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The stability of evolutionary $p(x)$-Laplacian equation

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Abstract

The paper studies the equation

$$u_t = \mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u),$$

with the boundary degeneracy coming from $a(x)|_{x\in\partial\Omega} = 0$. The paper introduces a new kind of weak solutions of the equation. One can study the stability of the new kind of weak solutions without any boundary value condition.

MSC: 35K55; 35K92; 35K85; 35R35

Keywords: electrorheological fluid equation; boundary degeneracy; well-posedness

1 Introduction

Consider the evolutionary $p(x)$-Laplacian equation

$$u_t = \mathrm{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

which comes from a new interesting kind of fluids: the so-called electrorheological fluids (see [1, 2]). Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x)$ is a measurable function, we assume that $a(x) > 0, x \in \Omega$, $a(x) = 0, x \in \partial\Omega$. If $a(x) = 1$, the initial boundary value problem of equation (1.1) has been widely studied [3–5]. If $a(x)|_{x\in\partial\Omega} = 0$, the situation may completely different from that of $a(x) \equiv 1$. To see that, let us suppose that u and v would be two classical solutions of equation (1.1) with the initial values $u(x,0)$ and $v(x,0)$, respectively. It is easy to show that

$$\int_\Omega \big|u(x,t) - v(x,t)\big|^2\,dx \le \int_\Omega \big|u_0(x) - v_0(x)\big|^2\,dx. \tag{1.2}$$

It implies that the classical solutions (if there are any) of equation (1.1) are controlled by the initial value completely. Certainly, since equation (1.1) is degenerate on the boundary and may be degenerate or singular at points where $|\nabla u| = 0$, it only has a weak solution generally, so whether the conclusion (1.1) is true or not remains to be verified. This is the main aim of the paper.

If $a(x) = d^\alpha(x)$, $d = \mathrm{dist}(x, \partial\Omega)$ is the distance from the boundary, the well-posedness of the solutions of the equation

$$u_t = \mathrm{div}\big(d^\alpha(x)|\nabla u|^{p-2}\nabla u\big), \quad (x,t) \in Q_T, \tag{1.3}$$

was first studied by Yin and Wang [6], and later by Yin and Wang [7], Zhan and Xie [8], *etc.* While the corresponding equation related to the $p(x)$-Laplacian

$$u_t = \mathrm{div}\big(d^\alpha(x)|\nabla u|^{p(x)-2}\nabla u\big), \quad (x,t) \in Q_T, \tag{1.4}$$

was studied by Zhan and Wen [9], and Zhan [10].

In this short paper, we will study the well-posedness of the solutions of equation (1.1) with the initial value

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.5}$$

but without any boundary value condition.

2 Basic functional space and a new kind of weak solution

Let us recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(a,\Omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(a,\Omega)$ according to [11]. Set

$$C_+(\overline{\Omega}) = \Big\{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\Big\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x), \qquad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, $L^{p(x)}(a,\Omega)$ consists of all measurable real-valued functions u such that

$$\int_\Omega a(x)\big|u(x)\big|^{p(x)}\,dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(a,\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

$W^{1,p(x)}(a,\Omega)$ is defined by

$$W^{1,p(x)}(a,\Omega) = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(a,\Omega)\big\},$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(a,\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(a,\Omega)}.$$

It is easy to see that the norm

$$\big|\|u\|\big| = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{u(x)}{\mu}\right|^{p(x)} + a(x)\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} dx\right) \le 1\right\}$$

is equivalent to $\|u\|_{W^{1,p(x)}(a,\Omega)}$.

Lemma 2.1 *Denote*

$$\rho(u) = \int_\Omega a(x)|u|^{p(x)}\,dx \quad \textit{for all } u \in L^{p(x)}(a,\Omega).$$

Then

(1) $\rho(u) > 1$ $(= 1; < 1)$ *if and only if* $\|u\|_{L^{p(x)}(a,\Omega)} > 1$ $(= 1; < 1)$, *respectively*;
(2) *if* $\|u\|_{L^{p(x)}(a,\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(a,\Omega)} \le \rho(u) \le \|u\|^{p^+}_{L^{p(x)}(a,\Omega)}$;
(3) *if* $\|u\|_{L^{p(x)}(a,\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(a,\Omega)} \le \rho(u) \le \|u\|^{p^-}_{L^{p(x)}(a,\Omega)}$

Remark 2.2 If we set

$$I(u) = \int_\Omega \big(|u|^{p(x)} + a(x)|u|^{p(x)}\big)\,dx,$$

then following the same argument we have

$$\min\big\{\big|\|u\|\big|^{p^-}, \big|\|u\|\big|^{p^+}\big\} \le I(u) \le \max\big\{\big|\|u\|\big|^{p^-}, \big|\|u\|\big|^{p^+}\big\}.$$

Let a be a measurable positive and a.e. finite function in $\mathbb{R}^N$ satisfying
(w1) $a \in L^1_{\mathrm{loc}}(\Omega)$ and $a^{-\frac{1}{p(x)-1}} \in L^1_{\mathrm{loc}}(\Omega)$;
(w2) $a^{-s(x)} \in L^1(\Omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.

It is worth pointing out that the condition (w1) is essential. Without it the space $W^{1,p(x)}(a,\Omega)$ is not necessarily a Banach space even though $p(x)$ is a constant; see [11]. There are several kinds functions which satisfy (w1), (w2), an obvious example is $a(x) = d^\alpha(x)$, $\alpha < p^- - 1$, $p^- = \min_{x\in\overline{\Omega}} p(x)$.

Lemma 2.3 *Under the condition* $1 < p_0 \le p(x) \le p_1 < \infty$ *for the conjugate space* $[L^{p(x)}(\Omega, a)]'$ *we have*

$$\big[L^{p(x)}(a,\Omega)\big]^* = L^{p'(x)}\big(\big[a(x)\big]^{\frac{1}{1-p(x)}}, \Omega\big), \qquad \frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

$$\left|\int_\Omega u(x)v(x)\,dx\right| \le k\|u\|_{L^{p'(x)}([a(x)]^{\frac{1}{1-p(x)}},\Omega)} \|v\|_{L^{p(x)}(a,\Omega)}.$$

Lemma 2.4 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set,* $p \in C_+(\overline{\Omega})$, *and let* Ω_0 *be a compact subset of* Ω. *If* (w1) *holds, then*

$$L^{p(x)}(a,\Omega) \hookrightarrow L^1(\Omega_0).$$

Here $\hookrightarrow$ *stands for a continuous embedding.*

Lemma 2.5 *Let $p \in C_+(\overline{\Omega})$. Then we have*

(i) *if a is a positive measurable and finite function, then $L^{p(x)}(a,\Omega)$ is a reflexive Banach space*;

(ii) *moveover, if* (w1) *holds, then $W^{1,p(x)}(a,\Omega)$ is a reflexive Banach space.*

Lemma 2.6 *Let $p, s \in C_+(\overline{\Omega})$ and let* (w1) *and* (w2) *be satisfied. Then we have the following compact embedding*:

$$W^{1,p(x)}(a,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$$

provided that $r \in C_+(\overline{\Omega})$ and $1 \le r(x) < p_s^*(x)$ for all $x \in \Omega$. Here,

$$p_s(x) = \frac{p(x)s(x)}{1+s(x)},$$

and

$$p_s^*(x) = \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)}, & \text{if } p_s(x) < N, \\ +\infty, & \text{if } p_s(x) \ge N. \end{cases}$$

All this heavily relies on [11].

Definition 2.7 A function $u(x,t)$ is said to be a weak solution of equation (1.1) with the initial condition (1.5), if

$$u \in L^\infty(Q_T), \qquad \frac{\partial u}{\partial t} \in L^2(Q_T), \qquad u \in L^\infty\bigl(0,T; W^{1,p(x)}(a,\Omega)\bigr), \tag{2.1}$$

and, for any function $\varphi_1 \in L^\infty(0,T;W^{1,p(x)}(a,\Omega))$, $\varphi_2 \in C_0^1(Q_T)$, the following integral equivalence holds:

$$\iint_{Q_T} \left(\frac{\partial u}{\partial t}\varphi_1\varphi_2 + a(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\varphi_1\varphi_2) \right) dx\,dt = 0. \tag{2.2}$$

The initial condition (1.5) is satisfied in the sense of

$$\lim_{t\to 0} \int_\Omega \bigl|u(x,t) - u_0(x)\bigr|\,dx = 0. \tag{2.3}$$

In our paper, we first study the existence of the weak solution.

Theorem 2.8 *If $a(x)$ satisfies the conditions* (w1), (w2),

$$u_0 \in L^\infty(\Omega), \qquad u_0 \in W^{1,p(x)}(a,\Omega), \tag{2.4}$$

then there is a solution of equation (1.1) *with the initial value* (1.5).

Then we will study the stability of the weak solutions.

Theorem 2.9 *If a satisfies* (w1)-(w2), *and for large enough n*,

$$n^{1-\frac{1}{p^+}}\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}}|\nabla a|^{p(x)}\,dx\right)^{\frac{1}{p^+}}\leq c, \tag{2.5}$$

let u, v be two solutions of equation (1.1) *with the initial values u_0, v_0, respectively. If u, v satisfy* (2.1), *then*

$$\int_{\Omega}\left|u(x,t)-v(x,t)\right|dx\leq c\int_{\Omega}\left|u(x,0)-v(x,0)\right|dx. \tag{2.6}$$

If $a(x)=d^{\alpha}(x)$ as (1.4), the condition (2.5) is equivalent to $\alpha\geq p^+-1$, then Theorem 2.9 is the same as the main result in [9].

Theorem 2.10 *If a satisfies* (w1)-(w2), *and for large enough n, let u, v be two solutions of equation* (1.1) *with the initial values u_0, v_0, respectively. If u, v satisfy* (2.1), *and*

$$n\left(\int_{\Omega_{\frac{1}{n}}\setminus\Omega_{\frac{2}{n}}}a(x)|\nabla u|^{p(x)}\,dx\right)^{\frac{1}{q^+}}\leq c,\qquad n\left(\int_{\Omega_{\frac{1}{n}}\setminus\Omega_{\frac{2}{n}}}a(x)|\nabla u|^{p(x)}\,dx\right)^{\frac{1}{q^+}}\leq c, \tag{2.7}$$

then the stability (2.6) *is true*.

By the way, the phenomenon that the solution of a degenerate parabolic equation may be free from the boundary condition also had been studied by Zhan [12] and others.

3 Proofs of Theorems 2.8-2.10

Proof of Theorem 2.8 Let $a(x)$ satisfy (w1), (w2). Consider the regularized equation

$$u_t=\operatorname{div}\left(\left(a(x)+\varepsilon\right)|\nabla u|^{p(x)-2}\nabla u\right),\quad (x,t)\in Q_T, \tag{3.1}$$

with the initial boundary conditions

$$u(x,0)=u_0(x),\quad x\in\Omega, \tag{3.2}$$

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T). \tag{3.3}$$

Similar to [9], we can easily prove that the solution u_ε of the initial boundary value problem (3.1)-(3.3), there is a constant c only dependent on $\|u_0\|_{L^\infty(\Omega)}$ but independent on ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)}\leq c,\qquad \|u_{\varepsilon t}\|_{L^2(Q_T)}\leq c. \tag{3.4}$$

Multiplying (3.1) by u_ε and integrating it over Q_T, we have

$$\frac{1}{2}\int_{\Omega}u_\varepsilon^2\,dx+\iint_{Q_T}\left(a(x)+\varepsilon\right)|\nabla u|^{p(x)-2}|\nabla u_\varepsilon|^2\,dx\,dt=\frac{1}{2}\int_{\Omega}u_0^2\,dx, \tag{3.5}$$

then

$$\frac{1}{2}\int_{\Omega} u_{\varepsilon}^{2}\,dx + \iint_{Q_T} \big(a(x)+\varepsilon\big)|\nabla u|^{p(x)-2}|\nabla u_{\varepsilon}|^{2}\,dx\,dt \le c \tag{3.6}$$

and

$$\iint_{Q_T} a(x)|\nabla u_{\varepsilon}|^{p(x)}\,dx\,dt \le \iint_{Q_T} \big(a(x)+\varepsilon\big)|\nabla u_{\varepsilon}|^{p(x)}\,dx\,dt \le c. \tag{3.7}$$

Hence, by (3.4), (3.7), using Lemma 2.3 and Lemma 2.6, there exist a function u and a n-dimensional vector $\overrightarrow{\zeta} = (\zeta_1,\ldots,\zeta_n)$ satisfying $\overrightarrow{\zeta} = (\zeta_1,\ldots,\zeta_n)$

$$u \in L^{\infty}(Q_T), \qquad \frac{\partial u}{\partial t} \in L^{2}(Q_T), \qquad |\overrightarrow{\zeta}| \in L^{1}\big(0,T;L^{\frac{p(x)}{p(x)-1}}\big(a^{\frac{1}{1-p(x)}},\Omega\big)\big), \tag{3.8}$$

and $u_{\varepsilon} \to u$ a.e. $\in Q_T$,

$$\begin{aligned}
&u_{\varepsilon} \rightharpoonup u, \quad \text{weakly star in } L^{\infty}(Q_T),\\
&u_{\varepsilon} \to u, \quad \text{in } L^{2}\big(0,T;L^{r(x)}(a,\Omega)\big),\\
&\frac{\partial u_{\varepsilon}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^{2}(Q_T),\\
&a(x)|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon} \rightharpoonup \overrightarrow{\zeta} \quad \text{in } L^{1}\big(0,T;L^{\frac{p(x)}{p(x)-1}}\big(a^{\frac{1}{1-p(x)}},\Omega\big)\big).
\end{aligned}$$

In order to prove u is the solution of equation (1.4), we notice that, for any function $\varphi \in C_0^{\infty}(Q_T)$,

$$\iint_{Q_T} \big[u_{\varepsilon t}\varphi + \big(a(x)+\varepsilon\big)|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}\cdot\nabla\varphi\big]\,dx\,dt = 0.$$

Since as $\varepsilon \to 0$, by the fact that $a(x)$ is a $C^1(\overline{\Omega})$ function with $a(x)|_{\partial\Omega} = 0$, $a(x) > 0, x \in \Omega$, we have $c > \sup_{\operatorname{supp}\varphi} \frac{|\nabla\varphi|}{a(x)} > 0$ due to $\varphi \in C_0^{\infty}(Q_T)$, and we have

$$\begin{aligned}
&\varepsilon\left|\iint_{Q_T} |\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}\cdot\nabla\varphi\,dx\,dt\right|\\
&\quad\le \varepsilon \sup_{\operatorname{supp}\varphi}\frac{|\nabla\varphi|}{a(x)}\iint_{Q_T} a(x)\big(|\nabla u_{\varepsilon}|^{p(x)}+c\big)\,dx\,dt \to 0,\\
&\iint_{Q_T} \vec{\zeta}\cdot\nabla\varphi\,dx\,dt = \lim_{\varepsilon\to 0}\iint_{Q_T} a(x)|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}\cdot\nabla\varphi\,dx\,dt\\
&\quad= \lim_{\varepsilon\to 0}\iint_{Q_T}\big(a(x)+\varepsilon\big)|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}\cdot\nabla\varphi\,dx\,dt\\
&\qquad- \lim_{\varepsilon\to 0}\varepsilon\iint_{Q_T}|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}\cdot\nabla\varphi\,dx\,dt\\
&\quad= \lim_{\varepsilon\to 0}\iint_{Q_T}\big(a(x)+\varepsilon\big)|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}\cdot\nabla\varphi\,dx\,dt.
\end{aligned}$$

Now, similar to the general evolutionary p-Laplacian equation, we are able to prove that (the details are omitted here)

$$\iint_{Q_T} (u\varphi_t + \vec{\varsigma} \cdot \nabla\varphi)\,dx\,dt = 0 \tag{3.9}$$

and

$$\iint_{Q_T} a(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\,dx\,dt = \iint_{Q_T} \overrightarrow{\zeta} \cdot \nabla\varphi\,dx\,dt, \tag{3.10}$$

for any function $\varphi \in C_0^\infty(Q_T)$. By the process of taking the limit, for $\varphi = \varphi_1\varphi_2$, (3.10) is still true. Here, $\varphi_1 \in L^\infty(0,T;W^{1,p(x)}(a,\Omega))$, $\varphi_2 \in C_0^1(Q_T)$. Then u satisfies equation (1.1) in the sense of Definition 2.7. □

Proof of Theorem 2.9 Let u and v be two weak solutions of equation (1.1) with the initial values $u(x,0)$, $v(x,0)$, respectively.

From the definition of the weak solution, we have $u,v \in L^\infty(0,T;W^{1,p(x)}(a,\Omega))$. For any given positive integer n, let $g_n(s)$ be an odd function, and

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2s^2\mathrm{e}^{1-n^2s^2}, & 0 \le s \le \frac{1}{n}. \end{cases}$$

Clearly,

$$\lim_{n\to 0} g_n(s) = \operatorname{sgn}(s), \quad s \in (-\infty,+\infty).$$

Denoting $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$, let

$$\phi_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_{\frac{1}{n}}, \\ na(x), & x \in \Omega \setminus \Omega_{\frac{1}{n}}. \end{cases}$$

Since $\varphi_1 = g_n(u-v) \in L^\infty(0,T;W^{1,p(x)}(a,\Omega))$, by the process of taking the limit, we can choose $\phi_n g_n(u-v)$ as the test function; then

$$\begin{aligned} &\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx \\ &\quad+ \int_\Omega a(x)\bigl(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\bigr)\cdot\nabla(u-v)g_n'(u-v)\phi_n(x)\,dx \\ &\quad+ \int_\Omega a(x)\bigl(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\bigr)\cdot\nabla(u-v)g_n(u-v)\nabla\phi_n\,dx = 0. \end{aligned} \tag{3.11}$$

Thus

$$\lim_{n\to\infty}\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx = \frac{d}{dt}\|u-v\|_{L^1(\Omega)}, \tag{3.12}$$

$$\int_\Omega a(x)\bigl(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\bigr)\cdot\nabla(u-v){g'}_n(u-v)\phi_n(x)\,dx \ge 0. \tag{3.13}$$

Denoting $q(x) = \frac{p(x)}{p(x)-1}$, by $|\nabla\phi_n| = n\nabla a$ when $x \in \Omega \setminus \Omega_{\frac{1}{n}}$, and in the other places, it being identical to zero, we have

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right| \\
&\quad = \left|\int_{\Omega\setminus\Omega_{\frac{1}{n}}} a(x)\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right| \\
&\quad \le n\int_{\Omega\setminus\Omega_{\frac{1}{n}}} a(x)|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\big|\nabla a g_n(u-v)\big|\,dx \\
&\quad \le cn\big\|\big(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\big)\big\|_{L^{q(x)}([a(x)]^{\frac{1}{1-p(x)}},\Omega\setminus\Omega_{\frac{1}{n}})}\big\|\nabla a g_n(u-v)\big\|_{L^{p(x)}(a,\Omega\setminus\Omega_{\frac{1}{n}})}. \qquad (3.14)
\end{aligned}$$

By (2.5), we have the following fact:

$$\begin{aligned}
&n\big\|g_n(u-v)\nabla a\big\|_{L^{p(x)}(a,\Omega\setminus\Omega_{\frac{1}{n}})} \\
&\quad \le n\|\nabla a\|_{L^{p(x)}(a,\Omega\setminus\Omega_{\frac{1}{n}})} \\
&\quad \le n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} a(x)|\nabla a|^{p(x)}\,dx\right)^{\frac{1}{p^+}} \le cn^{1-\frac{1}{p^+}}\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} |\nabla a|^{p(x)}\,dx\right)^{\frac{1}{p^+}} \le c. \qquad (3.15)
\end{aligned}$$

Then by (3.14)-(3.15), we have

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right| \\
&\quad \le c\left[\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} a(x)|\nabla u|^{p(x)}\right)^{\frac{1}{q^+}} + \left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} a(x)|\nabla v|^{p(x)}\right)^{\frac{1}{q^+}}\right], \qquad (3.16)
\end{aligned}$$

which goes to 0 as $n \to 0$.

Now, let $n \to \infty$ in (3.11). Then

$$\frac{d}{dt}\|u-v\|_{L^1(\Omega)} \le 0.$$

It implies that

$$\int_\Omega \big|u(x,t) - v(x,t)\big|\,dx \le \int_\Omega |u_0 - v_0|\,dx, \quad \forall t \in [0,T). \qquad \square$$

Proof of Theorem 2.10 Just as the proof of Theorem 2.9, we have (3.11)-(3.14). By the assumption (2.7),

$$\begin{aligned}
&n\big\|\big(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\big)\big\|_{L^{q(x)}([a(x)]^{\frac{1}{1-p(x)}},\Omega\setminus\Omega_{\frac{1}{n}})} \\
&\quad \le n\left(\int_{\Omega_{\frac{1}{n}}\setminus\Omega_{\frac{2}{n}}} a(x)|\nabla u|^{p(x)}\,dx\right)^{\frac{1}{q^+}} + n\left(\int_{\Omega_{\frac{1}{n}}\setminus\Omega_{\frac{2}{n}}} a(x)|\nabla v|^{p(x)}\,dx\right)^{\frac{1}{q^+}} \le c, \qquad (3.17)
\end{aligned}$$

from (3.14), we have

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right|\\
&\quad\le cn\big\|\big(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1}\big)\big\|_{L^{q(x)}([a(x)]^{\frac{1}{1-p(x)}},\Omega\setminus\Omega_{\frac{1}{n}})}\big\|\nabla a g_n(u-v)\big\|_{L^{p(x)}(a,\Omega\setminus\Omega_{\frac{1}{n}})}\\
&\quad\le c\big\|\nabla a g_n(u-v)\big\|_{L^{p(x)}(a,\Omega\setminus\Omega_{\frac{1}{n}})},
\end{aligned}$$

which goes to 0 as $n\to 0$.

Now, let $n\to\infty$ in (3.11), we have the conclusion. □

4 Another kind of weak solutions

In general, one may conjecture the conditions (w1)-(w2) to be necessary. Though beyond one's imagination, we still can prove the stability of the weak solutions without (w1)-(w2). In the following, we only assume that $a(x)>0$, when $x\in\Omega$, $a(x)=0$, when $x\in\partial\Omega$.

We will give a new kind of weak solution and study its stability without any boundary value condition. We denote

$$W_a^{1,p(x)}=\left\{u\in W_{\mathrm{loc}}^{1,p(x)}(\Omega):\int_\Omega a(x)|\nabla u|^{p(x)}\,dx<\infty\right\}\subseteq W_{\mathrm{loc}}^{1,p(x)}(\Omega). \tag{4.1}$$

Clearly,

$$W^{1,p(x)}(a,\Omega)\subseteq W_a^{1,p(x)}\subseteq W_{\mathrm{loc}}^{1,p(x)}(\Omega).$$

Here $W^{1,p(x)}(\Omega)$ is the variable exponent Sobolev space, one can refer to [13–15] for the details, also, roughly speaking, one can choose the weighted function $a(x)=1$ in the space $W^{1,p(x)}(a,\Omega)$ defined above.

Lemma 4.1

(i) *The spaces* $(L^{p(x)}(\Omega),\|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega),\|\cdot\|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) $p(x)$*-Hölder's inequality. Let* $q_1(x)$ *and* $q_2(x)$ *be real functions with* $\frac{1}{q_1(x)}+\frac{1}{q_2(x)}=1$ *and* $q_1(x)>1$. *Then the conjugate space of* $L^{q_1(x)}(\Omega)$ *is* $L^{q_2(x)}(\Omega)$. *And for any* $u\in L^{q_1(x)}(\Omega)$ *and* $v\in L^{q_2(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\,dx\right|\le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}.$$

(iii)

$$\begin{aligned}
&\textit{if } \|u\|_{L^{p(x)}(\Omega)}=1,\quad \textit{then } \int_\Omega \|u\|^{p(x)}\,dx=1,\\
&\textit{if } \|u\|_{L^{p(x)}(\Omega)}>1,\quad \textit{then } \|u\|_{L^{p(x)}(\Omega)}^{p^-}\le\int_\Omega |u|^{p(x)}\,dx\le\|u\|_{L^{p(x)}(\Omega)}^{p^+},\\
&\textit{if } \|u\|_{L^{p(x)}(\Omega)}<1,\quad \textit{then } \|u\|_{L^{p(x)}(\Omega)}^{p^+}\le\int_\Omega |u|^{p(x)}\,dx\le\|u\|_{L^{p(x)}(\Omega)}^{p^-}.
\end{aligned}$$

Definition 4.2 A function $u(x,t)$ is said to be a solution of equation (1.1) with the initial condition (1.5), if

$$u \in L^\infty(Q_T), \qquad u_t \in L^2(Q_T), \qquad a(x)|\nabla u|^{p(x)} \in L^\infty\big(0,T;L^1(\Omega)\big), \tag{4.2}$$

for any given t, $\varphi_1(x,t) \in W_\alpha^{1,p(x)}$ and for any given x, $|\varphi_1(x,t)| \le c$, $\varphi_2 \in C_0^1(Q_T)$,

$$\iint_{Q_T} \big(u_t(\varphi_1\varphi_2) + a(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\varphi_1\varphi_2)\big)\,dx\,dt = 0. \tag{4.3}$$

The initial condition (1.5) is satisfied in the sense of (2.3).

The definition is a minor version of Definition 2.7.

Theorem 4.3 *If $a(x) > 0, x \in \Omega$, $a(x) = 0, x \in \partial\Omega$, and*

$$u_0 \in L^\infty(\Omega), \qquad u_0 \in W^{1,p(x)}(a,\Omega),$$

then there is a solution of equation (1.1) *with the initial value* (1.5) *in the sense of Definition* 4.2.

We can prove Theorem 4.3 in a similar way to that of Theorem 2.8, so we omit the details here. Moreover, similar to the proof of Theorem 2.9, we can prove the stability of the solutions when the diffusion coefficient $a(x)$ does not obey the conditions (w1)-(w2).

Theorem 4.4 *Let u,v be two solutions of* (1.1) *with the initial* (1.5) *in the sense of Definition* 4.2, *$u,v \in L^\infty(Q_T)$, $a(x)|\nabla u|^p \in L^1(Q_T)$, $a(x)|\nabla v|^p \in L^1(Q_T)$, and $a(x)$ satisfy the condition* (2.5). *If the assumption* (2.6) *or* (2.7) *is true, then*

$$\int_\Omega \big|u(x,t) - v(x,t)\big|\,dx \le \int_\Omega |u_0 - v_0|\,dx, \quad a.e.\ t \in (0,T). \tag{4.4}$$

Proof Let u and v be two weak solutions of equation (1.1) with the initial values $u(x,0)$, $v(x,0)$, respectively.

From the definition of the weak solution, we have $a(x)|\nabla u|^{p(x)}, a(x)|\nabla v|^{p(x)} \in L^\infty(0,T; L^1(\Omega))$. For any given positive integer n, let $g_n(s)$ and $\phi_n(x)$ be as in the proof of Theorem 2.9.

Since $\varphi_1 = g_n(u-v) \in W_\alpha^{1,p(x)}$, by the process of taking the limit, we can choose $\phi_n g_n(u-v)$ as the test function, then

$$\begin{aligned}
&\int_\Omega \phi_n(x) g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx \\
&\quad + \int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla(u-v)g_n'(u-v)\phi_n(x)\,dx \\
&\quad + \int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla(u-v)g_n(u-v)\nabla\phi_n\,dx.
\end{aligned} \tag{4.5}$$

Thus, by the condition (2.5),

$$\lim_{n\to\infty}\int_\Omega \phi_n(x) g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx = \frac{d}{dt}\|u-v\|_{L^1(\Omega)}, \tag{4.6}$$

$$\int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla(u-v)g'_n(u-v)\phi_n(x)\,dx\geq 0. \tag{4.7}$$

We have

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right|\\
&\quad=\left|\int_{\Omega\setminus\Omega_{\frac{1}{n}}} a(x)\big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right|\\
&\quad\leq\left\|a(x)^{\frac{p(x)-1}{p(x)}}\big(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1}\big)\right\|_{L^{q(x)}(\Omega\setminus\Omega_{\frac{1}{n}})}\left\|na(x)^{\frac{1}{p(x)}}\nabla a\right\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})}
\end{aligned}\tag{4.8}$$

which goes to 0 as $n\to 0$ provided that the assumption (2.7) is true.

By these facts, we can deduce the conclusion of Theorem 2.9. □

Competing interests
The author declares that they have no competing interests.

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