^3}{\partial r}),\\ \frac{1}{r^2}\frac{\partial p}{\partial\theta}+\frac{\rho}{r^2}\frac{\partial\Psi}{\partial\theta}=\mu_0(H^1\frac{\partial H^2}{\partial r}+H^3\frac{\partial H^2}{\partial\varphi}+\frac{2}{r}H^1H^2-\frac{1}{r^2}H^1\frac{\partial H^1}{\partial\theta}\\ \qquad-2\sin\theta\cos\theta(H^3)^2-\sin^2\theta H^3\frac{\partial H^3}{\partial\theta}),\\ \frac{1}{r^2\sin^2\theta}\frac{\partial p}{\partial\varphi}+\frac{\rho}{r^2\sin^2\theta}\frac{\partial\Psi}{\partial\varphi}=\mu_0(H^1\frac{\partial H^3}{\partial r}+H^2\frac{\partial H^3}{\partial\theta}+\frac{2}{r}H^1H^3\\ \qquad+\frac{2\cos\theta}{\sin\theta}H^2H^3-\frac{1}{r^2\sin^2\theta}H^1\frac{\partial H^1}{\partial\varphi}-\frac{1}{\sin^2\theta}H^2\frac{\partial H^2}{\partial\varphi}).\end{cases}\tag{2.12}$$

Let $H_r=H^1$, $H_\theta=\frac{H^2}{r}$, $H_\varphi=\frac{H^3}{r\sin\theta}$. The spherical coordinate expressions of (2.12) can be rewritten as follows:

$$\begin{cases}\frac{\partial p}{\partial r}+\rho\frac{\partial\Psi}{\partial r}=\mu_0(\frac{H_\theta}{r}\frac{\partial H_r}{\partial\theta}+\frac{H_\varphi}{r\sin\theta}\frac{\partial H_r}{\partial\varphi}-\frac{H_\theta^2+H_\varphi^2}{r}\\ \qquad-H_\theta\frac{\partial H_\theta}{\partial r}-H_\varphi\frac{\partial H_\varphi}{\partial r}),\\ \frac{1}{r}\frac{\partial p}{\partial\theta}+\frac{\rho}{r}\frac{\partial\Psi}{\partial\theta}=\mu_0(H_r\frac{\partial H_\theta}{\partial r}+\frac{H_\varphi}{r\sin\theta}\frac{\partial H_\theta}{\partial\varphi}+\frac{H_rH_\theta}{r}-\frac{\cos\theta}{r\sin\theta}H_\varphi^2\\ \qquad-\frac{1}{r}H_r\frac{\partial H_r}{\partial\theta}-\frac{1}{r}H_\varphi\frac{\partial H_\varphi}{\partial\theta}),\\ \frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi}+\frac{\rho}{r\sin\theta}\frac{\partial\Psi}{\partial\varphi}=\mu_0(H_r\frac{\partial H_\varphi}{\partial r}+\frac{H_\theta}{r}\frac{\partial H_\varphi}{\partial\theta}+\frac{H_rH_\varphi}{r}\\ \qquad+\frac{\cos\theta}{r\sin\theta}H_\theta H_\varphi-\frac{1}{r\sin\theta}H_r\frac{\partial H_r}{\partial\varphi}-\frac{1}{r\sin\theta}H_\theta\frac{\partial H_\theta}{\partial\varphi}).\end{cases}\tag{2.13}$$

Remark 2.1 Let $H = (H_r, H_\theta, H_\varphi)$. Notice that (2.13) can be directly calculated by the definitions of gradient and of curl and cross products in spherical coordinates as follows:

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right), \tag{2.14}$$

$$\nabla \cdot H = \frac{1}{r^2}\frac{\partial(r^2 H_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta H_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial H_\varphi}{\partial\varphi}, \tag{2.15}$$

$$\begin{aligned}\nabla \times H = \frac{1}{r^2\sin\theta}\biggl(& r\cos\theta H_\varphi + r\sin\theta\frac{\partial H_\varphi}{\partial\theta} - r\frac{\partial H_\theta}{\partial\varphi}, \\ & r\frac{\partial H_r}{\partial\varphi} - r\sin\theta H_\varphi - r^2\sin\theta\frac{\partial H_\varphi}{\partial r}, \\ & r\sin\theta H_\theta + r^2\sin\theta\frac{\partial H_\theta}{\partial r} - r\sin\theta\frac{\partial H_r}{\partial\theta}\biggr).\end{aligned} \tag{2.16}$$

However, we give a new method to get (2.13) by using the definitions of curl and cross products (2.4) and (2.5) on the 3-D Riemanion manifold M, which only need a simple transformation $(H_r, H_\theta, H_\varphi) = (H^1, \frac{H^2}{r}, \frac{H^3}{r\sin\theta})$.

Noticing that the solutions we look for have the form (1.5) and omitting the terms $\frac{\partial}{\partial\varphi}$, we can rewrite (2.13) as follows:

$$\begin{cases} \frac{\partial p}{\partial r} + \rho\frac{\partial\Psi}{\partial r} = \mu_0(\frac{H_\theta}{r}\frac{\partial H_r}{\partial\theta} - \frac{H_\theta^2}{r} - H_\theta\frac{\partial H_\theta}{\partial r}), \\ \frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{\rho}{r}\frac{\partial\Psi}{\partial\theta} = \mu_0(H_r\frac{\partial H_\theta}{\partial r} + \frac{H_r H_\theta}{r} - \frac{1}{r}H_r\frac{\partial H_r}{\partial\theta}). \end{cases} \tag{2.17}$$

Let $H_\varphi = 0$. Omitting the terms $\frac{\partial}{\partial\varphi}$ in (2.15), it is easy to see that $\nabla \cdot H = 0$ implies that

$$r^2\frac{\partial H_r}{\partial r} + 2rH_r + r\frac{\partial H_\theta}{\partial\theta} + r\frac{\cos\theta}{\sin\theta}H_\theta = 0. \tag{2.18}$$

For simplicity, we denote

$$m = \left(\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r} - \frac{1}{r}\frac{\partial H_r}{\partial\theta}\right).$$

Then (2.17) can be rewritten as

$$\begin{cases} \frac{\partial p}{\partial r} + \rho\frac{\partial\Psi}{\partial r} = -\mu_0 m H_\theta, \\ \frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{\rho}{r}\frac{\partial\Psi}{\partial\theta} = \mu_0 m H_r. \end{cases} \tag{2.19}$$

2.2 The problem and formulation

Owing to (2.18), we get

$$\frac{\partial(r^2\sin\theta H_r)}{\partial r} + \frac{\partial(r\sin\theta H_\theta)}{\partial\theta} = 0, \tag{2.20}$$

which enables us to introduce a magnetic potential $\varphi_0(r,\theta)$ such that

$$\begin{cases} \frac{\partial\varphi_0}{\partial r} = r\sin\theta H_\theta, \\ \frac{\partial\varphi_0}{\partial\theta} = -r^2\sin\theta H_r. \end{cases} \tag{2.21}$$

Let

$$n(\rho)=\int_0^\rho \frac{p'(s)}{s}\,ds. \tag{2.22}$$

Then (2.19) and (2.21) imply

$$\begin{cases}\rho\frac{\partial(n(\rho)+\Psi)}{\partial r}=-\mu_0 m(\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\varphi_0),\\ \rho\frac{\partial(n(\rho)+\Psi)}{\partial\theta}=-\mu_0 m(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\varphi_0).\end{cases} \tag{2.23}$$

Let

$$\frac{\mu_0 m}{\rho r\sin\theta}=W. \tag{2.24}$$

In this paper, we only consider the case that W is a constant. Then it follows from (2.23) and (2.24) that

$$\begin{cases}\frac{\partial}{\partial r}(n(\rho)+\Psi)=-W\frac{\partial}{\partial r}\varphi_0,\\ \frac{\partial}{\partial\theta}(n(\rho)+\Psi)=-W\frac{\partial}{\partial\theta}\varphi_0.\end{cases} \tag{2.25}$$

Note that (2.25) implies that

$$\nabla\bigl(n(\rho)+\Psi+W\varphi_0\bigr)=0\quad\text{whenever }\rho>0. \tag{2.26}$$

Therefore,

$$n(\rho)+\Psi+W\varphi_0=\text{const.}:=\lambda,\quad\text{in the region }\rho>0, \tag{2.27}$$

where $n(\rho)$ is given by (2.22), and Ψ is given by (1.3).

In the following, we only need to solve problem (2.27) with the total mass constraint

$$\int_{\mathbb{R}^3}\rho(x)\,dx=M>0\quad\text{for some given }M. \tag{2.28}$$

3 Existence of star solution coupling to a magnetic field

3.1 The expression of magnetic potential

Combining (2.14) and $\frac{\partial}{\partial\varphi}=0$, we have

$$\nabla\varphi_0=\left(\frac{\partial\varphi_0}{\partial r},\frac{1}{r}\frac{\partial\varphi_0}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial\varphi_0}{\partial\varphi}\right)=\left(\frac{\partial\varphi_0}{\partial r},\frac{1}{r}\frac{\partial\varphi_0}{\partial\theta},0\right). \tag{3.1}$$

Using the divergence formula (2.15), from (2.21), (2.24), and (3.1) we can deduce that

$$\operatorname{div}\left(\frac{1}{r^2\sin^2\theta}\nabla\varphi_0\right)=\frac{m}{r\sin\theta}=\frac{W}{\mu_0}\rho. \tag{3.2}$$

Let

$$\varphi_0(x_1,x_2,x_3)=\varphi_0(r,\theta). \tag{3.3}$$

Then (3.2) is equivalent to

$$\frac{1}{x_1^2+x_2^2}\left(\frac{\partial^2\varphi_0}{\partial x_1^2}+\frac{\partial^2\varphi_0}{\partial x_2^2}+\frac{\partial^2\varphi_0}{\partial x_3^2}-\frac{2x_1}{x_1^2+x_2^2}\frac{\partial\varphi_0}{\partial x_1}-\frac{2x_2}{x_1^2+x_2^2}\frac{\partial\varphi_0}{\partial x_2}\right)=\frac{W}{\mu_0}\rho, \tag{3.4}$$

where

$$\rho(x_1,x_2,x_3)=\rho(r,\theta).$$

Define the operator $\mathcal{L}$ as follows:

$$\mathcal{L}=\frac{1}{x_1^2+x_2^2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-\frac{2x_1}{x_1^2+x_2^2}\frac{\partial}{\partial x_1}-\frac{2x_2}{x_1^2+x_2^2}\frac{\partial}{\partial x_2}\right). \tag{3.5}$$

It is easy to observe that $\mathcal{L}$ is a symmetric operator. Hence, it has a Green's function $G(x,y)$ for $x,y\in\mathbb{R}^3$. The following lemma gives an exact expression of $G(x,y)$.

Lemma 3.1 *The Green's function $G(x,y)$ for $\mathcal{L}$ has the expression*

$$G(x,y)=-\frac{\Lambda((x_1-y_1)^2+(x_2-y_2)^2)}{((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2)^{\frac{3}{2}}}, \tag{3.6}$$

where is a positive constant.

Proof We know that (3.2) is equivalent to (3.4). Let $\varphi_0=r^2\overline{\varphi}\sin^2\theta$. Inserting this into (3.2), we get

$$\frac{\partial^2\overline{\varphi}}{\partial r^2}+\frac{4}{r}\frac{\partial\overline{\varphi}}{\partial r}+\frac{3\cos\theta}{r^2\sin\theta}\frac{\partial\overline{\varphi}}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\overline{\varphi}}{\partial\theta^2}=\frac{W\rho}{\mu_0}, \tag{3.7}$$

and it is not difficult to observe that the Green's function of the operator

$$\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}+\frac{3\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

is given by

$$G(r,\theta)=-\frac{\Lambda}{r^3}, \tag{3.8}$$

where Λ is a positive constant.

Hence, the Green's function $G(x,y)$ of $\mathcal{L}$ can be expressed as

$$G(x,y)=-\frac{\Lambda r^2\sin^2\theta}{r^3}=\frac{\Lambda((x_1-y_1)^2+(x_2-y_2)^2)}{((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2)^{\frac{3}{2}}}. \tag{3.9}$$

□

3.2 Variational formulation

In this paper, we assume that the pressure function $p(\rho)$ satisfies the α-law (1.2) for some constant $\alpha>1$. Let

$$I(\rho)=\frac{\rho^\alpha}{\alpha-1}. \tag{3.10}$$

Then

$$n(\rho) = I'(\rho). \tag{3.11}$$

Moreover, the Newtonian potential operator is given by (1.3), and we can denote

$$\Psi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\,dy := -E(\rho). \tag{3.12}$$

Notice that the magnetic potential φ_0 satisfies (3.2). By Lemma 3.1 we see that $G(x,y)$ for $x, y \in \mathbb{R}^3$ is the Green's function for the operator $\mathcal{L}$ defined in (3.5), that is,

$$\mathcal{L}G = \delta(x-y), \tag{3.13}$$

where $\delta(x-y)$ is the Dirac measure giving the unit mass to the point x. Since $\mathcal{L}$ is symmetric, we have

$$\langle \mathcal{L}\varphi_0, G\rangle = \langle \varphi_0, \mathcal{L}G\rangle = \bigl\langle \varphi_0, \delta(x-y)\bigr\rangle = \varphi_0(y), \tag{3.14}$$

where the inner product $\langle\cdot,\cdot\rangle$ is taken in L^2. Thus, we have the following integral representation for φ_0:

$$\varphi_0(x) = D(\rho), \tag{3.15}$$

where the integral operator D is given by

$$D(\rho) = \frac{W}{\mu_0}\int_{\mathbb{R}^3} G(x,y)\rho(y)\,dy. \tag{3.16}$$

By (3.11), (3.12), and (3.16) it is obvious that equation (2.27) can be written as

$$I'(\rho) + E(\rho) + WD(\rho) = \lambda \quad \text{for } \rho > 0. \tag{3.17}$$

According to (3.17), we define the energy functional F as follows:

$$F(\rho) = \int_{\mathbb{R}^3}\left[I(\rho) + \frac{1}{2}\rho E(\rho) + \frac{1}{2}\rho WD(\rho)\right] dx, \tag{3.18}$$

where $I(\rho)$ is the function given in (3.10). The energy functional $F(\rho)$ means that solving (3.17) with the total mass constraint (2.28) is equivalent to proving that (3.18) has a minimizer in some function space X.

Now, we review the results for stationary solution (1.5). For $0 < M < \infty$, we define X_M by

$$\begin{aligned} X_M = \biggl\{ &\rho : \mathbb{R}^3 \to \mathbb{R}, \rho > 0 \text{ a.e.}, \int_{\mathbb{R}^3} \rho\,dx = M, \text{and} \\ &\int_{\mathbb{R}^3}\left[I(\rho) + \frac{1}{2}\rho E(\rho)\right] dx < \infty \biggr\}. \end{aligned} \tag{3.19}$$

For $\rho \in X_M$, we define the energy function F_1 for the nonrotating nonmagnetic by

$$F_1(\rho) = \int_{\mathbb{R}^3} \left[I(\rho) + \frac{1}{2}\rho E(\rho) \right] dx. \tag{3.20}$$

Thanks to the Lemma 2 in [1], Federbush et al. [4] obtained the following useful lemma for the minimizer of the functional F_1.

Lemma 3.2 *Suppose that the pressure $p(\rho) = \rho^\alpha$ ($\alpha > \frac{4}{3}$). Let ρ^* be a minimizer of* (3.20) *in X_M, and let*

$$\Gamma_M = \{ x \in \mathbb{R}^3 : \rho^* > 0 \}. \tag{3.21}$$

Then there exists a constant λ_1 such that

$$\begin{cases} I(\rho) + \frac{1}{2}\rho E(\rho) = \lambda_1, & x \in \Gamma_M, \\ E(\rho) \ge \lambda_1, & x \in \mathbb{R} - \Gamma_M. \end{cases} \tag{3.22}$$

Remark 3.3 The variational problem is unusual, in that a solution turns out to have compact support. The reason is that the functional one seeks to minimize is not lower semicontinuous on the class of all admissible functions. Auchmuty and Richard [1] first restrict their considerations to functions with support in the ball of radius R_M. Hence, we should find the radius R_M. Note that, for $\alpha > \frac{4}{3}$, Luo and Smoller [8] have proved that a local minimizer ρ^* of the function F_1 in X_M exists. Also, they showed that the minimizer ρ^* is actually radial and unique and has compact support, that is, for given total mass M, there exists a unique constant $R_M > 0$ such that

$$\begin{cases} \rho^*(x) > 0 & \text{if } |x| < R_M, \\ \rho^*(x) = 0 & \text{if } |x| \ge R_M. \end{cases} \tag{3.23}$$

Notice that ρ^* satisfying (3.22) in X_M is called a nonrotating nonmagnetic star solution, and R_M is called the radius of the nonrotating nonmagnetic star solution with total mass M.

Based on (3.23), we define the function spaces Y_M and $Y^\alpha_{MR_0}$ as follows:

$$\begin{aligned} Y^\alpha_M = \bigg\{ &\rho : \mathbb{R}^3 \to \mathbb{R}, \rho(x) = \rho(r,\theta), \rho \ge 0, \text{a.e.}, \rho \in L^1(\mathbb{R}^3) \cap L^\alpha(\mathbb{R}^3), \\ &\alpha > \frac{4}{3}, \int_{\mathbb{R}^3} \rho(x)\,dx = M \bigg\}, \end{aligned} \tag{3.24}$$

$$Y^\alpha_{MR_0} = \{ \rho \in Y^\alpha_M, \rho = 0 \text{ for } r \ge R_0 \}, \tag{3.25}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\theta = \arccos \frac{x_3}{r}$, $R_0 \ge R_M$ is a constant, and R_M is the radius of the nonrotating nonmagnetic star solution with prescribed total mass M.

We want to apply Theorem 3.2 in [4]. It is easy to see that a minimizer of the functional F as defined in(3.18) in $Y^\alpha_{MR_0}$ solves equation (3.17).

Theorem 3.4 *Let ρ_1^* be a minimizer of the energy functional F in $Y_{MR_0}^{\alpha}$, and*

$$\Gamma_M = \{x \in \mathbb{R}^3 : \rho_1^*(x) > 0\}. \tag{3.26}$$

If $\alpha > \frac{6}{5}$, then $\rho_1^* \in C(\mathbb{R}^3) \cap C^1(\Gamma_M)$. Moreover, there exists a constant λ_M^* such that

$$I'(\rho_1^*) + E(\rho_1^*) + WD(\rho_1^*) = \lambda_M^*, \quad x \in \Gamma_M. \tag{3.27}$$

Proof Let

$$F_2(\rho) = \frac{1}{2} W \int_{\mathbb{R}^3} \rho D(\rho)\,dx. \tag{3.28}$$

Then $F(\rho)$ can be written in two parts:

$$F(\rho) = F_1(\rho) + F_2(\rho), \tag{3.29}$$

where $F_1(\rho)$ is defined by (3.20).

Let $\rho + t\sigma \in Y_{MR_0}^{\alpha}$ for any $t \in \mathbb{R}$ under the condition $\int_{\mathbb{R}^3} \sigma\,dx = 0$. Let us note carefully that

$$\begin{aligned}
&\lim_{t\to 0} \frac{F_1(\rho + t\sigma) - F_1(\rho)}{t} \\
&\quad = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^3} \left(I(\rho + t\sigma) + \frac{1}{2}(\rho + t\sigma)E(\rho + t\sigma) - I(\rho) - \frac{1}{2}\rho E(\rho) \right) dx \\
&\quad = \lim_{t\to 0} \int_{\mathbb{R}^3} \frac{I(\rho + t\sigma) - I(\rho)}{t}\,dx + \lim_{t\to 0} \int_{\mathbb{R}^3} \frac{\rho E(\rho + t\sigma) - \rho E(\rho)}{2t}\,dx \\
&\qquad + \int_{\mathbb{R}^3} \frac{1}{2}\sigma E(\rho)\,dx \\
&\quad = \int_{\mathbb{R}^3} I'(\rho)\sigma\,dx + \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^3} \frac{\rho}{2} \left(\int_{\mathbb{R}^3} \frac{-\rho - t\sigma}{|x-y|}\,dy + \int_{\mathbb{R}^3} \frac{\rho}{|x-y|}\,dy \right) dx \\
&\qquad + \int_{\mathbb{R}^3} \frac{1}{2}\sigma E(\rho)\,dx \\
&\quad = \int_{\mathbb{R}^3} \left(I'(\rho)\sigma + \frac{1}{2}\sigma E(\rho) \right) dx + \int_{\mathbb{R}^3} \frac{\sigma(y)}{2} \left(\int_{\mathbb{R}^3} \frac{-\rho(x)}{|x-y|}\,dx \right) dy \\
&\quad = \int_{\mathbb{R}^3} \big(I'(\rho) + E(\rho)\big)\sigma\,dx
\end{aligned} \tag{3.30}$$

if $\rho \in Y_{MR_0}^{\alpha}$.

For $F_2(\rho)$, we get

$$\begin{aligned}
&\lim_{t\to 0} \frac{F_1(\rho + t\sigma) - F_1(\rho)}{t} \\
&\quad = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^3} \left(\frac{W}{2}(\rho + t\sigma)D(\rho + t\sigma) - \frac{W}{2}\rho D(\rho) \right) dx \\
&\quad = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^3} \frac{W\rho}{2}\big(D(\rho + t\sigma) - D(\rho)\big)\,dx + \int_{\mathbb{R}^3} \frac{W\sigma}{2} D(\rho)\,dx \\
&\quad = \lim_{t\to 0} \int_{\mathbb{R}^3} \frac{W\rho}{2t} \left(\int_{\mathbb{R}^3} \big(G(x,y)(\rho + t\sigma) - G(x,y)\rho\big)\,dy \right) dx + \int_{\mathbb{R}^3} \frac{W\sigma}{2} D(\rho)\,dx
\end{aligned}$$

$$= \int_{\mathbb{R}^3} \frac{W\rho}{2}\left(\int_{\mathbb{R}^3} G(x,y)\sigma\,dy\right)dx + \int_{\mathbb{R}^3} \frac{W\sigma}{2} D(\rho)\,dx$$
$$= \int_{\mathbb{R}^3} \frac{W\sigma}{2}\left(\int_{\mathbb{R}^3} G(x,y)\rho(x)\,dx\right)dy + \int_{\mathbb{R}^3} \frac{W\sigma}{2} D(\rho)\,dx$$
$$= \int_{\mathbb{R}^3} W\sigma D(\rho)\,dx, \tag{3.31}$$

where $G(x,y)$ is defined by (3.6).

Hence, from (3.30) and (3.31) we have

$$\lim_{t\to 0} \frac{F(\rho + t\sigma) - F(\rho)}{t} = \int_{\mathbb{R}^3} \left[I'(\rho) + E(\rho) + D(\rho)\right]\sigma\,dx = 0 \tag{3.32}$$

for all σ satisfying $\int_{\mathbb{R}^3} \sigma\,dx = 0$. Then we can prove the theorem using a similar argument as in [1]. □

Now, we give the main theorem of this paper.

Theorem 3.5 *Suppose that* $\alpha > \frac{4}{3}$. *Then the following statements hold*:

1. $\inf_{Y^\alpha_{MR_0}} F(\rho) < 0$,
2. $F(\rho) \geq C_1 \int_{\mathbb{R}^3} \rho^\alpha\,dx - C_2$, $\rho \in Y^\alpha_{MR_0}$, *for some positive constants* C_1 *and* C_2 *independent of* ρ,
3. F *has a minimizer* ρ^* *in* $Y^\alpha_{MR_0}$.

Remark 3.6 In comparison with Theorem 3.3 with adiabatic exponent $\alpha > 2$ in [4], we only need the adiabatic exponent $\alpha > \frac{4}{3}$.

3.3 The proof of Theorem 3.5

Before giving the proof of Theorem 3.5, we introduce the following lemma, which ensures that the functional F is bounded on the set $Y^\alpha_{MR_0}$ if $\alpha > \frac{4}{3}$.

Let

$$F_M = \inf_{\rho \in Y^\alpha_{MR_0}} F(\rho).$$

Lemma 3.7 *Let* $\alpha > \frac{4}{3}$. *If* $\rho \in Y^\alpha_{MR_0}$, *then there exist two positive constants* C_1 *and* C_2 *depending only on* α *and* M *such that*

$$C_1 \int_{\mathbb{R}^3} \rho^\alpha\,dx - C_2 < F(\rho) \quad \text{and} \quad F_M < 0.$$

Proof Let $F(\rho)$ be defined by (3.29). For $F_1(\rho)$, Lemma 2.4 in [8] implies that $C_1 \int_{\mathbb{R}^3} \rho^\alpha\,dx - C_2 < F_1(\rho)$ for two positive constants C_1 and C_2 depending only on α and M. Here, we only prove that $C_1 \int_{\mathbb{R}^3} \rho^\alpha\,dx - C_2 < F_2(\rho)$. By Hölder's inequality we have

$$F_2(\rho) = \frac{1}{2} W \int_{\mathbb{R}^3} \rho D(\rho)\,dx$$
$$\leq C\|\rho\|_{2-\varepsilon} \left\|D(\rho)\right\|_{\frac{2-\varepsilon}{1-\varepsilon}}.$$

Note that

$$D(\rho) = -\frac{\Lambda W}{\mu_0}\int_{\mathbb{R}^3}\frac{(x_1-y_1)^2+(x_2-y_2)^2}{((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2)^{\frac{3}{2}}}\rho(y)\,dy.$$

By the Riesz potential estimate [16], Lemma 7.12, p.159, we get

$$\big\|D(\rho)\big\|_{\frac{2-\varepsilon}{1-\varepsilon}} \le C|\Omega|^{\mu-\delta}\|\rho\|_p \quad \text{if } p < \frac{3(2-\varepsilon)}{7-5\varepsilon} \tag{3.33}$$

for $\mu = \frac{2}{3}$ and $\delta = \frac{1}{p} - \frac{1-\varepsilon}{2-\varepsilon}$, where Ω is the compact support of ρ.

By the interpolation inequality (Theorem 2.11 in [17]), if $f \in L^q \cap L^r$ $(1 \le q < p < r < \infty)$, then

$$\|f\|_p \le \|f\|_q^a \|f\|_r^{1-a} \tag{3.34}$$

for $a = \frac{p^{-1}-r^{-1}}{q^{-1}-r^{-1}}$.

Inserting $q = 1$, $r = 2-\varepsilon$, and $a = \frac{(2-\varepsilon)/p-1}{1-\varepsilon} < 1$ into (3.34), we get that (3.33) implies

$$\left|\frac{1}{2}\int_{\mathbb{R}^3}\rho D(\rho)\,dx\right| \le C\|\rho\|_{2-\varepsilon}^{2-a}\|\rho\|_1^a. \tag{3.35}$$

By the interpolation inequality [16], (7.6), p.145, we obtain

$$\int_{\mathbb{R}^3}\rho^{2-\varepsilon}\,dx \le \omega\int_{\mathbb{R}^3}\rho^\alpha\,dx + \omega^{-\frac{2-\varepsilon}{\alpha-2+\varepsilon}}|\Omega|^{\frac{\alpha-2+\varepsilon}{\alpha}} \le \omega\int_{\mathbb{R}^3}\rho^\alpha\,dx + \omega^{-\frac{2-\varepsilon}{\alpha-2+\varepsilon}}|\Omega|^{\frac{\alpha-2+\varepsilon}{\alpha}}, \tag{3.36}$$

where $(\alpha > 2-\varepsilon)$, and Ω is the compact support of ρ .

Together (3.35) with (3.36), we have

$$\left|\frac{1}{2}\int_{\mathbb{R}^3}\rho D(\rho)\,dx\right| \le \|\rho\|_1^a\left(\int_{\mathbb{R}^3}\omega\rho^\alpha\,dx + C(\omega)\right)^{\frac{2-a}{2-\varepsilon}},$$

where $C(\lambda) = \lambda^{-\frac{2-\varepsilon}{\alpha-2+\varepsilon}}|\Omega|^{\frac{\alpha-2+\varepsilon}{\alpha}}$.

Choosing $a = \varepsilon$, it is obvious that

$$p = \frac{2-\varepsilon}{1+\varepsilon-\varepsilon^2} > \frac{3(2-\varepsilon)}{7-5\varepsilon},$$

which implies that $7-5\varepsilon > 3+3\varepsilon-3\varepsilon^2$, $\varepsilon < \frac{2}{3}$.

It follows from $\alpha > 2-\varepsilon$ that

$$\alpha > \frac{4}{3} \quad \Longleftrightarrow \quad \varepsilon < \frac{2}{3}.$$

Hence, for $\alpha > \frac{4}{3}$, we have

$$\left|\frac{1}{2}\int_{\mathbb{R}^3}\rho D(\rho)\,dx\right| \le \omega M^a\int_{\mathbb{R}^3}\rho^\alpha\,dx + C(\omega)M^a.$$

Letting ω be sufficiently small, then we have

$$F(\rho) > C\int_{\mathbb{R}^3} \rho^\alpha \, dx - C_1. \qquad \square$$

Proof of Theorem 3.5 Note that Lemma 3.7 proves conclusion (2) of Theorem 3.5. Conclusion (3) in Theorem 3.5 can be proved by using the same method as in [8].

Notice that

$$\begin{aligned} F_2(\rho) &= -\frac{1}{2}W\int_{\mathbb{R}^3} \rho D(\rho)\,dx \\ &= -\frac{\Lambda W^2}{\mu_0}\rho\int_{\mathbb{R}^3} \frac{(x_1-y_1)^2+(x_2-y_2)^2}{((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2)^{\frac{3}{2}}}\rho(y)\,dy < 0. \end{aligned}$$

Also, by the argument in [8] we get

$$F_1(\rho) = \int_{\mathbb{R}^3}\left[I(\rho)+\frac{1}{2}\rho E(\rho)\right]dx < 0.$$

Hence, conclusion (1) of Theorem 3.5 is established. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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