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Spectral analysis of wave propagation on branching strings

Rakib F Efendiev[1], Hamzaga D Orudzhev[2] and Zaki FA El-Raheem[3*]

*Correspondence:
zaki55@hotmail.com
[3]Faculty of Education, Alexandria University, Alexandria, Egypt
Full list of author information is available at the end of the article

Abstract
In this paper the classical Hill problem with complex potentials are extended to the star graph. The definition of the Hill operator on such graph is discussed. The operator is defined with complex, periodic potentials and using special boundary conditions connecting values of the functions at the vertices. An explicit description of the resolvent is given and the spectrum is described exactly, the inverse problem with respect to the reflection coefficients is solved.

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1 Introduction

The purpose of the present paper is the spectral analysis of a wave propagation in a layered, inhomogeneous medium, such as a branching tube or a system of joined strings.

It is well known [1–4] that wave propagation in a one-dimensional non-conservative medium in a frequency domain is described by the Schrödinger equation,

$$-y''(x,\lambda)+\left[i\lambda p(x)+q(x)\right]y(x,\lambda)=\lambda^2 y(x,\lambda),\quad x\in R,$$

where R is the real axis, λ is the wave number (known as the momentum), λ^2 is the energy, $p(x)$ describes the joint effect of absorption and generation of energy, and $q(x)$ describes the regeneration of the force density.

As a model of layered, inhomogeneous medium we will use a special type of noncompact graph, star graph, that is, a flexible mathematical construction with single vertex in which a finite number of edges $N_k=[0,\infty)$, $k=1,2,3,\ldots,n$, are joined. The models which can be obtained by investigating differential operators on the graphs have both features of ordinary and partial differential operators.

In the problem of spectral analysis of a system of branching strings, we have the following correspondence: the strings correspond to the edges of the graphs and the points of the junctions of the strings correspond to the interior vertices.

Then for studying wave propagation on branching strings we must consider the system of equations

$$\begin{cases} -y_1''(x_1,\lambda) + [2\lambda p_1(x_1) + q_1(x_1)]y_1(x_1,\lambda) = \lambda^2 y_1(x_1,\lambda), \\ -y_2''(x_2,\lambda) + [2\lambda p_2(x_2) + q_2(x_2)]y_2(x_2,\lambda) = \lambda^2 y_2(x_2,\lambda), \\ -y_3''(x_3,\lambda) + [2\lambda p_3(x_3) + q_3(x_3)]y_3(x_3,\lambda) = \lambda^2 y_3(x_3,\lambda), \\ \cdots \\ -y_n''(x_n,\lambda) + [2\lambda p_n(x_n) + q_n(x_n)]y_n(x_n,\lambda) = \lambda^2 y_n(x_n,\lambda), \end{cases} \tag{1}$$

with the following boundary conditions at the initial points of the positive half axis satisfied:

$$y_1(0,\lambda) = y_2(0,\lambda) = y_3(0,\lambda) = \cdots = y_n(0,\lambda), \tag{2}$$

$$y_1'(0,\lambda) + y_2'(0,\lambda) + y_3'(0,\lambda) + \cdots + y_n'(0,\lambda) = 0, \tag{3}$$

in the space

$$L_2(G) = \bigoplus_{k=1}^{n} L_2[o_k,\infty),$$

where the notation o_k with subscript k to denote the initial point 0 of the kth positive half axis is used and the direct sum of the spaces is denoted by $\bigoplus$. The prime denotes the derivative with respect to space coordinate and λ is a complex number.

We assume that the potentials $p_k(x_k)$ and $q_k(x_k)$, $k = 1,2,3$, are of the form

$$p_k(x_k) = \sum_{n=1}^{\infty} p_{kn} e^{inx_k}, \quad \sum_{n=1}^{\infty} n|p_{kn}| < \infty; \tag{4}$$

$$q_k(x_k) = \sum_{n=1}^{\infty} q_{kn} e^{inx_k}, \quad \sum_{n=1}^{\infty} |q_{kn}| < \infty. \tag{5}$$

For simplicity in deriving the results, without any loss of generality, in the future we will consider the case $n = 3$.

In particular, spectral analyses of the operator with the periodic potential of the type $q(x) = \sum_{n=1}^{\infty} q_n e^{inx}$ in $L_2(-\infty,+\infty)$ have been studied by Gasymov [5], Shin [6], Carlson [7, 8], Guillemin and Uribe [9], and Pastur and Tkachenko [10]. As a final remark we mention [2, 11–15]. More information as regards the potentials can be found in [11].

Now we define the space $L_2(G)$

$$L_2(G) = \bigoplus_{k=1}^{3} L_2(N_k),$$

with scalar product

$$(f,g)_{L_2(G)} = \sum_{k=1}^{3} (f_k,g_k)_{L_2(N_k)},$$

and we consider the operator L_G

$$L_G = \bigoplus_{k=1}^{3} L_k,$$

where

$$L_k = -d^2/dx_k^2 + 2\lambda p_k(x_k) + q_k(x_k) \tag{6}$$

with domain

$$\begin{aligned} D(L_G) = \big\{ & y(x) | y_k(x), y_k'(x) \in AC[0,R] \text{ for all } R > 0, y_1(0) = y_2(0) = y_3(0) \\ & y_1'(0) + y_2'(0) + y_3'(0) = 0, y_k(x), y_k''(x) \in L_2(\mathbb{R}^+), k = 1, 2, 3 \big\}. \end{aligned}$$

Then the considered problem (1)-(5) can be interpreted as a study of the operator

$$L_G = \bigoplus_{k=1}^{3} L_k$$

on noncompact graph introduced as above.

The idea of investigation of quantum particles confined to a graph is rather old. The first justification of quasi-one-dimensional motion of electrons in aromatic compounds was given by Pauling [16] and worked out by Ruedenberg and Scherr [17] in 1953. Within the framework of the proposed approach each chemical bound is replaced by a narrow tube in which the electron moves and from which he cannot get away. Using honey graphs with the - Kirchhoff - boundary condition in combination with the Pauli principle, they reproduced the actual spectra with 10 percent accuracy.

As a result of that, the problem can be considered as a generalization of the classical inverse spectral problem for the Schrödinger operator on the line.

For studying the wave propagation on the graph a second order (ordinary) differential Hamiltonian is used. The Hamiltonian is a Schrödinger operator with zero Dirichlet condition on its boundary. The Dirichlet condition is responsible for confinement of electrons to the vicinity of graph.

The Schrödinger operator is defined on a graph in the following way. On each edge the wave function is a solution of the one-dimensional equation. At each vertex the wave equation must be uniquely defined.

The spectral problems on graphs arise in the investigation of processes in various domains of natural science; from complex molecules to neuron systems. Methods developed by mathematicians make it possible to describe such problems in terms of the differential equations by constructing for these problems an exact analogue of the Sturm-Liouville theory.

Without a claim to completeness of the investigation of inverse problems on graphs we list here the works of Carlson [8], Freiling and Yurko [18], Gerasimenko [19], Kostrykin and Schrader [20], Kuchment [21], and Pivovarchik [22, 23].

The paper consists of three sections.

In Section 1 we introduce the main notions and give a formulation of the direct problem.

In Section 2 the properties of the spectrum are studied. It is proved that the continuous spectrum of the operator fill out the $\operatorname{Im}\lambda = 0$ axis on which there may exist spectral singularities coinciding with the numbers $n/2, n = \pm1, \pm2, \pm3, \ldots$. Moreover, there may be a finite number of eigenvalues outside the interval $(-\infty, +\infty)$.

In Section 3 we give a formulation of the inverse problem and provide a constructive procedure for the solution of the inverse problem.

1.1 Formulation of the direct problem

The spectral problem can be described as follows:

Find the vector $y_k(x_k, \lambda) = (y_{k1}(x_1, \lambda), y_{k2}(x_2, \lambda), y_{k3}(x_3, \lambda))$ satisfying the Sturm-Liouville equation

$$-y_k''(x_k, \lambda) + 2\lambda p_k(x_k) y_k(x_k, \lambda) + q_k(x_k) y_k(x_k, \lambda) = \lambda^2 y_k(x_k, \lambda), \tag{7}$$

on N_k, $k = 1, 2, 3$, coupled at zero by the usual Kirchhoff conditions and complemented with initial conditions for the functions $y_k(x_k, \lambda)$, $k = 1, 2, 3$.

(a) y_k is continuous at the nodes of the graph, *i.e.*, in particular for our graph

$$y_{k1}(0, \lambda) = y_{k2}(0, \lambda) = y_{k3}(0, \lambda); \tag{8}$$

(b) the sum of the derivatives over all the branches emanating from a node, calculated for each node, is zero,

$$y_{k1}'(0, \lambda) + y_{k2}'(0, \lambda) + y_{k3}'(0, \lambda) = 0. \tag{9}$$

It is well known (see [2]) that, for each fixed $k = 1, 2, 3$ on the edge N_k, there exists a fundamental system of solutions of equations (7) $f_k^{\pm}(x_k, \lambda)$ for $\lambda \neq \pm n/2$, $n \in N$, and $\lambda \neq 0$ with the properties:

$$f_k^{\pm}(x_k, \lambda) = e^{\pm i\lambda x_k} \left(1 + \sum_{n=1}^{\infty} V_n^{(\pm k)} e^{inx_k} + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{(\pm k)}}{n \pm 2\lambda} e^{i\alpha x_k} \right), \tag{10}$$

where the numbers $V_n^{(\pm k)}$, $V_{n\alpha}^{(\pm k)}$ are defined by the following recurrent formulas:

$$\alpha^2 V_\alpha^{(\pm k)} + \alpha \sum_{n=1}^{\alpha} V_{n\alpha}^{(\pm k)} + \sum_{s=1}^{\alpha-1} \left(q_{k\alpha-s} V_s^{(\pm k)} \pm p_{k\alpha-s} \sum_{n=1}^{s} V_{ns}^{(\pm k)} \right) + q_{k\alpha} = 0, \tag{11}$$

$$\alpha(\alpha - n) V_{n\alpha}^{(\pm k)} + \sum_{s=n}^{\alpha-1} (q_{k\alpha-s} \mp n \cdot p_{k\alpha-s}) V_{ns}^{(\pm k)} = 0, \tag{12}$$

$$\alpha V_\alpha^{(\pm k)} \pm \sum_{s=1}^{\alpha-1} V_s^{(\pm k)} p_{k\alpha-s} \pm p_{k\alpha} = 0, \tag{13}$$

and the series

$$\sum_{n=1}^{\infty} n^2 \left|V_n^{\pm}\right|;$$
$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha(\alpha-n)\left|V_{n\alpha}^{\pm}\right|;$$
$$\sum_{n=1}^{\infty} n \cdot \left|V_{nn}^{\pm}\right|$$

are convergent.

Let us introduce

$$\begin{aligned} f_{nk}^{\pm}(x_k) &= \lim_{\lambda \to \mp \frac{n}{2}} (n \pm 2\lambda) f_k^{\pm}(x_k, \lambda) \\ &= \sum_{\alpha=n}^{\infty} V_{n\alpha}^{(\pm k)} e^{i\alpha x_k} e^{-i\frac{n}{2}x_k}. \end{aligned} \tag{14}$$

It follows from (12) that $f_{nk}^{\pm}(x_k) \neq 0$ is valid for $V_{n\alpha}^{(\pm k)} \neq 0$.

From (14) it follows that the linearly independent solutions of (7) according to $\lambda = \pm n/2$, $n \in N$ can be determined as

$$\tilde{f}_{nk}^{\pm}(x_k) = \lim_{\lambda \to \mp \frac{n}{2}} \left[f_k^{\pm}(x_k, \lambda) + \frac{V_{nn}^{\pm k} f_k^{\mp}(x_k, \lambda)}{n \pm 2\lambda} \right]. \tag{15}$$

According to the expressions for $f_k^{\pm}(x_k, \lambda)$ we can say that

$$\tilde{f}_{nk}^{\pm}(x_k) = e^{-i\frac{n}{2}x_k} \left(\varphi_{kn}^{\pm}(x_k) + x_k \tilde{\varphi}_{kn}^{\pm}(x_k) \right),$$

where $\varphi_{kn}^{\pm}(x_k)$, $\tilde{\varphi}_{kn}^{\pm}(x_k)$ are periodic functions. Obviously $\tilde{f}_{nk}^{\pm}(x_k)$ and $f_k^{\mp}(x_k, \lambda)$ are linearly independent solutions of (7) for $\lambda = \pm n/2$, $n \in N$.

Linearly independent solutions of equation (7) corresponding to $\lambda = 0$ are defined as

$$f_k^{+}(x_k, 0), \quad \text{and} \quad \frac{df_k^{+}(x_k, 0)}{d\lambda} = (ix_k)\left[f_k^{+}(x_k, 0) + o(1)\right], \quad |x_k| \to \infty. \tag{16}$$

As a solution of the problem we will understand a matrix

$$Y(x, \lambda) = \left[y_{jk}(x_k, \lambda)\right]_{k,j=1,2,3}$$

on the noncompact graph on the basis of the following requirements:

1.
$$L_G Y = \lambda^2 Y;$$

2. $y_{jk}(x_k, \lambda)$ is a solution on the ray $N_k = [0, \infty), k = 1, 2, 3;$

3.

$$y_{jk}(x_k,\lambda) = T_{jk}(\lambda) f_k^+(x_k,\lambda), \quad k \neq j; \tag{17}$$

and

$$y_{kk}(x_k,\lambda) = f_k^-(x_k,\lambda) + R_{kk}(\lambda) f_k^+(x_k,\lambda), \quad k = 1,2,3. \tag{18}$$

According to the physical meaning of the solutions $Y(x,\lambda) = [y_{jk}(x_k,\lambda)]_{k,j=1,2,3}$, it is natural to say that $T_{kj}(\lambda)$ are the transmission coefficients and $R_{kk}(\lambda)$ are the reflection coefficients for equation (7).

The coefficients $T_{kj}(\lambda)$ and $R_{kk}(\lambda)$ can be found by writing down the boundary conditions (8), (9) for the solution $y_{jk}(x_k,\lambda)$.

To be specific, suppose $k = 1$, then

$$f_1^-(0,\lambda) + R_{11}(\lambda) f_1^+(0,\lambda) = T_{12}(\lambda) f_2^+(0,\lambda) = T_{13}(\lambda) f_3^+(0,\lambda),$$
$$f_1^{-\prime}(0,\lambda) + R_{11}(\lambda) f_1^{+\prime}(0,\lambda) + T_{12}(\lambda) f_2^{+\prime}(0,\lambda) + T_{13}(\lambda) f_3^{+\prime}(0,\lambda) = 0.$$

We solve these equations for $R_{11}(\lambda)$, $T_{12}(\lambda)$ and $T_{13}(\lambda)$. We note that, for the Wronskian of the solutions, $W[f_1^+(0,\lambda), f_1^-(0,\lambda)] = 2i\lambda$, we obtain

$$\begin{aligned}
R_{11}(\lambda) &= \frac{\begin{vmatrix} -f_1^-(0,\lambda) & -f_2^+(0,\lambda) & 0 \\ -f_1^-(0,\lambda) & 0 & -f_3^+(0,\lambda) \\ -f_1^{-\prime}(0,\lambda) & f_2^{+\prime}(0,\lambda) & f_3^{+\prime}(0,\lambda) \end{vmatrix}}{\begin{vmatrix} f_1^+(0,\lambda) & -f_2^+(0,\lambda) & 0 \\ f_1^+(0,\lambda) & 0 & -f_3^+(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_2^{+\prime}(0,\lambda) & f_3^{+\prime}(0,\lambda) \end{vmatrix}} \\
&= \frac{[f_1^-(0,\lambda) f_2^+(0,\lambda) f_3^+(0,\lambda)]'}{[f_1^+(0,\lambda) f_2^+(0,\lambda) f_3^+(0,\lambda)]'} \\
&= -\frac{f_1^-(0,\lambda)}{f_1^+(0,\lambda)} + \frac{2i\lambda}{f_1^+(0,\lambda) f_1^+(0,\lambda) G(\lambda)}, \\
T_{12}(\lambda) &= \frac{\begin{vmatrix} f_1^+(0,\lambda) & -f_1^-(0,\lambda) & 0 \\ f_1^+(0,\lambda) & -f_1^-(0,\lambda) & -f_3^+(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & -f_1^{-\prime}(0,\lambda) & f_3^{+\prime}(0,\lambda) \end{vmatrix}}{\begin{vmatrix} f_1^+(0,\lambda) & -f_2^+(0,\lambda) & 0 \\ f_1^+(0,\lambda) & 0 & -f_3^+(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_2^{+\prime}(0,\lambda) & f_3^{+\prime}(0,\lambda) \end{vmatrix}} = \frac{2i\lambda}{f_1^+(0,\lambda) f_2^+(0,\lambda) G(\lambda)}, \\
T_{13}(\lambda) &= \frac{\begin{vmatrix} f_1^+(0,\lambda) & -f_2^+(0,\lambda) & -f_1^-(0,\lambda) \\ f_1^+(0,\lambda) & 0 & -f_1^-(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_2^{+\prime}(0,\lambda) & -f_1^{-\prime}(0,\lambda) \end{vmatrix}}{\begin{vmatrix} f_1^+(0,\lambda) & -f_2^+(0,\lambda) & 0 \\ f_1^+(0,\lambda) & 0 & -f_3^+(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_2^{+\prime}(0,\lambda) & f_3^{+\prime}(0,\lambda) \end{vmatrix}} = \frac{2i\lambda}{f_1^+(0,\lambda) f_3^+(0,\lambda) G(\lambda)},
\end{aligned}$$

where

$$G(\lambda) = \frac{f_1^{+'}(0,\lambda)}{f_1^+(0,\lambda)} + \frac{f_2^{+'}(0,\lambda)}{f_2^+(0,\lambda)} + \frac{f_3^{+'}(0,\lambda)}{f_3^+(0,\lambda)}.$$

2 The properties of the spectrum

To study the spectrum of the operator L_G at first we calculate the kernel of the resolvent of the operator $(L_G - \lambda^2 E)$. Note that every solution $\Psi_k(x_k,\lambda)$ of the problem on the edge $N_k = [0,\infty)$, $k = 1,2,3$, is a linear combination of the functions $y_{kk}(x_k,\lambda)$, $y_{jk}(x_k,\lambda)$, $j \neq k$, $j,k = 1,2,3$, and can be written in the form

$$\Psi_k(x_k,\lambda) = C_{0j}^{(k)}(x_k)y_{kk}(x_k,\lambda) + C_{1j}^{(k)}(x_k)y_{jk}(x_k,\lambda), \quad j \neq k, j,k = 1,2,3,$$

where $C_{0j}^{(k)}(x_k)$ and $C_{1j}^{(k)}(x_k)$ are such that conditions (8)-(9) hold for $\Psi_k(x_k,\lambda)$.

We will construct the resolvent of the operator L_G for $\operatorname{Im}\lambda > 0$.

To this aim, we solve the problem

$$\begin{aligned} &-y_k''(x_k,\lambda) + 2\lambda p_k(x_k)y_k(x_k,\lambda) + q_k(x_k)y_k(x_k,\lambda) = \lambda^2 y_k(x_k,\lambda) + \varphi_k(x_k), \\ &\quad k = 1,2,3, \end{aligned} \tag{19}$$

in the space $L_2[0_k,\infty)$. Here $\varphi_k(x_k)$ is an arbitrary function belonging to the space $L_2[0_k,\infty)$, $k = 1,2,3$.

By taking into account the relation

$$W\big[y_{kk}(x_k,\lambda), y_{jk}(x_k,\lambda)\big] = 2i\lambda T_{jk}(\lambda)$$

to find $C_{0j}^{(k)}(x_k)$ and $C_{1j}^{(k)}(x_k)$ we have

$$\begin{aligned} C_{0j}^{(k)}(x_k) &= \frac{1}{2i\lambda T_{jk}(\lambda)} \int_{x_k}^{\infty} y_{jk}(t_k,\lambda)\varphi_k(t_k)\,dt_k + C_{0j}^{(k)}(\infty), \\ C_{1j}^{(k)}(x_k) &= \frac{1}{2i\lambda T_{jk}(\lambda)} \int_{0_k}^{x_k} y_{kk}(t_k,\lambda)\varphi_k(t_k)\,dt_k + C_{1j}^{(k)}(o_k), \end{aligned}$$

where $x_k \in [0_k,\infty) = N_k$ and $C_{0j}^{(k)}(\infty)$, $C_{1j}^{(k)}(o_k)$ are arbitrary numbers.

Then

$$\begin{aligned} \Psi_k(x_k,\lambda) = {}& \frac{1}{2i\lambda T_{jk}(\lambda)} \int_{x_k}^{\infty} y_{kk}(x_k,\lambda)y_{jk}(t_k,\lambda)\varphi_k(t_k)\,dt_k + C_{0j}^{(k)}(\infty)y_{kk}(x_k,\lambda) \\ &+ \frac{1}{2i\lambda T_{jk}(\lambda)} \int_{0_k}^{x_k} y_{jk}(x_k,\lambda)y_{kk}(t_k,\lambda)\varphi_k(t_k)\,dt_k \\ &+ C_{1j}^{(k)}(o_k)y_{jk}(x_k,\lambda), \quad j \neq k, j,k = 1,2,3. \end{aligned}$$

By virtue of the condition $\Psi_k(\bullet,\lambda) \in L_2[0_k,\infty), y_{kk}(x_k,\lambda) \notin L_2[o_k,\infty)$, $y_{jk}(x_k,\lambda) \in L_2[o_k,\infty)$ we find that $C_{0j}^{(k)}(\infty) = 0$.

Then

$$\begin{aligned} \Psi_k(x_k,\lambda) = {}& \frac{1}{2i\lambda T_{jk}(\lambda)} \biggl[\int_{x_k}^{\infty} y_{kk}(x_k,\lambda)y_{jk}(t_k,\lambda)\varphi_k(t_k)\,dt_k \\ &+ \int_{o_k}^{x_k} y_{jk}(x_k,\lambda)y_{kk}(t_k,\lambda)\varphi_k(t_k)\,dt_k\biggr] \\ &+ C_{1j}^{(k)}(o_k)y_{jk}(x_k,\lambda), \quad j \neq k, j,k = 1,2,3, \end{aligned} \tag{20}$$

and

$$\Psi_k(o_k,\lambda) = \frac{1}{2i\lambda T_{jk}(\lambda)} \int_{0_k}^{\infty} y_{kk}(o_k,\lambda) y_{jk}(t_k,\lambda)\varphi_k(t_k)\,dt_k + C_{1j}^{(k)}(o_k) y_{jk}(o_k,\lambda),$$
$$j \neq k, j, k = 1, 2, 3.$$

Let us denote

$$I_k(\lambda) = \frac{1}{2i\lambda T_{jk}(\lambda)} \int_{0_k}^{\infty} y_{jk}(t_k,\lambda)\varphi_k(t_k)\,dt_k,$$
$$C_1^{(k)}(\lambda) = C_{1j}^{(k)}(o_k) T_{jk}(\lambda);$$

then by taking into account (17) we have

$$\Psi_k(o_k,\lambda) = y_{kk}(o_k,\lambda) I_k(\lambda) + C_1^{(k)}(\lambda) f_k^{+}(o_k,\lambda).$$

For finding the constants $C_1^{(k)}(\lambda)$, $k = 1, 2, 3$, we will use the boundary conditions (8)-(9) and obtain

$$C_1^1(\lambda) f_1^+(o_1,\lambda) - C_1^2(\lambda) f_2^+(o_2,\lambda) = y_{22}(o_2,\lambda) I_2(\lambda) - y_{11}(o_1,\lambda) I_1(\lambda),$$
$$C_1^1(\lambda) f_1^+(o_1,\lambda) - C_1^3(\lambda) f_3^+(o_3,\lambda) = y_{33}(o_3,\lambda) I_3(\lambda) - y_{11}(o_1,\lambda) I_1(\lambda),$$
$$C_1^1(\lambda) f_1'^+(o_1,\lambda) + C_1^2(\lambda) f_2'^+(o_2,\lambda) + C_1^3(\lambda) f_3'^+(o_3,\lambda)$$
$$= -y_{11}'(o_1,\lambda) I_1(\lambda) - y_{22}'(o_2,\lambda) I_2(\lambda) - y_{33}'(o_3,\lambda) I_3(\lambda).$$

The system of equations can be written as

$$F(\lambda) \cdot C(\lambda) = Y(\lambda)$$

where for $f_k^+ = f_k^+(o_k,\lambda)$, $y_{kk} = y_{kk}(o_k,\lambda)$

$$F(\lambda) = \begin{pmatrix} f_1^+ & -f_2^+ & 0 \\ f_1^+ & 0 & -f_3^+ \\ f_1'^+ & f_2'^+ & f_3'^+ \end{pmatrix},$$

$$C(\lambda) = \begin{pmatrix} C_1^1(\lambda) \\ C_1^2(\lambda) \\ C_1^3(\lambda) \end{pmatrix},$$

and

$$Y(\lambda) = -\begin{pmatrix} y_{22} I_2(\lambda) - y_{11} I_1(\lambda) \\ y_{33} I_3(\lambda) - y_{11} I_1(\lambda) \\ y_{11}' I_1(\lambda) + y_{22}' I_2(\lambda) + y_{33}' I_3(\lambda) \end{pmatrix}.$$

From this we find that

$$C(\lambda) = F^{-1}(\lambda) \cdot Y(\lambda)$$

with

$$F^{-1}(\lambda) = \frac{\begin{pmatrix} f_2'^+ f_3^+ & -f_2^+ f_3'^+ & f_2^+ f_3^+ \\ f_1'^+ f_3^+ - f_1^+ f_3'^+ & f_1^+ f_3'^+ & f_1^+ f_3^+ \\ f_1^+ f_2^+ & f_1'^+ f_2^+ - f_1^+ f_2'^+ & -f_1^+ f_2^+ \end{pmatrix}}{\Delta}$$

and

$$\Delta = \det F(\lambda) = \left[f_1^+ \cdot f_2^+ \cdot f_3^+\right]'.$$

By using the last relation we can find the coefficients $C_1^{(k)}(\lambda)$, $k = 1, 2, 3$, as

$$C_1^k(\lambda) = \beta_1^k(\lambda) I_1(\lambda) + \beta_2^k(\lambda) I_2(\lambda) + \beta_3^k(\lambda) I_3(\lambda).$$

To be specific, suppose $k = 1$, then

$$\beta_1^1(\lambda) = \frac{-y_{11} f_2^{+\prime} f_3^+ + y_{11} f_2^+ f_3^{+\prime} - y_{11}' f_2^+ f_3^+}{\Delta},$$
$$\beta_2^1(\lambda) = \frac{y_{22} f_2^{+\prime} f_3^+ - y_{22}' f_2^+ f_3^+}{\Delta},$$
$$\beta_3^1(\lambda) = \frac{y_{33} f_2^+ f_3^{+\prime} - y_{33}' f_2^+ f_3^+}{\Delta}.$$

By taking into account (17) we can rewrite equation (20) as

$$\Psi_k(x_k, \lambda) = \frac{1}{2i\lambda}\left[\int_{x_k}^{\infty} y_{kk}(x_k, \lambda) f_k^+(t_k, \lambda)\varphi_k(t_k)\,dt_k + \int_{o_k}^{x_k} f_k^+(x_k, \lambda) y_{kk}(t_k, \lambda)\varphi_k(t_k)\,dt_k\right]$$
$$+ \sum_{j=1}^{3} \frac{\beta_j^k(\lambda)}{2i\lambda} \int_{o_j}^{\infty} f_j^+(t_j, \lambda) f_k^+(x_k, \lambda)\varphi(t_j)\,dt_j, \quad k = 1, 2, 3.$$

It is readily seen that the function $\Psi(x, \lambda) = (\Psi_1(x_1, \lambda), \Psi_2(x_2, \lambda), \Psi_3(x_3, \lambda))$ where

$$\Psi_k(x_k, \lambda) = \sum_{j=1}^{3} \int_{o_j}^{\infty} G_{kj}(x_k, t_j, \lambda)\varphi(t_j)\,dt_j, \quad k = 1, 2, 3,$$

with

$$G_{kk}(x_k, t_k, \lambda) = \frac{1}{2i\lambda} \begin{cases} [y_{kk}(x_k, \lambda) + \beta_k^k(\lambda) f_k^+(x_k, \lambda)] f_k^+(t_k, \lambda), & o_k < x_k < t_k < \infty, \\ [y_{kk}(t_k, \lambda) + \beta_k^k(\lambda) f_k^+(t_k, \lambda)] f_k^+(x_k, \lambda), & o_k < t_k < x_k < \infty, \end{cases} \tag{21}$$

and

$$G_{jk}(x_k, t_j, \lambda) = \frac{\beta_j^k(\lambda)}{2i\lambda} f_k^+(x_k, \lambda) f_j^+(t_j, \lambda), \quad o_k < x_k < \infty, o_j < t_j < \infty, j \neq k, j, k = 1, 2, 3, \tag{22}$$

are sufficiently smooth and satisfy the boundary conditions (8) and (9), *i.e.* they are contained in the domain of the operator L_G. Thus, the constructed 'spectral' Green's function

$$G(x, t, \lambda) = \begin{cases} G_{kk}(x_k, t_k, \lambda), \\ G_{jk}(x_k, t_j, \lambda), \end{cases} \quad j \neq k, j, k = 1, 2, 3,$$

is the kernel of the resolvent $(L_G - \lambda^2 E)^{-1}$, which is an integral operator. The poles of the resolvent (poles of the Green's function) are eigenvalues of the operator L_G and can be found as zeros of the determinants of the matrices that participate in the construction of the Green's function.

A point $\lambda_0 \in \sigma(L_G)$ where $\sigma(L_G)$ is the set of spectrum of the operator L_G we call a spectral singularity of the operator L_G, in the sense of Naimark [24], if it is not an isolated eigenvalue of L_G, but $G(x,t,\lambda) \to \infty$ as $\lambda \in \rho(\lambda)$ ($\rho(\lambda)$ is the set of all regular points of the operator L_G) and $\lambda \to \lambda_0$.

Note that self-adjoint operators have no spectral singularities and for non-self-adjoint operators the spectral singularities correspond to resonance states with vanishing spectral width [25].

Thus, the procedure described above makes it possible to obtain explicitly the resolvent and calculate its poles.

So, we proved the following theorem.

Theorem 1 *Assume $F(\lambda)$ is nonsingular i.e. $F^{-1}(\lambda)$ exists, then for any*

$$\varphi = \{\varphi_1, \varphi_2, \varphi_3\}, \qquad \varphi_k \in L_2(N_k), \quad k = 1,2,3,$$

the unique solution $\Psi(x,\lambda) = (\Psi_1(x_1,\lambda), \Psi_2(x_2,\lambda), \Psi_3(x_3,\lambda))$ of (7), (8)-(9) *is given by*

$$\Psi_k(x_k,\lambda) = \sum_{j=1}^{3} \int_{o_j}^{\infty} G_{kj}(x_k,t_j,\lambda)\varphi(t_j)\,dt_j, \quad k = 1,2,3,$$

where $G_{jk}(x_k,t_j,\lambda), j,k = 1,2,3$, are determined by (21)-(22).

Theorem 2 *The operator L_G has no real eigenvalue.*

Proof We recall that equation (7) has fundamental solutions $f_k^{\pm}(x_k,\lambda)$ in the case $\lambda \neq 0, \pm n/2$. Then for the case $\lambda \neq 0, \pm n/2$ the solution of equation (7) on the edge N_k, $k = 1,2,3$, can be written in the form

$$y_k(x_k,\lambda) = C_1 f_k^{+}(x_k,\lambda) + C_2 f_k^{-}(x_k,\lambda).$$

So, the solution of equation (7) belonging to $L_2(G) = \bigoplus_{k=1}^{3} L_2[o_k,\infty)$ and satisfying the conditions (8)-(9) is necessarily has $C_1 = 0$ and $C_2 = 0$, $y_k(x_k,\lambda) = 0$. That shows that equation (7) has only a trivial solution belonging to $L_2(G) = \bigoplus_{k=1}^{3} L_2[o_k,\infty)$ for $\lambda \in (-\infty,+\infty)$, $\lambda \neq 0, \pm n/2$.

If as linearly independent solutions (15) and (16) of (7) according to $\lambda = \pm n/2$ or $\lambda = 0$ are taken instead of $f_k^{\pm}(x_k,\lambda)$ then a similar result also will be valid. So we proved that L_G has no real eigenvalue. □

Theorem 3 *The eigenvalues of operator L_G are finite and coincide with the zeros of the function $\Delta(\lambda)$.*

Proof From equation (10) it is easy to see that for $j = 1,2,3$

$$\begin{aligned}
\left|f_j^+(0,\lambda)\right| &= 1 + \sum_{n=1}^{\infty}\left|V_n^{(+j)}\right| + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\left|\frac{V_{\alpha n}^{(+j)}}{n+2\lambda}\right| \\
&< 1 + \sum_{n=1}^{\infty}\left|V_n^{(+j)}\right| + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{|V_{\alpha n}^{(+j)}|}{\sqrt{(n+2\operatorname{Re}\lambda)^2+4\operatorname{Im}^2\lambda}} \\
&< 1 + \sum_{n=1}^{\infty}\left|V_n^{(+j)}\right| + \frac{1}{|\operatorname{Im}\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\alpha}{n}\left|V_{\alpha n}^{(+j)}\right|.
\end{aligned}$$

Therefore, as $|\lambda| \to \infty$ we obtain $f_j^+(0,\lambda) = C_j + o(1)$, $j = 1,2,3$. Then for $\Delta = \det F(\lambda)$ we get the following asymptotic equalities:

$$\Delta(\lambda) = 3i\lambda C + o(1),$$

where C is a constant.

This asymptotic equality shows that the eigenvalues of the operator L_G are finite and coincide with the zeros of the function $\Delta(\lambda)$.

The theorem is proved. □

Theorem 4 *The spectrum of the operator L_G consists of the continuum spectrum filling the axis $-\infty < \lambda < +\infty$ on which there may exist spectral singularities coinciding with the numbers $\pm n/2$, $n \in N$.*

Proof In order for all numbers $\lambda \in (-\infty, +\infty)$ to belong to the continuous spectra it suffices to show that the operator has no real eigenvalue, the domain of $R_{L_G-\lambda^2 I}$ (the resolvent set) of the operator $(L_G - \lambda^2 I)$ is dense in $L_2(G)$, and the range of $R_{L_G-\lambda^2 I}$ is not equal to $L_2(G)$.

The absence of real spectra of L_G was proved above in Theorem 2.

To show that the domain of $R_{L_G-\lambda^2 I}$ (the resolvent set) of the operator $(L_G - \lambda^2 I)$ is dense in $L_2(G)$ we must prove that the orthogonal complement of the set $R_{L_G-\lambda^2 E}$ consists of only the zero element.

It is well known that the orthogonal complement of the set $R_{L_G-\lambda^2 E}$ coincides with the space of the solutions of the equation $L_G^* f = \lambda^2 f$ where the operator L_G^* is adjoint to the operator L_G.

Let $\psi_k(x_k) \in L_2[o_k, +\infty)$, $\psi_k(x_k) \neq 0$ and

$$\int_{0_k}^{+\infty}\left(L_G f_k - \lambda^2 f_k\right)\overline{\psi_k(x_k)}\,dx_k = 0, \quad k = 1,2,3, \tag{23}$$

be satisfied for any $f_k(x_k) \in D(L_G)$.

From (23) it follows that $\psi_k(x_k) \in D(L_G^*)$ and $\psi_k(x_k)$ are eigenfunctions of the operator L_G^* corresponding to the eigenvalues λ.

In fact $\overline{\psi_k(x_k)}$ is the solution of the equation

$$-z_k'' + \left[i\lambda p_k(x_k) + q_k(x_k)\right]z_k = \lambda^2 z_k \tag{24}$$

belonging to $L_2(G)$. We found that $\psi_k(x_k) = 0$, since the operator generated by the expression standing at the left hand side of (24) is an operator of type L_G.

This contradiction shows that the domain of $R_{L_G-\lambda^2 I}$ of the operator $(L_G - \lambda^2 I)$ is everywhere dense in $L_2(G)$.

Now let us prove that the range of $R_{L_G-\lambda^2 I}$ is not equal to $L_2(G)$. For this purpose we have to show that there is a function $f(x)$ from the space $L_2(G)$ for which the equation

$$L_G y = f \tag{25}$$

has no solution.

Indeed for the compact supported function $f(x) = (f_1(x_1), f_2(x_2), f_3(x_3))$ defined on $L_2(G)$ by

$$f(x) = \begin{cases} \varphi(x) & \text{if } 0 \le x \le a, \\ 0 & \text{if } x > a, \end{cases}$$

where

$$\varphi(x) = \bigl(\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3)\bigr)$$

is a solution of the following problem:

$$L_G \varphi = 0,$$
$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = 0$$

on $L_2(0,\infty)$ equation (25) has no solution. To prove this fact we assume the contrary. Let equation (25) have a solution belonging to $L_2(G)$. Then from Theorem 2 it follows that for $x > a$ the function $y(x,\lambda)$ will be a solution of (25) only under the condition

$$y(x,\lambda) = y'(x,\lambda) = 0.$$

Then from (25) we obtain

$$\begin{aligned}
(f,\bar{\varphi}) &= (L_G y, \bar{\varphi}) \\
&= \int_0^{+\infty} L_1 y_1(x_1)\varphi_1(x_1)\,dx_1 + \int_0^{+\infty} L_2 y_2(x_2)\varphi_2(x_2)\,dx_2 + \int_0^{+\infty} L_3 y_3(x_3)\varphi_3(x_3)\,dx_3 \\
&= \bigl[y_1'(x_1)\varphi_1(x_1) - y_1(x_1)\varphi_1'(x_1)\bigr]\big|_{x_1=0}^{a} + \bigl[y_2'(x_2)\varphi_2(x_2) - y_2(x_2)\varphi_2'(x_2)\bigr]\big|_{x_2=0}^{a} \\
&\quad + \bigl[y_3'(x_3)\varphi_3(x_3) - y_3(x_3)\varphi_3'(x_3)\bigr]\big|_{x_3=0}^{a} + \int_0^a y_1(x_1) L_1\varphi_1(x_1)\,dx_1 \\
&\quad + \int_0^a y_2(x_2) L_2\varphi_2(x_2)\,dx_2 + \int_0^a y_3(x_3) L_3\varphi_3(x_3)\,dx_3 \\
&= \bigl[y_1(0)\varphi_1'(0) + y_2(0)\varphi_2'(0) + y_3(0)\varphi_3'(0)\bigr] - \bigl[y_1'(0)\varphi_1(0) + y_2'(0)\varphi_2(0) + y_3'(0)\varphi_3(0)\bigr] \\
&= \bigl[y_1(0)\cdot 0 + y_2(0)\cdot 0 + y_3(0)\cdot 0\bigr] - \bigl[y_1'(0) + y_2'(0) + y_3'(0)\bigr] = 0;
\end{aligned}$$

on the other hand it is easy to see that

$$(f,\bar{\varphi}) = \sum_{k=1}^{3}\int_0^{+\infty}\bar{\varphi}_k(x_k)\varphi_k(x_k)\,dx_k = \sum_{k=1}^{3}\int_0^{a}\left|\varphi_k(x_k)\right|^2 dx_k > 0.$$

This contradiction shows that equation (25) has no solution belonging to $L_2(G)$. So it is proved that the range of $R_{L_G-\lambda^2 I}$ is not equal to $L_2(G)$.

From (14) we find that the functions

$$f_k^{\pm}(x_k,\lambda) = \frac{f_{nk}^{\pm}(x_k)}{n \pm 2\lambda} + \Phi_k^{\pm}(x_k,\lambda),$$

where $\Phi_k^{\pm}(x_k,\lambda)$, have no poles at the points $\mp n/2$, $n \in N$.

By using this fact it is easy to show that the Green's function $G(x,t,\lambda)$ has poles of first order at the points $\lambda_0 = \pm\frac{n}{2}, n \in N$. Therefore $\lambda = \pm\frac{n}{2}$, $n \in N$ is a spectral singularity of the operator L_G.

The theorem is proved. □

3 The inverse spectral problem on star graph

We will consider recovering the differential operator on each fixed edge.

Since the coefficients $R_{kk}(\lambda)$ may be found by the using matching conditions (8)-(9) at a central vertex, it is natural to formulate the inverse problem as: recovering of the potentials $p_k(x_k)$ and $q_k(x_k)$ at each edge by the reflection coefficients $R_{kk}(\lambda)$.

Inverse problem: Given the spectral data, the reflection coefficients $R_{kk}(\lambda)$ on each edge N_k, construct the potentials $p_k(x_k)$ and $q_k(x_k)$ where $k = 1,2,3$.

Theorem 5 *In each fixed edge* $k = 1,2,3$, $n \in N$,

$$\lim_{\lambda\to n/2}(n-2\lambda)R_{kk}(\lambda) = V_{nn}^{(-k)},$$

$$\lim_{\lambda\to -\frac{n}{2}}(n+2\lambda)\frac{1}{R_{kk}(\lambda)} = V_{nn}^{(k)},$$

are satisfied.

Proof It is well known that the functions $f_k^{+}(x_k,\lambda), f_k^{-}(x_k,\lambda)$ are linearly independent and their Wronskian is equal to $2i\lambda$.

Then from (14) it follows that the Wronskian of the functions $f_{nk}^{\pm}(x_k), f_k^{\mp}(x_k,\mp n/2)$ is equal to zero, and therefore they are linearly dependent. Thus

$$f_{nk}^{\pm}(x_k) = S_{nk}^{\pm} f_k^{\mp}(x_k,\mp n/2).$$

Comparing the formulas for these functions we see that $S_{nk}^{\pm} = V_{nn}^{(\pm k)}$.

Therefore

$$f_{nk}^{\pm}(x_k) = V_{nn}^{(\pm k)} f_k^{\mp}(x_k,\mp n/2). \tag{26}$$

Taking into account (26) it is easy to verify that

$$\lim_{\lambda\to\pm\frac{n}{2}}(n\mp 2\lambda)\frac{f_k^{\mp}(0,\lambda)}{f_k^{\pm}(0,\lambda)}=\frac{V_{nn}^{(\mp k)}f_k^{\pm}(0,\pm n/2)}{f_k^{\pm}(0,\pm n/2)}=V_{nn}^{(\mp k)}$$

and

$$\lim_{\lambda\to\pm\frac{n}{2}}(n\mp 2\lambda)\frac{2i\lambda}{f_1^{\pm}(0,\lambda)f_1^{\pm}(0,\lambda)[\frac{f_1^{\pm'}(0,\lambda)}{f_1^{\pm}(0,\lambda)}+\frac{f_2^{+'}(0,\lambda)}{f_2^{+}(0,\lambda)}+\frac{f_3^{+'}(0,\lambda)}{f_3^{+}(0,\lambda)}]}=0.$$

Consequently we can find all numbers $V_{nn}^{(\pm k)}$ from the relations

$$\begin{aligned}\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)R_{kk}(\lambda)&=\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)\frac{f_k^{-}(0,\lambda)}{f_k^{+}(0,\lambda)}\\&\quad+\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)\frac{2i\lambda}{f_1^{+}(0,\lambda)f_1^{+}(0,\lambda)[\frac{f_1^{+'}(0,\lambda)}{f_1^{+}(0,\lambda)}+\frac{f_2^{+'}(0,\lambda)}{f_2^{+}(0,\lambda)}+\frac{f_3^{+'}(0,\lambda)}{f_3^{+}(0,\lambda)}]}=V_{nn}^{(-k)}\end{aligned}$$

and

$$\begin{aligned}\lim_{\lambda\to-\frac{n}{2}}(n+2\lambda)\frac{1}{R_{kk}(\lambda)}&=\lim_{\lambda\to-\frac{n}{2}}(n+2\lambda)\frac{[f_1^{+}(0,\lambda)f_2^{+}(0,\lambda)f_3^{+}(0,\lambda)]'}{[f_1^{-}(0,\lambda)f_2^{+}(0,\lambda)f_3^{+}(0,\lambda)]'}\\&=\lim_{\lambda\to-\frac{n}{2}}(n+2\lambda)\frac{f_k^{+}(0,\lambda)}{f_k^{-}(0,\lambda)}\\&\quad+\lim_{\lambda\to-\frac{n}{2}}(n+2\lambda)\frac{2i\lambda}{f_1^{-}(0,\lambda)f_1^{-}(0,\lambda)[\frac{f_1^{-'}(0,\lambda)}{f_1^{-}(0,\lambda)}+\frac{f_2^{+'}(0,\lambda)}{f_2^{+}(0,\lambda)}+\frac{f_3^{+'}(0,\lambda)}{f_3^{+}(0,\lambda)}]}\\&=V_{nn}^{(k)}.\end{aligned}$$

The theorem is proved. □

Now to reconstruct of the potentials $p_k(x_k)$ and $q_k(x_k)$ for given $R_{kk}(\lambda)$, we first attempt to find explicit connections between the sequences $V_{n,n}^{(\pm k)}$, $V_{n,\alpha}^{(\pm k)}$ and $V_{\alpha}^{(\mp k)}$.

Taking into account (14) we get

$$\sum_{\alpha=n}^{\infty}V_{n\alpha}^{(\pm k)}e^{i\alpha x_k}e^{-i\frac{n}{2}x_k}=V_{nn}^{(\pm k)}e^{i\frac{n}{2}x_k}\left(1+\sum_{n=1}^{\infty}V_n^{(\mp k)}e^{inx_k}+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}^{(\mp k)}}{n+n}\right)$$

or

$$V_{m\alpha+m}^{(\pm k)}=V_{mm}^{(\pm k)}\left(V_{\alpha}^{(\mp k)}+\sum_{n=1}^{\alpha}\frac{V_{n\alpha}^{(\mp k)}}{n+m}\right),\quad m,\alpha=1,2,3,\ldots.\tag{27}$$

These relations are the fundamental equations for the reconstruction of $p_{nk}(x_k)$ and $q_{nk}(x_k)$ from the known $V_{n\alpha}^{\pm k}$, $V_n^{\pm k}$.

We propose to make the dependence of $V_{n\alpha}^{\pm k}$, on $V_{\alpha}^{\pm k}$ explicit.

The method applied in this paper is the synthesis of methods presented by Pastur and Tkachenko [10] and Jaulent [26], and for the benefit of the reader we reintroduce it here.

Let $\tilde{V}^{\pm k}_{m\alpha+m}$, $m, \alpha = 1, 2, 3, \ldots$, be a solution of equation (27) corresponding to $V^{\pm k}_{\alpha} = 1$ and $\hat{V}^{\pm k}_{m\alpha+m}$ corresponding to $V^{\pm k}_{\alpha} = \pm i$,

$$\begin{aligned} \tilde{V}^{\pm k}_{m\alpha+m} &= V^{\pm k}_{mm}\left(1 + \sum_{n=1}^{\alpha} \frac{\tilde{V}^{\mp k}_{n\alpha}}{n+m}\right), \\ \hat{V}^{\pm k}_{m\alpha+m} &= V^{\pm k}_{mm}\left(\pm i + \sum_{n=1}^{\alpha} \frac{\hat{V}^{\mp k}_{n\alpha}}{n+m}\right). \end{aligned} \tag{28}$$

Let $\gamma^{\pm k}_{m\alpha}$ and $\beta^{\mp k}_{m\alpha}$ be functions defined as

$$\begin{aligned} \gamma^{\pm k}_{m\alpha} &= \frac{1}{2}\left[\tilde{V}^{\mp k}_{m\alpha+m} \mp i\hat{V}^{\mp k}_{m\alpha+m}\right], \\ \beta^{\mp k}_{m\alpha} &= \frac{1}{2}\left[\tilde{V}^{\mp k}_{m\alpha+m} \pm i\hat{V}^{\mp k}_{m\alpha+m}\right]. \end{aligned}$$

Note that the quantities $\gamma^{\pm k}_{m\alpha}$ and $\beta^{\mp k}_{m\alpha}$ are uniquely determined by the recurrent equation (28) from the known $V^{\pm k}_{nn}$.

Then we easily obtain the following:

$$V^{\pm k}_{m\alpha+m} = V^{\mp k}_{\alpha} \cdot \gamma^{\pm k}_{m\alpha} + V^{\pm k}_{\alpha} \cdot \beta^{\mp k}_{m\alpha}. \tag{29}$$

Equation (29) shows that, if we can define the sequences $V^{(\pm k)}_{n,\alpha}$ and $V^{(\mp k)}_{\alpha}$ from the known $V^{\pm k}_{nn}$ then the potentials $p_k(x_k)$ and $q_k(x_k)$ may be reconstructed uniquely and effectively from (11)-(13).

Theorem 6 *All numbers $V^{\pm k}_{n\alpha}, n > \alpha$ and $V^{(\mp k)}_{\alpha}$ may be uniquely determined through the known numbers $V^{\pm k}_{nn}$.*

Proof In fact, if the given $V^{(\pm k)}_{n,n}$ uniquely determine all numbers $V^{(\mp k)}_{\alpha}$ then the numbers $V^{(\mp k)}_{n,\alpha}$ will be determined by (28).

Let us denote

$$x_k = it_k, \qquad \lambda = -i\mu, \qquad y_k(x) = Y_k(t), \tag{30}$$

then from (7) we obtain the equation

$$-Y_k{}''(t) + 2\mu\bar{p}_k(it_k)Y(t_k) + \bar{q}_k(it_k)Y_k(t_k) = \mu^2 Y_k(t), \tag{31}$$

in which

$$\overline{p_k}(t_k) = ip_k(it_k) = i\sum_{n=1}^{\infty} p_{nk}e^{-nt_k}, \qquad \bar{q}_k(t_k) = -q_k(it_k) = -\sum_{n=1}^{\infty} q_n k e^{-nt_k}. \tag{32}$$

As a result we obtain equation (31), whose potentials exponentially decrease as $t_k \to \infty$, $k = 1, 2, 3$.

The procedure of analytic continuation allows one to get corresponding results for equation (7) from the result of equation (31).

Equation (31) with potentials (32) has the solution

$$f_{\pm k}(t_k,\mu) = e^{\pm i\mu t_k}\left(1 + \sum_{n=1}^{\infty} V_n^{\pm k} e^{-nt_k} + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{\pm k}}{in \pm 2\mu} e^{-\alpha t_k}\right) \tag{33}$$

and the numbers $V_n^{\pm k}$, $V_{n\alpha}^{\pm k}$ are defined by equations (11)-(13).

Then with the help of the (33) we obtain

$$f_{\pm k}(t_k,\mu) = \Omega_k^{\pm}(t) e^{\pm i\mu t_k} + \int_{t_k}^{\infty} K_k^{\pm}(t_k,u_k) e^{\pm i\mu u_k}\, du_k, \tag{34}$$

where $K_k^{\pm}(t_k,u_k)$, $\Omega_k^{\pm}(t_k)$ have the form

$$K_k^{\pm}(t_k,u_k) = \frac{1}{2i}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} V_{n\alpha}^{\pm k} e^{-\alpha t_k}\cdot e^{-(u_k-t_k)n/2}, \qquad \Omega_k^{\pm}(t_k) = 1 + \sum_{n=1}^{\infty} V_n^{\pm k} e^{-nt_k}. \tag{35}$$

Rewriting the equality (27) in the form

$$\sum_{\alpha=n}^{\infty} V_{n\alpha}^{\pm k} e^{-\alpha t_k}\cdot e^{nt_k/2} = V_n k^{\pm} e^{-nt_k/2}\left(1 + \sum_{m=1}^{\infty} V_m^{\mp k} e^{-mt_k} + \sum_{m=1}^{\infty}\sum_{\alpha=m}^{\infty} \frac{V_{m\alpha}^{\mp k}}{i(m+n)} e^{-\alpha t_k}\right) \tag{36}$$

and denoting

$$z_k^{\pm}(t_k + s_k) = \sum_{m=1}^{\infty} V_m^{\pm k} e^{-(t_k+s_k)m/2}, \tag{37}$$

we obtain the Marchenko type equation

$$K_k^{\pm}(t_k,s_k) = \Omega_k^{\pm}(t_k) z_k^{\pm}(t_k+s_k) + \int_{t_k}^{\infty} K_k^{\mp}(t_k,u_k) z_k^{\pm}(u_k+s_k)\, du_k. \tag{38}$$

From the general theory of differential equations it is known that

$$\Omega_k^{\pm}(t_k) = e^{\mp i\int_{x_k}^{\infty} p_k(t_k)\, dt_k}.$$

By using it we get

$$\Omega_k^{+}(t_k)\cdot\Omega_k^{-}(t_k) = 1. \tag{39}$$

On the other hand, we easily derive the relation from (34),

$$\Omega_k^{+}(t_k) - \Omega_k^{-}(t_k) = \int_{t_k}^{\infty} \left[K^{-}(t_k,u) - K^{+}(t_k,u)\right] du. \tag{40}$$

The last relations (39)-(40) give us the following system of equations for finding the dependence of $V_{n,\alpha}^{(\pm k)}$ and $V_{\alpha}^{(\mp k)}$. We have

$$\begin{aligned} &V_{\alpha}^{(k)} + V_{\alpha}^{(-k)} + \sum_{s=1}^{\alpha-1} V_s^{(k)} V_{\alpha-s}^{(-k)} = 0, \\ &V_{\alpha}^{(k)} - V_{\alpha}^{(-k)} + \sum_{n=1}^{\alpha} \frac{V_{n\alpha}^{(k)} - V_{n\alpha}^{(-k)}}{n} = 0. \end{aligned} \tag{41}$$

Then by using (28) we get

$$\begin{aligned} \sum_{n=1}^{\alpha} \frac{V_{n\alpha}^{(k)} - V_{n\alpha}^{(-k)}}{n} &= \sum_{n=1}^{\alpha} \frac{V_{\alpha}^{(-k)}\gamma_{n\alpha-n}^{+} + V_{\alpha}^{(k)}\beta_{n\alpha-n}^{-} - V_{\alpha}^{(k)}\gamma_{n\alpha-n}^{-} - V_{\alpha}^{(-k)}\beta_{n\alpha-n}^{+}}{n} \\ &= V_{\alpha}^{(-k)} \sum_{n=1}^{\alpha} \frac{\gamma_{n\alpha-n}^{+} - \beta_{n\alpha-n}^{+}}{n} + V_{\alpha}^{(k)} \sum_{n=1}^{\alpha} \frac{\beta_{n\alpha-n}^{-} - \gamma_{n\alpha-n}^{-}}{n}. \end{aligned}$$

Finally from (41) we obtain

$$V_{\alpha}^{(k)} \left(1 - \sum_{n=1}^{\alpha} \frac{\beta_{n\alpha-n}^{-} - \gamma_{n\alpha-n}^{-}}{n}\right) - V_{\alpha}^{(-k)} \left(1 - \sum_{n=1}^{\alpha} \frac{\gamma_{n\alpha-n}^{+} - \beta_{n\alpha-n}^{+}}{n}\right) = 0. \tag{42}$$

Let

$$\Lambda_{\alpha} = \frac{1 - \sum_{n=1}^{\alpha} \frac{\beta_{n\alpha-n}^{-} - \gamma_{n\alpha-n}^{-}}{n}}{1 - \sum_{n=1}^{\alpha} \frac{\gamma_{n\alpha-n}^{+} - \beta_{n\alpha-n}^{+}}{n}};$$

then from (42) we obtain

$$V_{\alpha}^{(k)} = V_{\alpha}^{(-k)} \Lambda_{\alpha} \tag{43}$$

and

$$V_{\alpha}^{(-k)}(1 + \Lambda_{\alpha}) + \sum_{s=1}^{\alpha-1} V_s^{(-k)} V_{\alpha-s}^{(-k)} \Lambda_s = 0. \tag{44}$$

Equations (43) and (44) uniquely determined all numbers $V_{\alpha}^{(\pm k)}$. Then from (28) all numbers $V_{n\alpha}^{(\pm k)}$ are defined.

The theorem is proved. □

Theorem 7 *The specification of the spectral data uniquely determines potentials $p_k(x_k)$, $q_k(x_k)$ on each edge N_k, $k = 1, 2, 3$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors typed read and approved the final manuscript; also they contributed to each part of this work equally.

Author details
[1]Department of Mathematics Teaching, Qafqaz University, Khirdalan city, H. Aliyev str. 120, Baku, Absheron AZ0101, Azerbaijan. [2]Department of Applied Mathematics, Baku State University, Z. Khalilov, 23, Baku, AZ1148, Azerbaijan. [3]Faculty of Education, Alexandria University, Alexandria, Egypt.

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