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Dirichlet type problems for Dunkl-Poisson equations

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Abstract

In this paper, using the intertwine relations of differential operators, we study one representation of real analytic functions by Dunkl-harmonic functions, which is a generalization of the well-known Almansi formula. As an application of the representation, we construct a solution of the Dunkl-Poisson equations in Clifford analysis. Then we investigate solutions of homogeneous and inhomogeneous Dirichlet type problems for Dunkl-Poisson's equation, and inhomogeneous Dirichlet problems for Dunkl-Laplace's equation.

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Keywords: Dirichlet problem; Dunkl-Poisson equation; Dunkl-Laplace equation; Clifford analysis

1 Introduction

In mathematics, a Dirichlet problem for Laplace's equation can be stated as follows: Given a function f that has values everywhere on the boundary of a region in $\mathbf{R}^m$, there is a unique function u twice continuously differentiable in the interior and continuous to the boundary, such that u is harmonic in the interior and $u = f$ on the boundary. The Dirichlet problem [1] can be investigated for many PDEs, although originally it was posed for Laplace's equation. In this paper, we consider Dirichlet type problems for Dunkl-Poisson equations.

Dunkl operators T_j ($j = 1, \ldots, m$) introduced by Dunkl in [2, 3] are combinations of differential and difference operators, associated to a finite reflection group. These operators have the property of being invariant under reflections and, additionally, they are pairwise commuting. Also, they are very important in pure mathematics and physics. They provide a useful tool in the study of special functions with root systems and they are closely related to certain representations of degenerate affine Hecke algebras (see [4, 5]). Moreover, the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models (see [6–8]). One of the most important properties of Dunkl operators is that they are mutually commute. This allowed Cerejeiras *et al.* in [9] to introduce a Dirac operator, called the Dunkl-Dirac operator, based on differential-difference operators which are invariant under reflection groups and also construct the Dunkl-Laplacian which is a combination of the classical Laplacian in $\mathbf{R}^m$ with some difference terms. The theory of Dunkl-Clifford analysis is further developed in [10–14].

Using Almansi representations, Karachik constructed solutions of initial and boundary value problems for partial differential equations in real analysis, such as Dirichlet problems, Neumann problems, and Riqurie problems, etc. (see [15–18]). However, the study of boundary value problems for partial differential equations in Clifford analysis is a very difficult task. Clifford analysis is the study of functions defined in Euclidean space $\mathbf{R}^m$ and taking values in a Clifford algebra. Functions in Clifford analysis are not mutually commuting (see [19]). Using the intertwining relations of differential operators (*i.e.*, differential operators satisfy the defining relations of the Lie algebra (see [20])), we overcome the noncommutative properties between functions. In this paper, we investigate solutions of the homogeneous and inhomogeneous Dirichlet problem for Dunkl-Poisson's equation and the inhomogeneous Dirichlet problem for Dunkl-Laplace's equation in Clifford analysis.

The paper is organized as follows. In Section 2, we introduce the definition of Dunkl operators and review some results on the theory of Dunkl-Clifford analysis. In Section 3, applying the intertwine relations of differential operators, we study one representation of real analytic functions by Dunkl-harmonic functions. Using the representation, we construct solutions for Dunkl-Poisson's equation. In Section 4, we first consider solutions of the homogeneous Dirichlet problem for Dunkl-Poisson's equation. Then we investigate solutions of the inhomogeneous Dirichlet problem for Dunkl-Laplace's equation and the inhomogeneous Dirichlet problem for Dunkl-Poisson's equation.

2 Preliminaries

2.1 Dunkl operators

Let $\mathbf{R}^m$ be the Euclidean space equipped with a scalar product $\langle,\rangle$ and let $\|x\| = \sqrt{\langle x,x\rangle}$. For $\alpha \in \mathbf{R}^m\setminus\{0\}$, let σ_α be the reflection in the hyperplane orthogonal to α *i.e.* for $x \in \mathbf{R}^m$,

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha.$$

A finite set $\mathsf{R} \subset \mathbf{R}^m \setminus \{0\}$ is called a root system if $\alpha R \cap \mathsf{R} = \{\alpha, -\alpha\}$ and $\sigma_\alpha \mathsf{R} = \mathsf{R}$ for all $\alpha \in \mathsf{R}$. Each root system can be written as a disjoint union $\mathsf{R} = \mathsf{R}_+ \cup (-\mathsf{R}_+)$, where R_+ and $-\mathsf{R}_+$ are separated by a hyperplane through the origin. The subgroup $G \subset O(m)$ generated by the reflections $\{\sigma_\alpha \mid \alpha \in \mathsf{R}\}$ is called the finite reflection group associated with R.

A multiplicity function κ on the root system R is a G-invariant function $\kappa : \mathsf{R} \to \mathbf{C}$ *i.e.* $\kappa(\alpha) = \kappa(g\alpha)$ for all $g \in G$. We will denote $\kappa(\alpha)$ by κ_α. For abbreviation, we introduce the index

$$\gamma = \gamma_\kappa = \sum_{\alpha\in\mathsf{R}_+} \kappa_\alpha$$

and the weight function

$$h_\kappa(x) = \prod_{\alpha\in\mathsf{R}_+} \bigl|\langle \alpha, x\rangle\bigr|^{\kappa_\alpha}.$$

Throughout this paper, we will assume that $\kappa_\alpha \geq 0$ for all $\alpha \in \mathsf{R}$ and $\gamma_\kappa > 0$.

For each subsystem R_+ and multiplicity function κ_α we have the Dunkl operators

$$T_i f(x) = \frac{\partial f(x)}{\partial x_i} + \sum_{\alpha\in\mathsf{R}_+} \kappa_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle x, \alpha\rangle}\alpha_i, \quad i = 1, \ldots, m,$$

for $f \in C^1(\mathbf{R}^m)$. An important consequence is that the operators T_i are mutually commuting, that is, $T_iT_j = T_jT_i$ (see [2]).

2.2 Dunkl-Clifford analysis

Let $\{e_1, e_2, \dots, e_m\}$ be an orthogonal basis of the Euclidean space $\mathbf{R}^m$. We consider a function $f : \mathbf{R}^m \to \mathbf{R}_{0,m}$. Hereby $\mathbf{R}_{0,m}$ denotes the 2^m-dimensional real Clifford algebra over $\mathbf{R}^m$ with basis given by $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \dots, h_k\} \subset \{1, \dots, n\}$ for $1 \le h_1 \le \cdots, \le h_k \le m$. The function f can be written as $f = \sum_A e_A f_A(x)$, where $f_A(x)$ is a real-valued function. An element $x = (x_1, \dots, x_m)$ of $\mathbf{R}^m$ can be identified with $x = \sum_{i=1}^m x_i e_i$. By direct calculation, we have $x^2 = -|x|^2$.

A Dunkl-Dirac operator in $\mathbf{R}^m$ for the corresponding reflection group G is defined as $D_h = \sum_{i=1}^m e_i T_i$, where T_i are Dunkl operators. Functions belonging to the kernel of the Dunkl-Dirac operator D_h are called Dunkl-monogenic functions.

If we let D_h act on x, we see that

$$\mu := \frac{1}{2} D_h x = \frac{m}{2} + \gamma,$$

where μ is a complex number in contrast to the non-Dunkl case of the dimension m. In this paper, we assume that $\mu \ge 0$.

The Dunkl Laplacian is defined as

$$\Delta_h f(x) = -D_h^2 f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathrm{R}_+} \kappa_\alpha \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} |x|^2 \right),$$

where Δ and ∇ are the usual Laplacian and gradient operator. When $\kappa_\alpha = 0$, the Dunkl Laplacian Δ_h is just the ordinary Laplacian. Functions belonging to the kernel of the Dunkl Laplacian Δ_h are called Dunkl-harmonic functions.

3 Dunkl-Poisson equation in Clifford analysis

3.1 One representation of real analytic functions by Dunkl-harmonic functions

Definition 3.1 ([21]) An open connected set $\Omega \subset \mathbf{R}^m$ is a star domain with center 0 if any $\underline{x} \in \Omega$ and $0 \le t \le 1$ imply that $t\underline{x} \in \Omega$. The set is denoted by $\Omega^\star$.

Definition 3.2 Let $\Omega^\star$ be a star domain in $\mathbf{R}^m$ with center 0. Then the generalized Euler operator on domain $\Omega^\star$ is defined by

$$\mathbf{E}_t = t\mathbf{I} + \mathbf{E} = t\mathbf{I} + \sum_{i=1}^m x_i \partial_{x_i},$$

where t is a real number, $\mathbf{I}$ is the identity operator, and $\mathbf{E}$ is the Euler operator.

Now we can see the most important intertwining relations concerning the operators x^2, Δ_h, E_μ.

Lemma 3.3 ([20]) *The operators*

$$E := \frac{x^2}{2}, \qquad F := \frac{-\Delta_h}{2}, \qquad H := \mathbf{E}_\mu$$

generate the lie algebra

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H,$$

where $\mu = \frac{m}{2} + \gamma$ *and the Lie bracket* $[x,y]$ *is the commutator* $[x,y] = xy - yx$.

Lemma 3.4 *Let* $\Omega^\star$ *be a star domain in* $\mathbf{R}^m$ *with center* 0. *If* $f(x) \in C^2(\Omega^\star) \otimes R_{0,m}$ *and* $\mu \geq 0$, *then*

$$\Delta_h\big(x^{2s}f(x)\big) = x^{2s}\Delta_h f(x) + 4sx^{2s-2}\mathbf{E}_{\mu+s-1}f(x). \tag{1}$$

Proof By Lemma 3.3 and the definition of $\mathbf{E}_t$, we have

$$\begin{aligned}
\Delta_h\big[x^{2s}f(x)\big] &= \Delta_h x^2\big[x^{2s-2}f(x)\big] \\
&= \big(x^2\Delta_h + 4\mathbf{E}_\mu\big)\big[x^{2s-2}f(x)\big] \\
&= x^2\Delta_h x^{2s-2}f(x) + 4\mathbf{E}_\mu x^{2s-2}f(x) \\
&= x^2\big(x^2\Delta_h + 4\mathbf{E}_\mu\big)x^{2s-4}f(x) + 4\big(x^2\mathbf{E}_\mu + 2x^2\big)x^{2s-4}f(x) = \cdots \\
&= x^{2s}\Delta_h f(x) + 4sx^{2s-2}\mathbf{E}_{\mu+s-1}f(x).
\end{aligned}$$

Thus, we finish the proof. □

Lemma 3.5 *Let* $g(x) \in C^1(\Omega^\star) \otimes R_{0,m}$. *Then*

$$(\mathbf{E} + l + 1)\int_0^1 \alpha^l g(\alpha x)\,d\alpha = g(x) \tag{2}$$

and

$$(\mathbf{E} + l + 1)\int_0^1 \frac{(1-\alpha)^q}{q!}\alpha^l g(\alpha x)\,d\alpha = \int_0^1 \frac{(1-\alpha)^{q-1}}{(q-1)!}\alpha^{l+1}g(\alpha x)\,d\alpha \tag{3}$$

for $q \in \mathbf{N}$ *and* $l \geq 0$.

Proof The proof can be referred to in the literature [22]. □

In this paper, we assume the following infinite series converges absolutely and uniformly in Ω^*.

Theorem 3.6 *Let* $G(x) \in C^\infty(\Omega^*) \otimes R_{0,m}$. *Then*

$$G(x) = f_0 + \sum_{s=1}^{\infty}\frac{x^{2s}}{4^s s!(s-1)!}\int_0^1 (1-\alpha)^{s-1}\alpha^{\mu-1}f_s(\alpha x)\,d\alpha, \tag{4}$$

where $\Delta_h f_s(x) = 0$ *and*

$$f_s(x) = \Delta_h^s G(x) + \sum_{l=1}^{\infty}\frac{(-1)^l x^{2l}}{4^l l!}\int_0^1 \frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha. \tag{5}$$

Proof First we prove that the functions $f_s(x)$ satisfy (5). Substituting $f_s(x)$ into the right-hand side of the identity (5), we have

$$
\begin{aligned}
& f_0+\sum_{s=1}^{\infty}\frac{x^{2s}}{4^s s!(s-1)!}\int_0^1(1-\alpha)^{s-1}\alpha^{\mu-1}f_s(\alpha x)\,d\alpha \\
&\quad = G(x)+\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s}}{4^s s!}\int_0^1\frac{(1-\alpha)^{s-1}\alpha^{s-1}}{(s-1)!}\alpha^{\mu-1}\Delta_h^s G(\alpha x)\,d\alpha \\
&\qquad +\sum_{s=1}^{\infty}\frac{x^{2s}}{4^s s!(s-1)!}\int_0^1(1-\alpha)^{s-1}\alpha^{\mu-1}\Delta_h^s G(\alpha x)\,d\alpha \\
&\qquad +\sum_{s=1}^{\infty}\frac{x^{2s}}{4^s s!(s-1)!}\int_0^1(1-\alpha)^{s-1}\alpha^{\mu-1} \\
&\qquad \times\sum_{l=1}^{\infty}\frac{(-1)^l(\alpha x)^{2l}}{4^l l!}\int_0^1\frac{(1-\beta)^{l-1}\beta^{l-1}}{(l-1)!}\beta^{\frac{\mu}{2}-1}\Delta_h^{l+s}G(\alpha\beta x)\,d\beta\,d\alpha.
\end{aligned}
\tag{6}
$$

Denote by $A_1(x)$ the fourth term on the right side of equation (6). Then

$$
\begin{aligned}
A_1(x) &= \sum_{s=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^l x^{2s+2l}}{4^{l+s}s!l!}\int_0^1\frac{(1-\alpha)^{s-1}\alpha^{2l+\mu-1}}{(s-1)!}\int_0^1\frac{(1-\beta)^{l-1}\beta^{l+\mu-2}}{(l-1)!}\Delta_h^{l+s}G(\alpha\beta x)\,d\beta\,d\alpha \\
&= \sum_{s=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^l x^{2s+2l}}{4^{l+s}s!l!}\int_0^1\frac{\alpha^2(1-\alpha)^{s-1}}{(s-1)!}\int_0^1\frac{(\alpha-\alpha\beta)^{l-1}(\alpha\beta)^{l+\mu-2}}{(l-1)!}\Delta_h^{l+s}G(\alpha\beta x)\,d\beta\,d\alpha.
\end{aligned}
$$

Denote by $A_2(x)$ the integral on the above expression. Let $t=\alpha\beta$. Then $dt=\alpha\,d\beta$. We calculate

$$
\begin{aligned}
A_2(x) &= \int_0^1\frac{\alpha(1-\alpha)^{s-1}}{(s-1)!}\int_0^{\alpha}\frac{(\alpha-t)^{l-1}t^{l+\mu-2}}{(l-1)!}\Delta_h^{l+s}G(tx)\,dt\,d\alpha \\
&= \int_0^1\int_0^{\alpha}\frac{\alpha(1-\alpha)^{s-1}(\alpha-t)^{l-1}}{(s-1)!(l-1)!}t^{l+\mu-2}\Delta_h^{l+s}G(tx)\,dt\,d\alpha \\
&= \int_0^1\int_t^1\frac{\alpha(1-\alpha)^{s-1}(\alpha-t)^{l-1}}{(s-1)!(l-1)!}t^{l+\mu-2}\Delta_h^{l+s}G(tx)\,d\alpha\,dt \\
&= \int_0^1\frac{t^{l+\mu-2}}{(s-1)!(l-1)!}\Delta_h^{l+s}G(tx)\,dt\int_t^1\alpha(1-\alpha)^{s-1}(\alpha-t)^{l-1}\,d\alpha.
\end{aligned}
$$

Let $\alpha=\beta+t$. Then we have

$$
A_3(t)=\int_t^1\alpha(1-\alpha)^{s-1}(\alpha-t)^{l-1}\,d\alpha=\int_0^{1-t}(\beta+t)(1-\beta-t)^{s-1}\beta^{l-1}\,d\beta.
$$

Let $\beta=\alpha(1-t)$. It follows that

$$
\begin{aligned}
A_3(t) &= \int_0^1(\alpha-\alpha t+t)(1-\alpha)^{s-1}(1-t)^{s-1}\alpha^{l-1}(1-t)^l\,d\alpha \\
&= (1-t)^{s+l-1}\int_0^1(\alpha-\alpha t+t)(1-\alpha)^{s-1}\alpha^{l-1}\,d\alpha.
\end{aligned}
$$

We calculate

$$A_3(t) = (1-t)^{l+s}\int_0^1 \alpha^l(1-\alpha)^{s-1}\,d\alpha + t(1-t)^{l+s-1}\int_0^1 \alpha^{l-1}(1-\alpha)^{s-1}\,d\alpha$$
$$= (1-t)^{l+s}B(l+1,s) + t(1-t)^{s+l-1}B(l,s),$$

where the beta functions

$$B(l,s) = \int_0^1 \alpha^{l-1}(1-\alpha)^{s-1}\,d\alpha. \tag{7}$$

Using the properties of beta functions and gamma functions:

$$B(l,s) = \frac{\Gamma(l)\Gamma(s)}{\Gamma(s+l)} \tag{8}$$

and

$$\Gamma(s) = (s-1)!, \tag{9}$$

we have

$$A_3(t) = \frac{\Gamma(l+1)\Gamma(s)}{\Gamma(l+s+1)}(1-t)^{s+l} + \frac{\Gamma(l)\Gamma(s)}{\Gamma(l+s)}t(1-t)^{l+s-1}$$
$$= \frac{l!(s-1)!}{(l+s)!}(1-t)^{s+l} + \frac{(l-1)!(s-1)!}{(l+s-1)!}t(1-t)^{l+s-1}.$$

By substituting $A_3(t)$ into $A_1(x)$, we have

$$\sum_{s=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^l x^{2s+2l}}{4^{l+s}}\int_0^1\left[\frac{(1-t)^{s+l}}{s!(l-1)!(l+s)!} + \frac{t(1-t)^{l+s-1}}{s!l!(l+s-1)!}\right]t^{l+\mu-2}\Delta_h^{l+s}G(tx)\,dt$$
$$= \sum_{s=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^l x^{2s+2l}}{4^{l+s}}\int_0^1\left[\frac{t^{l-1}(1-t)^{s+l}}{s!(l-1)!(l+s)!} + \frac{t^l(1-t)^{l+s-1}}{s!l!(l+s-1)!}\right]t^{\mu-1}\Delta_h^{l+s}G(tx)\,dt$$
$$= \sum_{i=2}^{\infty}\sum_{l=1}^{i-1}\frac{(-1)^l x^{2i}}{4^i}\int_0^1\left[\frac{t^{i-l-1}(1-t)^i}{l!(i-l-1)!i!} + \frac{t^{i-l}(1-t)^{i-1}}{l!(i-l)!(i-1)!}\right]t^{\mu-1}\Delta_h^i G(tx)\,dt$$
$$= \sum_{i=2}^{\infty}\frac{x^{2i}}{4^i}\int_0^1\left[\frac{(1-t)^i}{i!}\sum_{l=1}^{i-1}\frac{(-1)^l t^{i-l-1}}{l!(i-l-1)!} + \frac{(1-t)^{i-1}}{(i-1)!}\sum_{l=1}^{i-1}\frac{(-1)^l t^{i-l}}{l!(i-l)!}\right]t^{\mu-1}\Delta_h^i G(tx)\,dt.$$

We calculate

$$\sum_{l=1}^{i-1}\frac{(-1)^l t^{i-l-1}}{l!(i-l-1)!} = \sum_{s=0}^{i-1}\frac{(-1)^l t^{i-l-1}}{l!(i-l-1)!} - \frac{t^{i-1}}{(i-1)!}$$
$$= \frac{(t-1)^{i-1}}{(i-1)!} - \frac{t^{i-1}}{(i-1)!}$$

and

$$\sum_{l=1}^{i-1}\frac{(-1)^l t^{i-l}}{l!(i-l)!} = \sum_{s=0}^{i-1}\frac{(-1)^l t^{i-l}}{l!(i-l)!} - \frac{t^i}{i!} - \frac{(-1)^i}{i!} = \frac{(t-1)^i}{i!} - \frac{t^i}{i!} - \frac{(-1)^i}{i!}.$$

Thus, we have

$$\begin{aligned} A_1(x) &= \sum_{i=2}^{\infty}\frac{x^{2i}}{4^i}\int_0^1\left[-\frac{(1-t)^{i-1}t^{i-1}}{(i-1)!i!} - \frac{(-1)^i(1-t)^{i-1}}{(i-1)!i!}\right]t^{\mu-1}\Delta_h^i G(tx)\,dt \\ &= \sum_{i=1}^{\infty}\frac{x^{2i}}{4^i}\int_0^1\left[-\frac{(1-t)^{i-1}t^{i-1}}{(i-1)!i!} - \frac{(-1)^i(1-t)^{i-1}}{(i-1)!i!}\right]t^{\mu-1}\Delta_h^i G(tx)\,dt. \end{aligned}$$

By substituting $A_1(x)$ into (6), we have (5).

Next, we prove that $\Delta_h f_s(x) = 0$. By Lemma 3.4, we have

$$\begin{aligned} &\Delta_h f_s(x) \\ &\quad= \Delta_h^{s+1}G(x) + \sum_{l=1}^{\infty}\frac{(-1)^l}{4^l l!}\Delta_h\left(x^{2l}\int_0^1\frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha\right) \\ &\quad= \Delta_h^{s+1}G(x) + \sum_{l=1}^{\infty}\frac{x^{2l}}{4^l l!}\int_0^1\frac{(1-\alpha)^{l-1}\alpha^{l+1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l+1}G(\alpha x)\,d\alpha \\ &\qquad - \sum_{l=1}^{\infty}\frac{x^{2(l-1)}}{4^{l-1}(l-1)!}(\mathbf{E}+\mu+l-1)\int_0^1\frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha. \end{aligned}$$

Denote by $B_1(x)$ the third term of the above equality. From Lemma 3.5, we have

$$\begin{aligned} B_1(x) &= \sum_{l=1}^{\infty}\frac{(-1)^l x^{2(l-1)}}{4^{l-1}(l-1)!}(\mathbf{E}+\mu+l-1)\int_0^1\frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha \\ &= -(\mathbf{E}+\mu)\int_0^1\alpha^{\mu-1}\Delta_h^{s+1}G(\alpha x)\,d\alpha \\ &\quad - \sum_{l=2}^{\infty}\frac{x^{2(l-1)}}{4^{l-1}(l-1)!}(\mathbf{E}+\mu+l-1)\int_0^1\frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha \\ &= -\Delta_h^{s+1}G(x) - \sum_{l=2}^{\infty}\frac{(-1)^l x^{2(l-1)}}{4^{l-1}(l-1)!}\int_0^1\frac{(1-\alpha)^{l-2}\alpha^{l}}{(l-2)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha \\ &= -\Delta_h^{s+1}G(x) - \sum_{l=2}^{\infty}\frac{x^{2(l-1)}}{4^{l-1}(l-1)!}\int_0^1\frac{(1-\alpha)^{l-2}\alpha^{l}}{(l-2)!}\alpha^{\mu-1}\Delta_h^{s+l}G(\alpha x)\,d\alpha \\ &= -\Delta_h^{s+1}G(x) - \sum_{l=1}^{\infty}\frac{x^{2l}}{4^l l!}\int_0^1\frac{(1-\alpha)^{l-1}\alpha^{l+1}}{(l-1)!}\alpha^{\mu-1}\Delta_h^{s+l+1}G(\alpha x)\,d\alpha, \end{aligned}$$

which implies that $\Delta_h f_s(x) = 0$. Thus, we finish the proof. □

Corollary 3.7 *Let $P_l(x)$ be a homogeneous polynomial of degree l. Then*

$$P_l(x) = R_l(x) + x^2 R_{l-2}(x) + \cdots + x^{2k} R_{l-2k}(x), \tag{10}$$

where $R_{l-2k}(x)$ are homogeneous Dunkl-harmonic polynomials and

$$R_{l-2k}(x) = \frac{1}{4^k k!(l-k+\mu)\cdots(l-2k+\mu)} \sum_{s=0}^{[l-k]} \frac{(-1)^s x^{2s} \Delta_h^{s+k} P_l(x)}{4^s s!(l-2k-s+\mu-1)_s}. \tag{11}$$

Proof Let $P_l(x)$ be a homogeneous polynomial of degree l. By Theorem 3.6, we have

$$P_l(x) = R_l(x) + x^2 R_{l-2}(x) + \cdots + x^{2k} R_{l-2k}(x),$$

where

$$R_{l-2k}(x) = \frac{f_k(x)}{4^k k!(k-1)!} \int_0^1 (1-\alpha)^{k-1} \alpha^{l-2k+\mu-1}\, d\alpha = \frac{f_k(x) B(k, l-2k+\mu)}{4^k k!(k-1)!}. \tag{12}$$

Using equation (5), we have

$$\begin{aligned} f_0(x) &= P_l(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s}}{4^s s!} \int_0^1 \frac{(1-\alpha)^{s-1}\alpha^{s-1}}{(s-1)!} \alpha^{\mu-1} \Delta_h^s P_l(\alpha x)\, d\alpha \\ &= P_l(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^s P_l(x)}{4^s s!(s-1)!} \int_0^1 (1-\alpha)^{s-1} \alpha^{l-s+\mu-2}\, d\alpha \end{aligned}$$

for the case $G(x) = P_l(x)$. Using equations (7) and (8), we have

$$\int_0^1 (1-\alpha)^{s-1} \alpha^{l-s+\mu-2}\, d\alpha = B(s, l-s+\mu-1) = \frac{(s-1)!}{(l-s+\mu-1)_s},$$

where $(m)_s = m(m+1)\cdots(m+s-1)$ is the Pochhammer symbol. Therefore,

$$f_0(x) = P_l(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^s P_l(x)}{4^s s!(l-s+\mu-1)_s} = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^s P_l(x)}{4^s s!(l-s+\mu-1)_s}.$$

By equation (5), we have

$$f_k(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^{s+k} P_l(x)}{4^s s!(l-2k-s+\mu-1)_s}.$$

Thus, it follows from (12) that

$$\begin{aligned} R_{l-2k}(x) &= \frac{f_k(x) B(k, l-2k+\mu)}{4^k k!(k-1)!} \\ &= \frac{1}{4^k k!(l-k+\mu)\cdots(l-2k+\mu)} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^{s+k} P_l(x)}{4^s s!(l-2k-s+\mu-1)_s}, \end{aligned}$$

which completes the proof. □

3.2 Solutions of the Dunkl-Poisson equation in Clifford analysis

In this section, we study the Dunkl-Poisson equation in Clifford analysis,

$$\Delta_h G(x) = f(x), \tag{13}$$

where $f(x) \in C^\infty(\Omega) \otimes R_{0,m}$ is a real analytic function.

Theorem 3.8 *Let $f(x) \in C^\infty(\Omega^*) \otimes R_{0,m}$. A real analytic solution of equation* (13) *can be found in the form*

$$G(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \Delta_h^s f(\alpha x)\, d\alpha. \tag{14}$$

Proof Let $f(x) \in C^\infty(\Omega^*) \otimes R_{0,m}$. Then it follows by Theorem 3.6 that

$$G(x) = f_0 + \sum_{s=1}^{\infty} \frac{x^{2s}}{4^s s!(s-1)!} \int_0^1 (1-\alpha)^{s-1} \alpha^{\mu-1} f_s(\alpha x)\, d\alpha, \tag{15}$$

where $f_s(x)$ are Dunkl-harmonic in Ω^* given by the relation

$$f_s(x) = \Delta_h^s G(x) + \sum_{l=1}^{\infty} \frac{(-1)^l x^{2l}}{4^l l!} \int_0^1 \frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!} \alpha^{\mu-1} \Delta_h^{s+l} G(\alpha x)\, d\alpha. \tag{16}$$

Note that $\Delta_h^{s+1} G(x) = \Delta_h^s f$. Thus, we have

$$\begin{aligned} G(x) - f_0(x) &= -\sum_{l=1}^{\infty} \frac{(-1)^l x^{2l}}{4^l l!} \int_0^1 \frac{(1-\alpha)^{l-1}\alpha^{l-1}}{(l-1)!} \alpha^{\mu-1} \Delta_h^l G(\alpha x)\, d\alpha \\ &= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \Delta_h^s f(\alpha x)\, d\alpha. \end{aligned} \tag{17}$$

Since $\Delta_h[G(x) - f_0(x)] = f(x)$, it implies that $[G(x) - f_0(x)]$ is a solution of the Poisson equation (13). Therefore, the right-hand of equation (17) is a solution of equation (13). □

Corollary 3.9 *The solution of the Poisson equation $\Delta_h G(x) = P_l(x)$ can be represented in the form*

$$G(x) = \sum_{s=0}^{[\frac{l}{2}]} \frac{(-1)^s x^{2(s+1)} \Delta_h^s P_l(x)}{4^{s+1}(s+1)!(l+\mu)\cdots(l-s+\mu)}, \tag{18}$$

where $[\frac{l}{2}]$ is the integer part of $\frac{l}{2}$ and $(a,b)_k = a(a+b)\cdots(a+kb-b)$ is the generalized Pochhammer symbol with the convention that $(a,b)_0 = 1$.

Proof Let $P_l(x)$ be a homogeneous polynomial of degree l. Then we have $\Delta_h^k P_l(\alpha x) = \alpha^{l-2k}\Delta_h^k P_l(x)$. Therefore, (14) can be transformed into

$$G(x) = \sum_{s=0}^{[\frac{l}{2}]} \frac{(-1)^s x^{2(s+1)} \Delta_h^s P_l(x)}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \alpha^{l-2s}\, d\alpha.$$

Furthermore, we can write

$$G(x) = \sum_{s=0}^{[\frac{l}{2}]} \frac{(-1)^s x^{2(s+1)} \Delta_h^s P_l(x)}{4^{s+1}(s+1)!s!} B(s+1, \mu + l - s),$$

where $B(s,k) = \int_0^1 \alpha^{s-1}(1-\alpha)^{k-1}\,d\alpha$. Then by the relation

$$B(s,k) = \frac{\Gamma(s)\Gamma(k)}{\Gamma(s+k)},$$

we have

$$u(x) = \sum_{s=0}^{[\frac{l}{2}]} \frac{(-1)^s \Gamma(l-s+\mu) x^{2(s+1)} \Delta_h^s P_l(x)}{4^{s+1}(s+1)!\Gamma(l+\mu+1)}.$$

Using the property $\Gamma(s+1) = s\Gamma(s)$ of the gamma function, we find that

$$\Gamma(l+\mu+1) = (l+\mu)(l+\mu-1)\cdots(l+\mu-s)\Gamma(l+\mu-s).$$

It follows that

$$G(x) = \sum_{s=0}^{[\frac{l}{2}]} \frac{(-1)^s x^{2(s+1)} \Delta_h^s P_l(x)}{4^{s+1}(s+1)!(l+\mu)\cdots(l-s+\mu)},$$

which completes the proof. □

Corollary 3.10 *Let $P_l(x)$ be a homogeneous harmonic polynomial of degree l. The solution of the equation $\Delta_h G(x) = x^{2k}P_l(x)$ is given by*

$$G(x) = x^{2k+2}P_l(x) \sum_{s=0}^{k} \frac{(-1)^s (2k-2s+2,2)_s (2l+2\mu+2k-2s,2)_s}{4^{s+1}(s+1)!(\mu+2k+l)\cdots(\mu+2k+l-s)}.$$

Proof Let $f(x) = x^{2k}P_l(x)$. We calculate this solution using equation (14) to obtain

$$G(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \Delta_h^s \bigl[x^{2k}P_l(x)\bigr]\,d\alpha.$$

Let us derive an expression for $\Delta_h^s[x^{2k}P_l(x)]$. By Lemma 3.5, we have

$$\Delta_h\bigl[x^{2k}P_l(x)\bigr] = 4kx^{2k-2}(l+\mu+k-1)P_l(x).$$

Therefore, for $2s \leq 2k + l$, we have

$$\begin{aligned}
&\Delta_h^s\bigl[x^{2k}P_l(x)\bigr] \\
&\quad = 2k(2k-2)\cdots(2k-2s+2)(2l+2\mu+2k-2)\cdots(2l+2\mu+2k-2s)x^{2k-2s}P_l(x) \\
&\quad = (2k-2s+2,2)_s(2l+2\mu+2k-2s,2)_s x^{2k-2s}P_l(x).
\end{aligned}$$

Thus, we find

$$G(x) = \sum_{s=0}^{k} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \times \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} (2k-2s+2,2)_s (2l+2\mu+2k-2s,2)_s (\alpha x)^{2k-2s} P_l(\alpha x)\, d\alpha$$

and, since $P_l(\alpha x) = \alpha^l P_l(x)$,

$$G(x) = x^{2k+2} P_l(x) \sum_{s=0}^{k} \frac{(-1)^s (2k-2s+2,2)_s (2l+2\mu+2k-2s,2)_s}{4^{s+1}(s+1)!s!} \times \int_0^1 (1-\alpha)^s \alpha^{\mu+2k+l-s-1}\, d\alpha.$$

The integral in this expression is evaluated as

$$\begin{aligned}\int_0^1 (1-\alpha)^s \alpha^{\mu+2k+l-s-1}\, d\alpha &= B(s+1, \mu+2k+l-s) \\ &= \frac{s!}{(\mu+2k+l)\cdots(\mu+2k+l-s)},\end{aligned}$$

where $B(m,n)$ is the Euler beta function. Then $G(x)$ is transformed into

$$G(x) = x^{2k+2} P_l(x) \sum_{s=0}^{k} \frac{(-1)^s (2k-2s+2,2)_s (2l+2\mu+2k-2s,2)_s}{4^{s+1}(s+1)!(\mu+2k+l)\cdots(\mu+2k+l-s)}.$$

Thus, we complete the proof. □

4 Dirichlet type problems for Dunkl-Poisson's equation

In [23–25], a weak solution of the Dirichlet problem of the Poisson equation with homogeneous boundary data in variable exponent space was obtained. In [17], Karachik used the Almansi representation for Laplace operator to construct a polynomial solution of the inhomogeneous Dirichlet problem for Poisson's equation in harmonic analysis. Inspired by the above-mentioned results, we develop further these ideas for Dunkl-Poisson's equation in Clifford analysis.

4.1 Homogeneous Dirichlet problem for Dunkl-Poisson's equation

In this section, we consider the following boundary value problem for the Dunkl-Poisson equation in the unit ball $B = \{x \in R^m : |x| < 1\}$:

$$\begin{cases} \Delta_h G(x) = f(x), & x \in B, \\ G(x)|_{\partial B} = 0, \end{cases} \tag{19}$$

where $f(x)$ is a polynomial.

In order to obtain solutions of the homogeneous Dirichlet problem for Dunkl-Poisson's equation (19), we first consider the following boundary value problem for the Dunkl-Poisson equation in the unit ball $B = \{x \in R^m : |x| < 1\}$:

$$\begin{cases} \Delta_h u(x) = P_l(x), & x \in B, \\ u(x)|_{\partial B} = 0, & x \in \partial B = \{|x| = 1\}, \end{cases} \tag{20}$$

where $P_l(x)$ is a homogeneous polynomial of degree l.

Theorem 4.1 *Let $P_l(x)$ be a homogeneous polynomial of degree l. The solution of the Dirichlet problem* (20) *can be written as*

$$u(x) = \frac{x^2+1}{2} \sum_{k=0}^{[\frac{l}{2}]} \frac{(-1)^k \Delta_h^k P_l(x)}{4^{k+1}(k+1)!k!} \int_0^1 \left(1+\alpha x^2\right)^k (1-\alpha)^k \alpha^{l-2k+\mu-1}\, d\alpha. \tag{21}$$

Proof By Corollary 3.7, it follows that the solution of the equation $\Delta_h G(x) = P_l(x)$ becomes

$$G(x) = \sum_{k=0}^{[\frac{l}{2}]} \frac{x^{2k+2} R_{l-2k}(x)}{4(k+1)(l-k+\mu)}. \tag{22}$$

Note that $R_{l-2k}(x)$ are Dunkl-harmonic polynomials. Then the polynomial

$$G_0(x) = \sum_{k=0}^{[\frac{l}{2}]} \frac{R_{l-2k}(x)}{4(k+1)(l-k+\mu)} \tag{23}$$

is Dunkl-harmonic.

Thus, we have $\Delta_h[G(x) + G_0(x)] = P_l(x)$ and the property $G(x) + G_0(x) = 0$ for $|x| = 1$. Therefore, the polynomial $G(x) + G_0(x)$ solves the Dirichlet problem (20).

Using equations (22) and (23), we find

$$\begin{aligned} &G(x) + G_0(x) \\ &\quad= \sum_{k=0}^{[\frac{l}{2}]} \frac{x^{2k+2} R_{l-2k}(x)}{4(k+1)(l-k+\mu)} + \sum_{k=0}^{[\frac{l}{2}]} \frac{R_{l-2k}(x)}{4(k+1)(l-k+\mu)} \\ &\quad= \sum_{k=0}^{[\frac{l}{2}]} \frac{(x^{2k+2}+1)}{4(k+1)(l-k+\mu)} R_{l-2k}(x) \\ &\quad= \sum_{k=0}^{[\frac{l}{2}]} \frac{(-1)^k \Delta^k P_l(x)}{4^{k+1}(k+1)!} \sum_{i=0}^{k+1} \frac{(l+\mu-2k+2i-1)x^{2i}}{i!(k-i+1)!(l+\mu-2k+i-1)_{k+2}} \\ &\quad= \sum_{k=0}^{[\frac{l}{2}]} \frac{(-1)^k \Delta^k P_l(x)}{4^{k+1}(k+1)!} \sum_{i=0}^{k+1} \frac{(l+\mu-2k+i-1)\Gamma(l+\mu-2k+i-1)x^{2i}}{i!(k-i+1)!\Gamma(l+\mu-k+i+1)}. \end{aligned}$$

Using equations (7), (8), and (9), we transform the inner sum in this expression as

$$\begin{aligned}\frac{\Gamma(l+\mu-2k+i-1)}{\Gamma(l+\mu-k+i+1)} &= \frac{B(i+2,l+\mu-2k+i-1)}{\Gamma(i+2)}\\ &= \frac{1}{(i+1)!}\int_0^1 (1-\alpha)^{i+1}\alpha^{l+\mu-2k+i-2}\,d\alpha.\end{aligned}$$

Using $i\alpha^i = \alpha(\alpha^i)'$ and the binomial theorem, we have

$$\begin{aligned}&(l+\mu-2k-1)\int_0^1 (1-\alpha)^{k+1}\alpha^{l+\mu-2k-2}\sum_{i=0}^{k+1}\frac{x^{2i}\alpha^i}{i!(k-i+1)!}\,d\alpha\\ &\quad+2\int_0^1 (1-\alpha)^{i+1}\alpha^{l+\mu-2k-2}\sum_{i=0}^{k+1}\frac{x^{2i}\alpha^i}{i!(k-i+1)!}\,d\alpha\\ &=\int_0^1 (1-\alpha)^{i+1}(l+\mu-2k-1)\alpha^{l+\mu-2k-2}\bigl(1+\alpha x^2\bigr)^{i+1}\,d\alpha\\ &\quad+2\int_0^1 (1-\alpha)^{i+1}\alpha^{l+\mu-2k-2}\bigl[\bigl(1+\alpha x^2\bigr)^{i+1}\bigr]'\,d\alpha\\ &=\int_0^1 (1-\alpha)^{i+1}d\bigl(\alpha^{l+\mu-2k-1}\bigl(1+\alpha x^2\bigr)^{i+1}\bigr)\\ &\quad+(i+1)x^2\int_0^1 (1-\alpha)^{i+1}\alpha^{l+\mu-2k-1}\bigl(1+\alpha x^2\bigr)^{i}\,d\alpha.\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}&\int_0^1 (1-\alpha)^{i+1}d\bigl(\alpha^{l+\mu-2k-1}\bigl(1+\alpha x^2\bigr)^{i+1}\bigr)\\ &=(i+1)\int_0^1 (1-\alpha)^{i}\alpha^{l+\mu-2k-1}\bigl(1+\alpha x^2\bigr)^{i}\bigl[\bigl(1+\alpha x^2\bigr)+x^2(1-\alpha)\bigr]\,d\alpha\\ &=(i+1)\bigl(1+x^2\bigr)\int_0^1 (1-\alpha)^{i}\alpha^{l+\mu-2k-1}\bigl(1+\alpha x^2\bigr)^{i}\,d\alpha.\end{aligned}$$

Therefore, the polynomial $G(x)+G_0(x)$ can be rewritten as

$$G(x)+G_0(x)=\frac{x^2+1}{2}\sum_{k=0}^{[\frac{l}{2}]}\frac{(-1)^k\Delta_h^k P_l(x)}{4^{k+1}(k+1)!k!}\int_0^1\bigl(1+\alpha x^2\bigr)^k(1-\alpha)^k\alpha^{l-2k+\mu-1}\,d\alpha. \qquad\square$$

Theorem 4.2 *Let $f(x)$ be an arbitrary polynomial. Then the solution of the Dirichlet problem* (19) *can be written as*

$$u(x)=\frac{x^2+1}{2}\int_0^1\sum_{k=0}^{\infty}\frac{(1+\alpha x^2)^k(1-\alpha)^k}{4^{k+1}(k+1)!k!}\Delta_h^k f(\alpha x)\alpha^{\mu-1}\,d\alpha. \tag{24}$$

Proof Let $f(x)$ be an arbitrary polynomial. Then $f(x)=\sum_l P_l(x)$, where $P_l(x)$ is a homogeneous polynomial of degree l. Using (21), we see that the solution of the Dirichlet problem

(19) is

$$
\begin{aligned}
u(x) &= \sum_l u_l(x) \\
&= \sum_l \frac{x^2+1}{2} \sum_{k=0}^{\infty} \int_0^1 \frac{(1+\alpha x^2)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^k P_l(\alpha x) \alpha^{\mu-1}\, d\alpha \\
&= \frac{x^2+1}{2} \int_0^1 \sum_{k=0}^{\infty} \frac{(1+\alpha x^2)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^k f(\alpha x) \alpha^{\mu-1}\, d\alpha.
\end{aligned}
$$

□

4.2 Inhomogeneous Dirichlet problem for Dunkl-Laplace's equation

Now we consider the following Dirichlet problem for Dunkl-Laplace's equation in the unit ball B:

$$
\begin{cases} \Delta_h v(x) = 0, & x \in B, \\ v(x)|_{\partial B} = P(x)|_{\partial B}, \end{cases} \tag{25}
$$

with a polynomial boundary value $P(x)$.

Theorem 4.3 *Let $P(x)$ be a polynomial. Then the solution of problem* (25) *can be written as*

$$
v(x) = P(x) - \frac{x^2+1}{2} \int_0^1 \sum_{k=0}^{\infty} \frac{(1+\alpha x^2)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^{k+1} P(\alpha x) \alpha^{\mu-1}\, d\alpha. \tag{26}
$$

Proof Using equation (24), we find the solution of the Dirichlet problem

$$
\begin{cases} \Delta_h u(x) = \Delta_h P(x), & x \in B, \\ u(x)|_{\partial B} = 0, \end{cases} \tag{27}
$$

as follows:

$$
u(x) = \frac{x^2+1}{2} \int_0^1 \sum_{k=0}^{\infty} \frac{(1+\alpha x^2)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^{k+1} P(\alpha x) \alpha^{\mu-1}\, d\alpha.
$$

Then the function

$$
v(x) = P(x) - u(x) = P(x) - \frac{x^2+1}{2} \int_0^1 \sum_{k=0}^{\infty} \frac{(1+\alpha x^2)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^{k+1} P(\alpha x) \alpha^{\mu-1}\, d\alpha
$$

is Dunkl-harmonic, because $\Delta_h v(x) = \Delta_h P(x) - \Delta_h u(x) = 0$. And the function $\upsilon(x)$ satisfies the boundary condition $v(x)|_{\partial B} = P(x)|_{\partial B}$. Therefore, the function $\upsilon(x)$ is a solution of problem (25).

Combining Theorems 4.2 and 4.3 yields the following result. □

4.3 Inhomogeneous Dirichlet problem for Dunkl-Poisson's equation

Now we consider the following Dirichlet problem for Dunkl-Laplace's equation in the unit ball B:

$$\begin{cases} \Delta_h u(x) = f(x), & x \in B, \\ u(x)|_{\partial B} = P(x)|_{\partial B}, \end{cases} \tag{28}$$

with a polynomial boundary value $P(x)$.

Theorem 4.4 *Let $f(x)$ and $P(x)$ be polynomials. Then the solution of the Dirichlet problem* (28) *can be written as*

$$u(x) = P(x) - \frac{x^2+1}{2} \int_0^1 \sum_{k=0}^{\infty} \frac{(1-\alpha x^2)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^k (f - \Delta_h P)(\alpha x)\alpha^{\mu-1}\, d\alpha. \tag{29}$$

Proof The solution of problem (28) can be decomposed into the sum of solutions of two problems: (19) and (25). It follows by solutions (24) and (26) that the solution of the problem (28) is the function (29). □

Competing interests
The author declares to have no competing interests.

Author's contributions
The author read and approved the final manuscript.

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