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Inverse inhomogeneous penetrable obstacle scattering problems in a stratified medium

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Abstract

In this paper we study the inverse inhomogeneous penetrable obstacle scattering problems in a stratified medium. On the basis of the uniqueness and existence of solutions for the direct scattering by an inhomogeneous penetrable obstacle in a stratified medium, we first establish an *a priori* estimate of the solution on some part of the penetrable interfaces S_i $(i = 1, 2)$, which plays an important role in the inverse scattering problems, and then we prove that both the penetrable interfaces S_i $(i = 1, 2)$ and the refractive index $n(x)$ of the inhomogeneous penetrable obstacle Ω_2 can be uniquely determined from knowledge of the far-field pattern $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S}, \mathbb{S}$ is the unit sphere of $\mathbb{R}^3)$ for an incident plane wave $u^i(x) = e^{ik_1 x\cdot d}$ $(x \in \mathbb{R}^3)$.

Keywords: inverse scattering problems; inhomogeneous penetrable obstacle; stratified medium; Helmholtz equation

1 Introduction

In this paper, we study the inverse inhomogeneous penetrable obstacle scattering problems in a stratified medium. In many branches of science and engineering such as radar and sonar, remote sensing, geophysics, geological exploration, nondestructive testing and medical imaging, the background medium may be described as a stratified medium rather than a homogeneous medium. Consequently, one possible model would be an inhomogeneous penetrable obstacle buried in a stratified medium. For simplicity and without loss of generality, in this paper we consider the case where the inhomogeneous penetrable obstacle is buried in a stratified medium with two layers. To be specific, let $\Omega_3 \subset \mathbb{R}^3$ be a bounded homogeneous medium with a closed C^2 boundary surface S_2 such that $\mathbb{R}^3\backslash\overline{\Omega_3}$ is divided into two connected domains Ω_2 and Ω_1 by a closed C^2 boundary surface S_1, where Ω_2 is an inhomogeneous penetrable obstacle and Ω_1 is an unbounded homogeneous medium.

The problems of scattering by an inhomogeneous penetrable obstacle in a stratified medium with two layers in $\mathbb{R}^3$ can be described as the following Helmholtz equations with transmission boundary conditions on their interfaces and Sommerfeld radiation condition, that is, the following boundary value problem:

$$\Delta u(x) + k_1^2 u(x) = 0, \quad \text{in } \Omega_1, \tag{1.1}$$

$$\Delta v(x) + k_2^2 n(x)v(x) = 0, \quad \text{in } \Omega_2, \tag{1.2}$$

$$\Delta w(x) + k_3^2 w(x) = 0, \quad \text{in } \Omega_3, \tag{1.3}$$

$$u(x) - v(x) = 0, \qquad \frac{\partial u(x)}{\partial \nu} - \lambda_1 \frac{\partial v(x)}{\partial \nu} = 0, \quad \text{on } S_1, \tag{1.4}$$

$$v(x) - w(x) = 0, \qquad \frac{\partial v(x)}{\partial \nu} - \lambda_2 \frac{\partial w(x)}{\partial \nu} = 0, \quad \text{on } S_2, \tag{1.5}$$

$$\lim_{r\to\infty} r\left[\frac{\partial u^s(x)}{\partial r} - ik_1 u^s(x)\right] = 0, \tag{1.6}$$

where $r = |x| = \sqrt{x_1^2 + x_2^2 + z^2}$, $x = (x_1, x_2, z) \in \mathbb{R}^3$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian, $u(x) = u(x_1, x_2, z)$, $v(x) = v(x_1, x_2, z)$ and $w(x) = w(x_1, x_2, z)$ are the time-harmonic velocity potentials, ν is the unit outward normal to the penetrable interface S_i $(i = 1, 2)$ and $n(x) = n(x_1, x_2, z) \in C^{0,\alpha}(\overline{\Omega_2}), 0 < \alpha < 1$, is the refractive index of an inhomogeneous penetrable obstacle with $\Re[n(x)] > 0, \Im[n(x)] \geq 0$. Here, the total field $u(x) = u^i(x) + u^s(x)$ is given as the sum of the unknown scattered wave $u^s(x) = u^s(x_1, x_2, z)$ which is required to satisfy the Sommerfeld radiation condition (1.6) and the incident wave $u^i(x) = u^i(x_1, x_2, z)$, $k_j > 0$ $(j = 1, 2, 3)$ is the wave number given by $k_j = \frac{\omega_j}{c_j}$ $(j = 1, 2, 3)$ in terms of the frequency ω_j $(j = 1, 2, 3)$ and the wave speed c_j $(j = 1, 2, 3)$ in the corresponding medium Ω_j $(j = 1, 2, 3)$. The distinct wave numbers k_j $(j = 1, 2, 3)$ correspond to the fact that the medium consists of several physically different materials. On the penetrable interfaces S_i $(i = 1, 2)$, the so-called transmission conditions (1.4)-(1.5) with two constants $\lambda_1 > 0$ and $\lambda_2 > 0$ are imposed, respectively, which represent the continuity of the medium and equilibrium of the forces acting on them.

For the uniqueness results for inverse scattering by an inhomogeneity with compact support in a homogeneous medium, see Hähner [1], Nachman [2], Novikov [3], and Ramm [4, 5], or see Colton and Kress [6], Isakov [7], and Kirsch [8] for a comprehensive discussion. In the case when the obstacle is impenetrable, Liu *et al.* have proved the unique determination of some inverse scattering problems, see [9, 10], or see Hähner [11], and when the obstacle is penetrable, some results concerned with the unique determination of the inverse scattering problems can be found in Athanasiadis and Stratis [12], Kirsch and Päivärinta [13], Liu and Zhang [14], Nachman, Päivärinta and Teirlilä [15], and Yan [16]. Moreover, Giorgi, Brignone, Aramini, and Piana [17] have presented a hybrid approach, which merges a qualitative and a quantitative method to optimize the way of exploiting the *a priori* information on the background within the inversion procedure, to numerically solving two-dimensional electromagnetic inverse scattering problems, whereby the unknown scatterer is hosted by a possibly inhomogeneous background.

The rest of the paper is organized as follows: in Section 2, we recall the uniqueness and existence of solutions for the direct scattering by an inhomogeneous obstacle in a stratified medium, which will be useful in the rest of the paper. In Section 3, we will establish *a priori* estimate of the solution on some part of the penetrable interfaces S_i $(i = 1, 2)$, which plays an important role in the inverse scattering problem. In Section 4, we will prove that both the penetrable interfaces S_i $(i = 1, 2)$ and the refractive index $n(x)$ of the inhomogeneous penetrable obstacle Ω_2 can be uniquely determined from knowledge of the far-field pattern $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S}, \mathbb{S}$ is the unit sphere of $\mathbb{R}^3)$ for an incident plane wave $u^i(x) = e^{ik_1 x\cdot d}$.

2 Preliminaries

In this section, we recall the uniqueness and existence of solutions for the direct scattering by an inhomogeneous obstacle in a stratified medium, which we have addressed in [18]. These results will be useful in the rest of the paper. From now on, we assume that $k_1, k_2, k_3, \lambda_1, \lambda_2$ are given positive numbers and that k_2^2 is not a Neumann eigenvalue of $\Delta v(x) + k_2^2 n(x)v(x) = 0$ in Ω_2.

The incident wave fields $u^i(x) = u^i(x_1, x_2, z)$ may be an incident plane wave $e^{ik_1 x\cdot d}$ or point source $\Phi_j(\cdot, z_j)$ $(j = 1, 3)$, which will be given below, where $d \in \mathbb{S}$ is the incident direction, $z_j \in \Omega_j$ $(j = 1, 3)$. Denote by $u^s(\cdot, d)$ the scattered field for an incident plane wave $u^i(\cdot, d)$ and by $u^\infty(\cdot, d)$ the corresponding far-field pattern, and denote by $u^s(\cdot, z_j)$ $(j = 1, 3)$ the scattered field for an incident point source $\Phi_j(\cdot, z_j)$ $(j = 1, 3)$ and by $\Phi^\infty(\cdot, z_j)$ $(j = 1, 3)$ the corresponding far-field pattern.

We will look for the solution $u(x) \in C^2(\Omega_1) \cap C^{1,\alpha}(\overline{\Omega_1})$, $v(x) \in C^2(\Omega_2) \cap C^{1,\alpha}(\overline{\Omega_2})$ and $w(x) \in C^2(\Omega_3) \cap C^{1,\alpha}(\overline{\Omega_3})$ satisfying the following Helmholtz equations with transmission boundary conditions on their interfaces and Sommerfeld radiation condition, that is, the following boundary value problem:

$$\Delta u(x) + k_1^2 u(x) = 0, \quad \text{in } \Omega_1, \tag{2.1}$$

$$\Delta v(x) + k_2^2 n(x) v(x) = 0, \quad \text{in } \Omega_2, \tag{2.2}$$

$$\Delta w(x) + k_3^2 w(x) = 0, \quad \text{in } \Omega_3, \tag{2.3}$$

$$u(x) - v(x) = f(x), \qquad \frac{\partial u(x)}{\partial \nu} - \lambda_1 \frac{\partial v(x)}{\partial \nu} = g(x), \quad \text{on } S_1, \tag{2.4}$$

$$v(x) - w(x) = p(x), \qquad \frac{\partial v(x)}{\partial \nu} - \lambda_2 \frac{\partial w(x)}{\partial \nu} = q(x), \quad \text{on } S_2, \tag{2.5}$$

$$\lim_{r\to\infty} r\left[\frac{\partial u^s(x)}{\partial r} - ik_1 u^s(x)\right] = 0, \tag{2.6}$$

where $f(x) \in C^{1,\alpha}(S_1)$, $g(x) \in C^{0,\alpha}(S_1)$, $p(x) \in C^{1,\alpha}(S_2)$ and $q(x) \in C^{0,\alpha}(S_2)$ are given functions from Hölder spaces with $0 < \alpha < 1$. For the scattering problem, if the incident field $u^i(x) = u^i(x_1, x_2, z)$ is the incident plane wave $e^{ik_1 x\cdot d}$ or the point source $\Phi_1(\cdot, z_1)$ with $z_1 \in \Omega_1$, then $f(x) = -u^i(x), g(x) = -\frac{\partial u^i(x)}{\partial \nu}, p = 0, q = 0$, and if the incident field $u^i(x) = u^i(x_1, x_2, z)$ is the point source $\Phi_3(\cdot, z_3)$ with $z_3 \in \Omega_3$, then $f(x) = u^i(x), g(x) = \lambda_1\lambda_2 \frac{\partial u^i(x)}{\partial \nu}, p(x) = -u^i(x), q(x) = -\lambda_2 \frac{\partial u^i(x)}{\partial \nu}$.

So we can recall the following three lemmas.

Lemma 1 *There exists at most one solution for the Helmholtz equations* (2.1)-(2.3) *with transmission boundary conditions* (2.4)-(2.5) *and Sommerfeld radiation condition* (2.6), *that is, the boundary value problem* (2.1)-(2.6).

Proof The proof is analogous to the proof of Theorem 1 in [18] and, hence, is omitted. □

Lemma 2 *There exists a unique solution for the Helmholtz equations* (2.1)-(2.3) *with transmission boundary conditions* (2.4)-(2.5) *and Sommerfeld radiation condition* (2.6), *that is,* *the boundary value problem* (2.1)-(2.6). *In particular, such a solution satisfies the estimate*

$$\begin{aligned} &\|u(x)\|_{1,\alpha,\overline{\Omega_1}} + \|v(x)\|_{1,\alpha,\overline{\Omega_2}} + \|w(x)\|_{1,\alpha,\overline{\Omega_3}} \\ &\quad \le C\big(\|f(x)\|_{1,\alpha,S_1} + \|g(x)\|_{0,\alpha,S_1} + \|p(x)\|_{1,\alpha,S_2} + \|q(x)\|_{0,\alpha,S_2}\big), \end{aligned} \tag{2.7}$$

for some positive constant $C = C(\alpha)$.

Proof The proof is analogous to the proof of Theorem 2 in [18] and, hence, is omitted. □

Lemma 3 *For the scattering of the incident plane wave* $u^i(\cdot, d)$ *with the incident direction* $d \in \mathbb{S}$ *and the incident point source* $\Phi(\cdot, z)$ *from the inhomogeneous penetrable obstacle* Ω_2, *we have*

$$\Phi^{\infty}(\widehat{x}, z) = \begin{cases} \frac{1}{4\pi} u^s(z, -\widehat{x}), & z \in \Omega_1, \\ \frac{\lambda_1\lambda_2}{4\pi} w(z, -\widehat{x}), & z \in \Omega_3, \end{cases} \tag{2.8}$$

where $\widehat{x} = \frac{x}{|x|} \in \mathbb{S}$ *is the observation direction*.

Proof The proof is analogous to the proof of Theorem 3 in [18] and, hence, is omitted. □

3 Results and discussion

In this section, we will establish an *a priori* estimate of the solution on the penetrable interfaces S_i $(i = 1, 2)$, which plays an important role in the inverse scattering problem.

To prove the next two lemmas, we first need the fundamental solution Φ_j $(j = 1, 2, 3)$ to the Helmholtz equation with wave number k_j $(j = 1, 2, 3)$ given by

$$\Phi_j(x, y) = \frac{e^{ik_j|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y, j = 1, 2, 3. \tag{3.1}$$

Define the single-layer and double-layer potentials $\widetilde{S}_{i,j}$ $(i = 1, 2, j = 1, 2, 3)$ and $\widetilde{K}_{i,j}$ $(i = 1, 2, j = 1, 2, 3)$, respectively, by

$$(\widetilde{S}_{i,j}\phi)(x) \triangleq \int_{S_i} \Phi_j(x, y)\phi(y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash S_i, i = 1, 2, j = 1, 2, 3, \tag{3.2}$$

$$(\widetilde{K}_{i,j}\phi)(x) \triangleq \int_{S_i} \frac{\partial \Phi_j(x, y)}{\partial \nu(y)} \phi(y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash S_i, i = 1, 2, j = 1, 2, 3, \tag{3.3}$$

and the normal derivative operators $\widetilde{K}'_{i,j}$ $(i = 1, 2, j = 1, 2, 3)$ and $\widetilde{T}_{i,j}$ $(i = 1, 2, j = 1, 2, 3)$ by

$$\big(\widetilde{K}'_{i,j}\phi\big)(x) \triangleq \frac{\partial}{\partial \nu(x)} \int_{S_i} \Phi_j(x, y)\phi(y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash S_i, i = 1, 2, j = 1, 2, 3, \tag{3.4}$$

$$(\widetilde{T}_{i,j}\phi)(x) \triangleq \frac{\partial}{\partial \nu(x)} \int_{S_i} \frac{\partial \Phi_j(x, y)}{\partial \nu(y)} \phi(y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash S_i, i = 1, 2, j = 1, 2, 3. \tag{3.5}$$

These operators restricted on the penetrable interfaces S_i $(i = 1, 2)$ will be denoted by $S_{i,j}$, $K_{i,j}$, $K'_{i,j}$ and $T_{i,j}$ $(i = 1, 2, j = 1, 2, 3)$, respectively. For the proof of the next two lemmas, we also need the volume potential

$$(V\phi)(x) \triangleq k_2^2 \int_{\Omega_2} \Phi_2(x,y)\big[n(y)-1\big]\phi(y)\,dy, \quad x \in \mathbb{R}^3, \tag{3.6}$$

and its normal derivative operator denoted by V'. For mapping properties of these operators in the classical spaces of continuous and Hölder continuous functions, see the monographs of Colton and Kress [6, 19].

Let Ω be the complement of Ω_1, that is, $\Omega \triangleq \mathbb{R}^3 \backslash \overline{\Omega_1}$. Choose a large ball B_R centered at the origin such that $\overline{\Omega} \subset B_R$ and let $\Omega_R = B_R \backslash \overline{\Omega}$. Denote by D any of $\overline{\Omega_R}$, $\overline{\Omega_2}$, $\overline{\Omega_3}$, S_1 or S_2. Let $x' \in D$ be an arbitrarily fixed point and denote by the weight space $C_0(D)$ consisting of all continuous functions $h(x) \in C(D\backslash\{x'\})$, such that

$$\lim_{x \to x'} \big|(x-x')h(x)\big| \tag{3.7}$$

exists. It can easily be verified that $C_0(D)$ is a Banach space equipped with the weighted maximum norm

$$\big\|h(x)\big\|_{0,D} \triangleq \sup_{x \neq x', x \in D} \big|(x-x')h(x)\big|. \tag{3.8}$$

Let B_1, B_2 be two small balls with center $x^{(1)}$ and radii r_1, r_2, respectively, satisfying that $x^{(1)} \in S_1$, $r_1 < r_2$, $B_2 \cap \overline{\Omega_3} = \emptyset$. Now we consider the scattering problem of the incident point source $\Phi_1(x, z_1)$ with $z_1 \in B_1 \cap \Omega_1$. For the proof of the unique determination of the boundary interface S_1 in the inverse scattering problem in the next section, we first study the behavior of the solution v on some part of the boundary interface S_1.

Lemma 4 *Let $u(x) \in C^2(\Omega_1) \cap C^{1,\alpha}(\overline{\Omega_1})$, $v(x) \in C^2(\Omega_2) \cap C^{1,\alpha}(\overline{\Omega_2})$ and $w(x) \in C^2(\Omega_3) \cap C^{1,\alpha}(\overline{\Omega_3})$ be a solution of the boundary value problem* (2.1)-(2.6) *with $f(x)$, $g(x)$, $p(x)$, $q(x)$ given below, then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\big\|v(x)\big\|_{\infty,S_1\backslash B_2} + \left\|\frac{\partial v(x)}{\partial \nu}\right\|_{\infty,S_1\backslash B_2} \\
&\quad \le C\big(\big\|f_1(x)\big\|_{0,S_1} + \big\|g_1(x)\big\|_{0,S_1} + \big\|f_1(x)\big\|_{1,\alpha,S_1\backslash B_1} \\
&\qquad + \big\|g_1(x)\big\|_{0,\alpha,S_1\backslash B_1} + \big\|p_1(x)\big\|_{1,\alpha,S_2} + \big\|q_1(x)\big\|_{0,\alpha,S_2} \\
&\qquad + \big\|\Phi_2(x,z_1)\big\|_{L^2(\Omega_2)}\big),
\end{aligned} \tag{3.9}$$

where $f(x) = f_1(x) = \frac{1}{\lambda_1}\Phi_2(x,z_1) - \Phi_1(x,z_1)$, $g(x) = g_1(x) = \frac{\partial \Phi_2(x,z_1)}{\partial \nu} - \frac{\partial \Phi_1(x,z_1)}{\partial \nu}$, $p(x) = p_1(x) = -\frac{1}{\lambda_1}\Phi_2(x,z_1)$, $q(x) = q_1(x) = -\frac{\partial \Phi_2(x,z_1)}{\partial \nu}$ and $z_1 \in B_1 \cap \Omega_1$.

Proof From [19] and [6], we look for the unique solution in the form

$$u(x) = \lambda_1(\widetilde{K}_{1,1}\varphi_1)(x) + (\widetilde{S}_{1,1}\phi_1)(x), \quad \text{in } \Omega_1, \tag{3.10}$$

$$v(x) = (\widetilde{K}_{1,2}\varphi_1)(x) + (\widetilde{S}_{1,2}\phi_1)(x) + \lambda_2(\widetilde{K}_{2,2}\varphi_2)(x) + (\widetilde{S}_{2,2}\phi_2)(x) + (Vv)(x), \quad \text{in } \Omega_2, \tag{3.11}$$

$$w(x) = (\widetilde{K}_{2,3}\varphi_2)(x) + (\widetilde{S}_{2,3}\phi_2)(x), \quad \text{in } \Omega_3, \tag{3.12}$$

with four densities $\varphi_1(x) \in C^{1,\alpha}(S_1)$, $\phi_1(x) \in C^{0,\alpha}(S_1)$, $\varphi_2(x) \in C^{1,\alpha}(S_2)$, $\phi_2(x) \in C^{0,\alpha}(S_2)$. Then, from the transmission boundary conditions (2.4)-(2.5), we can see that the velocity potentials $u(x)$, $v(x)$ and $w(x)$ given by (3.10)-(3.12) solve the boundary value problem (2.1)-(2.6) if the four mentioned densities satisfy the following system of integral equations:

$$\begin{aligned}
&\frac{\lambda_1+1}{2}\varphi_1(x) + \big[(\lambda_1 K_{1,1} - K_{1,2})\varphi_1\big](x) + \big[(S_{1,1} - S_{1,2})\phi_1\big](x) - (\lambda_2 K_{2,2}\varphi_2 + S_{2,2}\phi_2)(x) \\
&\quad - (Vv)(x) = f_1(x), \quad \text{on } S_1, \qquad (3.13)\\
&-\frac{\lambda_1+1}{2}\phi_1(x) + \lambda_1\big[(T_{1,1} - T_{1,2})\varphi_1\big](x) + \big[\big(K'_{1,1} - \lambda_1 K'_{1,2}\big)\phi_1\big](x) \\
&\quad - \lambda_1\big(\lambda_2 T_{2,2}\varphi_2 + K'_{2,2}\phi_2\big)(x) - \lambda_1\big(V'v\big)(x) = g_1(x), \quad \text{on } S_1, \qquad (3.14)\\
&\frac{\lambda_2+1}{2}\varphi_2(x) + \big[(\lambda_2 K_{2,2} - K_{2,3})\varphi_2\big](x) + \big[(S_{2,2} - S_{2,3})\phi_2\big](x) + (K_{1,2}\varphi_1 + S_{1,2}\phi_1)(x) \\
&\quad + (Vv)(x) = p_1(x), \quad \text{on } S_2, \qquad (3.15)\\
&-\frac{\lambda_2+1}{2}\phi_2(x) + \lambda_2\big[(T_{2,2} - T_{2,3})\varphi_2\big](x) + \big[\big(K'_{2,2} - \lambda_2 K'_{2,3}\big)\phi_2\big](x) \\
&\quad + \big(T_{1,2}\varphi_1 + K'_{1,2}\phi_1\big)(x) + \lambda_2\big(V'v\big)(x) = q_1(x), \quad \text{on } S_2. \qquad (3.16)
\end{aligned}$$

From the integral equations (3.13) and (3.15) and by using Theorem 3.4 in [6], we can see that $\varphi_1(x) \in C^{1,\alpha}(S_1)$ and $\varphi_2(x) \in C^{1,\alpha}(S_2)$. Define the product space

$$X \triangleq C^{1,\alpha}(\overline{\Omega_1}) \times C^{1,\alpha}(\overline{\Omega_2}) \times C^{1,\alpha}(\overline{\Omega_3}) \times C^{1,\alpha}(S_1) \times C^{0,\alpha}(S_1) \times C^{1,\alpha}(S_2) \times C^{0,\alpha}(S_2), \qquad (3.17)$$

then X can be chosen as the solution space of the above system (3.10)-(3.16). Assume that the operator $A : X \to X$ is given in the following matrix form:

$$A = \begin{pmatrix}
0 & 0 & 0 & -\lambda_1\widetilde{K}_{1,1} & -\widetilde{S}_{1,1} & 0 & 0 \\
0 & -V & 0 & -\widetilde{K}_{1,2} & -\widetilde{S}_{1,2} & -\lambda_2\widetilde{K}_{2,2} & -\widetilde{S}_{2,2} \\
0 & 0 & 0 & 0 & 0 & -\widetilde{K}_{2,3} & -\widetilde{S}_{2,3} \\
0 & -\frac{2V}{\lambda_1+1} & 0 & \frac{2(\lambda_1 K_{1,1}-K_{1,2})}{\lambda_1+1} & \frac{2(S_{1,1}-S_{1,2})}{\lambda_1+1} & -\frac{2\lambda_2 K_{2,2}}{\lambda_1+1} & -\frac{2S_{2,2}}{\lambda_1+1} \\
0 & -\frac{2\lambda_1 V'}{\lambda_1+1} & 0 & \frac{2\lambda_1(T_{1,2}-T_{1,1})}{\lambda_1+1} & \frac{2(\lambda_1 K'_{1,2}-K'_{1,1})}{\lambda_1+1} & \frac{2\lambda_1\lambda_2 T_{2,2}}{\lambda_1+1} & \frac{2\lambda_1 K'_{2,2}}{\lambda_1+1} \\
0 & \frac{2V}{\lambda_2+1} & 0 & \frac{2K_{1,2}}{\lambda_2+1} & \frac{2S_{1,2}}{\lambda_2+1} & -\frac{2(\lambda_2 K_{2,2}-K_{2,3})}{\lambda_2+1} & -\frac{2(S_{2,2}-S_{2,3})}{\lambda_2+1} \\
0 & -\frac{2\lambda_2 V'}{\lambda_2+1} & 0 & -\frac{2T_{1,2}}{\lambda_2+1} & -\frac{2K'_{1,2}}{\lambda_2+1} & \frac{2\lambda_2(T_{2,3}-T_{2,2})}{\lambda_2+1} & \frac{2(\lambda_2 K'_{2,3}-K'_{2,2})}{\lambda_2+1}
\end{pmatrix}.$$

Hence, the above system (3.10)-(3.16) can be rewritten in the abbreviated form

$$(I + A)U(x) = R(x), \qquad (3.18)$$

where I is the identity operator, $U(x) = (u(x), v(x), w(x), \varphi_1(x), \phi_1(x), \varphi_2(x), \phi_2(x))^T$, and

$$R(x) = \left(0, 0, 0, \frac{2f_1(x)}{\lambda_1+1}, -\frac{2g_1(x)}{\lambda_1+1}, \frac{2p_1(x)}{\lambda_2+1}, -\frac{2q_1(x)}{\lambda_2+1}\right)^T. \qquad (3.19)$$

Also, we define another weighted product space

$$Y \triangleq C_0(\overline{\Omega_R}) \times C^{1,\alpha}(\overline{\Omega_2}) \times C_0(\overline{\Omega_3}) \times C_0(S_1) \times C_0(S_1) \times C^{1,\alpha}(S_2) \times C^{0,\alpha}(S_2). \tag{3.20}$$

From [20] and [13], we know that all entries of the matrix operator A are compact, hence, we can easily see that the matrix operator A is compact in the weighted product space Y. By using Theorem 2 in [18], we know that the operator $I + A$ has a trivial null space in the weighted product space X. Consequently, by applying the Riesz-Fredholm theory to the dual system $\langle X, Y\rangle$ with the L^2 bilinear form, we can easily see that the adjoint operator $I + A'$ has a trivial null space in the weighted product space Y. Then, by applying the Riesz-Fredholm theory again to the dual system $\langle Y, Y\rangle$ with the L^2 bilinear form, we can easily see that the operator $I + A$ has a trivial null space in the weighted product space Y. Hence, system (3.18) is uniquely solvable in the weighted product space Y, and the solution depends continuously on the right-hand side:

$$\begin{aligned}
\|U(x)\|_0 &\triangleq \|u(x)\|_{0,\overline{\Omega_R}} + \|v(x)\|_{1,\alpha,\overline{\Omega_2}} + \|w(x)\|_{0,\overline{\Omega_3}} \\
&\quad + \|\varphi_1(x)\|_{0,S_1} + \|\phi_1(x)\|_{0,S_1} + \|\varphi_2(x)\|_{1,\alpha,S_2} + \|\phi_2(x)\|_{0,\alpha,S_2} \\
&\leq C\big(\|f_1(x)\|_{0,S_1} + \|g_1(x)\|_{0,S_1} + \|p_1(x)\|_{1,\alpha,S_2} + \|q_1(x)\|_{0,\alpha,S_2} \\
&\quad + \|\Phi_2(x,z_1)\|_{L^2(\Omega_2)}\big).
\end{aligned} \tag{3.21}$$

In particular, this implies that

$$\begin{aligned}
\|v(x)\|_{\infty,S_1\backslash B_2} &\leq C\big(\|f_1(x)\|_{0,S_1} + \|g_1(x)\|_{0,S_1} + \|p_1(x)\|_{1,\alpha,S_2} + \|q_1(x)\|_{1,\alpha,S_2} \\
&\quad + \|\Phi_2(x,z_1)\|_{L^2(\Omega_2)}\big).
\end{aligned} \tag{3.22}$$

Let B_{12} be a ball of radius r_{12} and centered at $x^{(1)}$ with $r_1 < r_{12} < r_2$ and assume that $\rho_1(x) \in C^2(S_1)$ is a function satisfying $\rho_1(x) = 0$ for $x \in S_1\backslash B_2$ and $\rho_1(x) = 1$ in the neighborhood of B_{12}, $\rho_2(x) \in C^2(S_1)$ is another function satisfying $\rho_2(x) = 0$ for $x \in S_1\backslash B_{12}$ and $\rho_2(x) = 1$ in the neighborhood of B_1.

We rewrite $U(x)$ in the form

$$U(x) = \begin{pmatrix} u(x) \\ v(x) \\ w(x) \\ \varphi_1(x) \\ \phi_1(x) \\ \varphi_2(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} \rho_1(x)u(x) \\ \rho_1(x)v(x) \\ w(x) \\ \rho_1(x)\varphi_1(x) \\ \rho_1(x)\phi_1(x) \\ \varphi_2(x) \\ \phi_2(x) \end{pmatrix} + \begin{pmatrix} (1-\rho_1(x))u(x) \\ (1-\rho_1(x))v(x) \\ 0 \\ (1-\rho_1(x))\varphi_1(x) \\ (1-\rho_1(x))\phi_1(x) \\ 0 \\ 0 \end{pmatrix} \triangleq U_{\rho_1}(x) + U_{1-\rho_1}(x), \tag{3.23}$$

and for a matrix M, denote M_ρ by the same matrix but with its first, second, fourth, and fifth rows multiplied by $\rho(x)$. Hence, from (3.18), we have

$$U_{\rho_2}(x) = R_{\rho_2}(x) - A_{\rho_2}U_{\rho_1}(x) - A_{\rho_2}U_{1-\rho_1}(x). \tag{3.24}$$

The mapping operator $U(x) \to A_{\rho_2} U_{\rho_1}(x)$ is bounded from the weighted product space Y into the weighted product space X as its kernel vanishes in a neighborhood of the diagonal element. Moreover, by using Theorems 2.30 and 2.31 in [19], we can see that

$$\left\| A_{\rho_2} U_{1-\rho_1}(x) \right\|_{0,\alpha} \le C \left\| A U_{1-\rho_1}(x) \right\|_{0,\alpha} \le C \left\| U_{1-\rho_1}(x) \right\|_{\infty} \le C \left\| U(x) \right\|_{0}, \tag{3.25}$$

where the norms $\|U(x)\|_{0,\alpha}$ and $\|U(x)\|_{\infty}$ are defined as follows: the first, second, fourth, and fifth components of $U(x)$ are defined by the corresponding norms but its third, sixth, and seventh components are equipped with $C^{1,\alpha}(\overline{\Omega_3})$, $C^{1,\alpha}(S_2)$ and $C^{0,\alpha}(S_2)$ norms, respectively. From (3.21) and (3.24), we can see that

$$\begin{aligned}
\left\| U(x) \right\|_{0,\alpha} &\triangleq \left\| u(x) \right\|_{0,\alpha,\overline{\Omega_R}\setminus B_{12}} + \left\| v(x) \right\|_{1,\alpha,\overline{\Omega_2}\setminus B_{12}} + \left\| w(x) \right\|_{0,\alpha,\overline{\Omega_3}} \\
&\quad + \left\| \varphi_1(x) \right\|_{0,\alpha,S_1\setminus B_{12}} + \left\| \phi_1(x) \right\|_{0,\alpha,S_1\setminus B_{12}} + \left\| \varphi_2(x) \right\|_{1,\alpha,S_2} + \left\| \phi_2(x) \right\|_{0,\alpha,S_2} \\
&\le C \left\| U_{\rho_2}(x) \right\|_{0,\alpha} \\
&\le C \left(\left\| R_{\rho_2}(x) \right\|_{0,\alpha} + \left\| U(x) \right\|_{0} \right) \\
&\le C \left(\left\| f_1(x) \right\|_{0,S_1} + \left\| g_1(x) \right\|_{0,S_1} + \left\| f_1(x) \right\|_{0,\alpha,S_1\setminus B_1} + \left\| g_1(x) \right\|_{0,\alpha,S_1\setminus B_1} \right. \\
&\quad \left. + \left\| p_1(x) \right\|_{1,\alpha,S_2} + \left\| q_1(x) \right\|_{0,\alpha,S_2} + \left\| \Phi_2(x,z_1) \right\|_{L^2(\Omega_2)} \right).
\end{aligned} \tag{3.26}$$

Then we estimate $\|\varphi_1(x)\|_{1,\alpha,S_1\setminus B_{12}}$. Multiplying (3.13) by $\rho_2(x)$, using (3.24), and noting the fact that the integral operators mapping $C^{0,\alpha}$-functions into $C^{1,\alpha}$-functions are bounded and the fact that $\varphi_1(x) = \rho_1(x)\varphi_1(x) + [1 - \rho_1(x)]\varphi_1(x)$ and $\phi_1(x) = \rho_1(x)\phi_1(x) + [1 - \rho_1(x)]\phi_1(x)$, we can see that

$$\begin{aligned}
\left\| \varphi_1(x) \right\|_{1,\alpha,S_1\setminus B_{12}} &\le \left\| \rho_2(x)\varphi_1(x) \right\|_{1,\alpha,S_1} \\
&\le C \left(\left\| \rho_2(x) \left[(\lambda_1 K_{1,1} - K_{1,2})\varphi_1 \right](x) \right\|_{1,\alpha} + \left\| \rho_2(x) \left[(S_{1,1} - S_{1,2})\phi_1 \right](x) \right\|_{1,\alpha} \right. \\
&\quad + \left\| \rho_2(x)\lambda_2 (K_{2,2}\varphi_2)(x) \right\|_{1,\alpha} + \left\| \rho_2(x)(S_{2,2}\phi_2)(x) \right\|_{1,\alpha} \\
&\quad \left. + \left\| \rho_2(x)(Vv)(x) \right\|_{1,\alpha} + \left\| \rho_2(x) f_1(x) \right\|_{1,\alpha} \right) \\
&\le C \left(\left\| U(x) \right\|_{0} + \left\| \left[1 - \rho_1(x) \right] U(x) \right\|_{0,\alpha} + \left\| \rho_2(x) f_1(x) \right\|_{1,\alpha} \right) \\
&\le C \left(\left\| U(x) \right\|_{0} + \left\| U(x) \right\|_{0,\alpha} + \left\| f_1(x) \right\|_{1,\alpha,S_1\setminus B_1} \right).
\end{aligned} \tag{3.27}$$

From (3.21) and (3.26)-(3.27), we can establish the following estimate in the spaces of Hölder continuous functions for $(u(x), v(x), w(x), \varphi_1(x), \phi_1(x), \varphi_2(x), \phi_2(x))$:

$$\begin{aligned}
\left\| U(x) \right\|_{1,\alpha} &\triangleq \left\| u(x) \right\|_{0,\alpha,\overline{\Omega_R}\setminus B_{12}} + \left\| v(x) \right\|_{1,\alpha,\overline{\Omega_2}\setminus B_{12}} + \left\| w(x) \right\|_{0,\alpha,\overline{\Omega_3}} \\
&\quad + \left\| \varphi_1(x) \right\|_{1,\alpha,S_1\setminus B_{12}} + \left\| \phi_1(x) \right\|_{0,\alpha,S_1\setminus B_{12}} + \left\| \varphi_2(x) \right\|_{1,\alpha,S_2} + \left\| \phi_2(x) \right\|_{0,\alpha,S_2} \\
&\le C \left(\left\| f_1(x) \right\|_{0,S_1} + \left\| g_1(x) \right\|_{0,S_1} + \left\| f_1(x) \right\|_{1,\alpha,S_1\setminus B_1} + \left\| g_1(x) \right\|_{0,\alpha,S_1\setminus B_1} \right. \\
&\quad \left. + \left\| p_1(x) \right\|_{1,\alpha,S_2} + \left\| q_1(x) \right\|_{0,\alpha,S_2} + \left\| \Phi_2(x,z_1) \right\|_{L^2(\Omega_2)} \right).
\end{aligned} \tag{3.28}$$

Next, we estimate $\| \frac{\partial v(x)}{\partial \nu} \|_{0,\alpha,S_1 \backslash B_2}$. From (3.11) and the jump relation, we can obtain, on S_1,

$$\begin{aligned}\frac{\partial v(x)}{\partial \nu} &= \frac{1}{2}\phi_1(x) + (T_{1,2}\varphi_1)(x) + \big(K'_{1,2}\phi_1\big)(x) + \lambda_2(T_{2,2}\varphi_2)(x) \\ &\quad + \big(K'_{2,2}\phi_2\big)(x) + \big(V'v\big)(x).\end{aligned} \tag{3.29}$$

From (3.29) and by using the fact that $\varphi_1(x) = \rho_1(x)\varphi_1(x) + [1-\rho_1(x)]\varphi_1(x)$ and $\phi_1(x) = \rho_1(x)\phi_1(x) + [1-\rho_1(x)]\phi_1(x)$ again, we can see that

$$\begin{aligned}\left\| \frac{\partial v(x)}{\partial \nu} \right\|_{0,\alpha,S_1\backslash B_2} &\le \left\| \big[1-\rho_1(x)\big] \frac{\partial v(x)}{\partial \nu} \right\|_{0,\alpha,S_1} \\ &\le C\big(\| U(x) \|_0 + \| [1-\rho_1(x)]\varphi_1(x) \|_{1,\alpha,S_1} + \| [1-\rho_1(x)]\phi_1(x) \|_{0,\alpha,S_1} \\ &\quad + \| \varphi_2(x) \|_{1,\alpha,S_2} + \| \phi_2(x) \|_{0,\alpha,S_2} \big) \\ &\le C\big(\| U(x) \|_0 + \| U(x) \|_{1,\alpha} \big).\end{aligned} \tag{3.30}$$

From the estimate (3.30) with (3.21) and (3.28), we can see that

$$\begin{aligned}\left\| \frac{\partial v(x)}{\partial \nu} \right\|_{0,\alpha,S_1\backslash B_2} &\le C\big(\| f_1(x) \|_{0,S_1} + \| g_1(x) \|_{0,S_1} + \| f_1(x) \|_{1,\alpha,S_1\backslash B_1} + \| g_1(x) \|_{0,\alpha,S_1\backslash B_1} \\ &\quad + \| p_1(x) \|_{1,\alpha,S_2} + \| q_1(x) \|_{0,\alpha,S_2} + \| \Phi_2(x,z_1) \|_{L^2(\Omega_2)} \big).\end{aligned} \tag{3.31}$$

It finishes the proof of the lemma. □

Similarly, let B_3, B_4 be two small balls with center $x^{(2)}$ and radii r_3, r_4, respectively, satisfying that $x^{(2)} \in S_2$, $r_3 < r_4$, $B_4 \cap \overline{\Omega_1} = \emptyset$. Now we consider the scattering problem of the incident point source $\Phi_3(x,z_3)$ with $z_3 \in B_3 \cap \Omega_3$. To prove the unique determination of the boundary interface S_2 in the inverse scattering problem in the next section, we need to study the behavior of the solution w on some part of the boundary interface S_2.

Lemma 5 *Assume that* $z_3 \in B_3 \cap \Omega_3$ *and* $u(x) \in C^2(\Omega_1) \cap C^{1,\alpha}(\overline{\Omega_1})$, $v(x) \in C^2(\Omega_2) \cap C^{1,\alpha}(\overline{\Omega_2})$ *and* $w(x) \in C^2(\Omega_3) \cap C^{1,\alpha}(\overline{\Omega_3})$ *is a solution of the boundary value problem* (2.1)-(2.6) *with* $f_2(x)$, $g_2(x)$, $p_2(x)$, $q_2(x)$ *given below, then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}&\| w(x) \|_{\infty,S_2\backslash B_4} + \left\| \frac{\partial w(x)}{\partial \nu} \right\|_{\infty,S_2\backslash B_4} \\ &\quad \le C\big(\| f_2(x) \|_{1,\alpha,S_1} + \| g_2(x) \|_{0,\alpha,S_1} + \| p_2(x) \|_{0,S_2} + \| q_2(x) \|_{0,S_2} \\ &\qquad + \| p_2(x) \|_{1,\alpha,S_2\backslash B_3} + \| q_2(x) \|_{0,\alpha,S_2\backslash B_3} \big),\end{aligned} \tag{3.32}$$

where $f_2(x) = \frac{1}{\lambda_1\lambda_2}\Phi_3(x,z_3)$, $g_2(x) = \frac{\partial \Phi_3(x,z_3)}{\partial \nu}$, $p_2(x) = -\frac{1}{\lambda_2}\Phi_3(x,z_3)$, $q_2(x) = -\frac{\partial \Phi_3(x,z_3)}{\partial \nu}$.

Proof Arguing similarly as in the above lemma, in addition to the weighted product space X, Y, we also consider the weighted product space

$$Z \stackrel{\Delta}{=} C_0(\overline{\Omega_R}) \times C_0(\overline{\Omega_2}) \times C^{1,\alpha}(\overline{\Omega_3}) \times C^{1,\alpha}(S_1) \times C^{0,\alpha}(S_1) \times C_0(S_2) \times C_0(S_2). \tag{3.33}$$

From [20] and [13], we know that all entries of the matrix operator A are compact, hence, we can easily see that the matrix operator A is compact in the weighted product space Z. By using Theorem 2 in [18], we know that the operator $I + A$ has a trivial null space in the weighted product space X. Consequently, by applying the Fredholm alternative to the dual system $\langle X, Z\rangle$ with the L^2 bilinear form, we can see that the adjoint operator $I + A'$ has a trivial null space in the weighted product space Z. Then, by applying the Fredholm alternative again to the dual system $\langle Z, Z\rangle$ with the L^2 bilinear form, we can see that the operator $I + A$ has a trivial null space in the weighted product space Z. Hence, by using the Riesz-Fredholm theory, system (3.18) is uniquely solvable in the weighted product space Z, and the solution depends continuously on the right-hand side:

$$\begin{aligned}\|U(x)\|_0 &\triangleq \|u(x)\|_{0,\overline{\Omega_R}} + \|v(x)\|_{0,\overline{\Omega_2}} + \|w(x)\|_{1,\alpha,\overline{\Omega_3}} \\ &\quad + \|\varphi_1(x)\|_{1,\alpha,S_1} + \|\phi_1(x)\|_{0,\alpha,S_1} + \|\varphi_2(x)\|_{0,S_2} + \|\phi_2(x)\|_{0,S_2} \\ &\le C\big(\|f_2(x)\|_{1,\alpha,S_1} + \|g_2(x)\|_{0,\alpha,S_1} + \|p_2(x)\|_{0,S_2} + \|q_2(x)\|_{0,S_2}\big). \end{aligned} \tag{3.34}$$

In particular, this implies that

$$\|w(x)\|_{\infty,S_2\backslash B_4} \le C\big(\|f_2(x)\|_{1,\alpha,S_1} + \|g_2(x)\|_{0,\alpha,S_1} + \|p_2(x)\|_{0,S_2} + \|q_2(x)\|_{0,S_2}\big). \tag{3.35}$$

Let B_{34} be a ball of radius r_{34} and centered at $x^{(2)}$ with $r_3 < r_{34} < r_4$, and assume that $\rho_3(x) \in C^2(S_2)$ is a function satisfying $\rho_3(x) = 0$ for $x \in S_2\backslash B_4$ and $\rho_3(x) = 1$ in the neighborhood of B_{34}, $\rho_4(x) \in C^2(S_2)$ is another function satisfying $\rho_4(x) = 0$ for $x \in S_2\backslash B_{34}$ and $\rho_4(x) = 1$ in the neighborhood of B_3.

We rewrite $U(x)$ in the form

$$\begin{aligned} U = \begin{pmatrix} u(x) \\ v(x) \\ w(x) \\ \varphi_1(x) \\ \phi_1(x) \\ \varphi_2(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} u(x) \\ \rho_3(x)v(x) \\ \rho_3(x)w(x) \\ \varphi_1(x) \\ \phi_1(x) \\ \rho_3(x)\varphi_2(x) \\ \rho_3(x)\phi_2(x) \end{pmatrix} + \begin{pmatrix} 0 \\ (1-\rho_3(x))v(x) \\ (1-\rho_3(x))w(x) \\ 0 \\ 0 \\ (1-\rho_3(x))\varphi_2(x) \\ (1-\rho_3(x))\phi_2(x) \end{pmatrix} \\ \triangleq U_{\rho_3}(x) + U_{1-\rho_3}(x), \end{aligned} \tag{3.36}$$

and for a matrix N, denote N_ρ by the same matrix but with its second, third, sixth, and seventh rows multiplied by $\rho(x)$. Hence, from (3.18), we have

$$U_{\rho_4}(x) = R_{\rho_4}(x) - A_{\rho_4}U_{\rho_3}(x) - A_{\rho_4}U_{1-\rho_3}(x). \tag{3.37}$$

The mapping operator $U(x) \to A_{\rho_4}U_{\rho_3}(x)$ is bounded from the weighted product space Z into the weighted product space X since its kernel vanishes in a neighborhood of the diagonal element. Moreover, by using theorems 2.30 and 2.31 in [19], we can see that

$$\|A_{\rho_4}U_{1-\rho_3}(x)\|_{0,\alpha} \le C\|AU_{1-\rho_3}(x)\|_{0,\alpha} \le C\|U_{1-\rho_3}(x)\|_\infty \le C\|U(x)\|_0, \tag{3.38}$$

where the norms $\|U(x)\|_{0,\alpha}$ and $\|U(x)\|_{\infty}$ are defined as follows: the second, third, sixth, and seventh components of $U(x)$ are defined by the corresponding norms but its first, fourth, and fifth components are equipped with $C^{1,\alpha}(\overline{\Omega_R})$, $C^{1,\alpha}(S_1)$ and $C^{0,\alpha}(S_1)$ norms, respectively. From (3.34) and (3.37), we can see that

$$\begin{aligned}\|U(x)\|_{0,\alpha} &\triangleq \|u(x)\|_{0,\alpha,\overline{\Omega_R}} + \|v(x)\|_{1,\alpha,\overline{\Omega_2}\setminus B_{34}} + \|w(x)\|_{0,\alpha,\overline{\Omega_3}\setminus B_{34}} \\ &\quad + \|\varphi_1(x)\|_{1,\alpha,S_1} + \|\phi_1(x)\|_{0,\alpha,S_1} + \|\varphi_2(x)\|_{0,\alpha,S_2\setminus B_{34}} + \|\phi_2(x)\|_{0,\alpha,S_2\setminus B_{34}} \\ &\le C\|U_{\rho_4}(x)\|_{0,\alpha} \\ &\le C\big(\|R_{\rho_4}(x)\|_{0,\alpha} + \|U(x)\|_0\big) \\ &\le C\big(\|f_2(x)\|_{1,\alpha,S_1} + \|g_2(x)\|_{0,\alpha,S_1} + \|p_2(x)\|_{1,\alpha,S_2\setminus B_3} + \|q_2(x)\|_{0,\alpha,S_2\setminus B_3} \\ &\quad + \|p_2(x)\|_{0,S_2} + \|q_2(x)\|_{0,S_2}\big). \end{aligned} \tag{3.39}$$

Then we estimate $\|\varphi_2(x)\|_{1,\alpha,S_2\setminus B_{34}}$. Multiplying (3.15) by $\rho_4(x)$, using (3.37), and noting the fact that the integral operators mapping $C^{0,\alpha}$-functions into $C^{1,\alpha}$-functions are bounded and the fact that $\varphi_2(x) = \rho_3(x)\varphi_2(x) + [1-\rho_3(x)]\varphi_2(x)$ and $\phi_2(x) = \rho_3(x)\phi_2(x) + [1-\rho_3(x)]\phi_2(x)$, we can see that

$$\begin{aligned}\|\varphi_2(x)\|_{1,\alpha,S_2\setminus B_{34}} &\le \|\rho_4(x)\varphi_2(x)\|_{1,\alpha,S_2} \\ &\le C\big(\|\rho_4(x)[(\lambda_2K_{2,2}-K_{2,3})\varphi_2](x)\|_{1,\alpha} + \|\rho_4(x)[(S_{2,2}-S_{2,3})\phi_2](x)\|_{1,\alpha} \\ &\quad + \|\rho_4(x)(K_{1,2}\varphi_1)(x)\|_{1,\alpha} + \|\rho_4(x)(S_{1,2}\phi_1)(x)\|_{1,\alpha} \\ &\quad + \|\rho_4(x)(Vv)(x)\|_{1,\alpha} + \|\rho_4(x)p_2(x)\|_{1,\alpha}\big) \\ &\le C\big(\|U(x)\|_0 + \|[1-\rho_3(x)]U(x)\|_{0,\alpha} + \|\rho_4(x)p_2(x)\|_{1,\alpha}\big) \\ &\le C\big(\|U(x)\|_0 + \|U(x)\|_{0,\alpha} + \|p_2(x)\|_{1,\alpha,S_2\setminus B_3}\big). \end{aligned} \tag{3.40}$$

From (3.34) and (3.39)-(3.40), we can establish the following estimate in the spaces of Hölder continuous functions for $(u(x), v(x), w(x), \varphi_1(x), \phi_1(x), \varphi_2(x), \phi_2(x))$:

$$\begin{aligned}\|U(x)\|_{1,\alpha} &\triangleq \|u(x)\|_{0,\alpha,\overline{\Omega_R}} + \|v(x)\|_{1,\alpha,\overline{\Omega_2}\setminus B_{34}} + \|w(x)\|_{0,\alpha,\overline{\Omega_3}\setminus B_{34}} \\ &\quad + \|\varphi_1(x)\|_{1,\alpha,S_1} + \|\phi_1(x)\|_{0,\alpha,S_1} + \|\varphi_2(x)\|_{1,\alpha,S_2\setminus B_{34}} + \|\phi_2(x)\|_{0,\alpha,S_2\setminus B_{34}} \\ &\le C\big(\|f_2(x)\|_{1,\alpha,S_1} + \|g_2(x)\|_{0,\alpha,S_1} + \|p_2(x)\|_{1,\alpha,S_2\setminus B_3} + \|q_2(x)\|_{0,\alpha,S_2\setminus B_3} \\ &\quad + \|p_2(x)\|_{0,S_2} + \|q_2(x)\|_{0,S_2}\big). \end{aligned} \tag{3.41}$$

Next, we estimate $\|\frac{\partial w(x)}{\partial \nu}\|_{0,\alpha,S_2\setminus B_4}$. From (3.12) and the jump relation, we can see that, on S_2,

$$\frac{\partial w(x)}{\partial \nu} = \frac{1}{2}\phi_2(x) + (T_{2,3}\varphi_2)(x) + \big(K'_{2,3}\phi_2\big)(x). \tag{3.42}$$

From (3.42), and by using the fact that $\varphi_2(x) = \rho_3(x)\varphi_2(x) + [1 - \rho_3(x)]\varphi_2(x)$ and $\phi_2(x) = \rho_3(x)\phi_2(x) + [1 - \rho_3(x)]\phi_2(x)$ again, we can see that

$$\begin{aligned}\left\| \frac{\partial w(x)}{\partial \nu} \right\|_{0,\alpha,S_2\setminus B_4} &\leq \left\| \bigl[1-\rho_3(x)\bigr] \frac{\partial w(x)}{\partial \nu} \right\|_{0,\alpha,S_2} \\ &\leq C\bigl(\|U(x)\|_0 + \bigl\|\bigl[1-\rho_3(x)\bigr]\varphi_2(x)\bigr\|_{1,\alpha,S_2} + \bigl\|\bigl[1-\rho_3(x)\bigr]\phi_2(x)\bigr\|_{0,\alpha,S_2} \bigr) \\ &\leq C\bigl(\|U(x)\|_0 + \|U(x)\|_{1,\alpha} \bigr). \end{aligned} \tag{3.43}$$

From the estimate (3.43) with (3.34) and (3.41), we can see that

$$\begin{aligned}\left\| \frac{\partial w(x)}{\partial \nu} \right\|_{0,\alpha,S_2\setminus B_4} &\leq C\bigl(\|f_2(x)\|_{1,\alpha,S_1} + \|g_2(x)\|_{0,\alpha,S_1} + \|p_2(x)\|_{1,\alpha,S_2\setminus B_3} + \|q_2(x)\|_{0,\alpha,S_2\setminus B_3} \\ &\quad + \|p_2(x)\|_{0,S_2} + \|q_2(x)\|_{0,S_2} \bigr). \end{aligned} \tag{3.44}$$

It finishes the proof of the lemma. □

4 Conclusions

In this section, we will prove that both the penetrable interfaces S_i $(i = 1, 2)$ and the refractive index $n(x)$ of the inhomogeneous penetrable obstacle Ω_2 can be uniquely determined from knowledge of the far-field pattern $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S})$ for incident plane waves $u^i(x) = e^{ik_1 x\cdot d}$.

4.1 Unique determination of the penetrable interfaces S_i $(i = 1, 2)$

Following the transmission boundary value problems in a homogeneous medium [20], the transmission boundary value problems in an inhomogeneous medium [13] and the inhomogeneous impenetrable obstacle scattering in a stratified medium [9], we prove in this subsection that the penetrable interfaces S_i $(i = 1, 2)$ can be uniquely determined by the far-field pattern $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S})$ for incident plane waves $u^i(x) = e^{ik_1 x\cdot d}$. To achieve this, we first give the following two lemmas.

Lemma 6 *Assume that $h_1(x) \in L^2(\Omega_2)$, $g_3(x) \in C(S_1)$, $p_3(x) \in C(S_2)$, $q_3(x) \in C(S_2)$, $\eta_1(x) \in C(S_1)$ with $\eta_1(x) \neq 0$ and $\eta_1(x) \leq 0$ on S_1, then the following boundary value problem has a unique solution $v(x) \in C^2(\Omega_2) \cap C(\overline{\Omega_2})$ and $w(x) \in C^2(\Omega_3) \cap C(\overline{\Omega_3})$:*

$$\Delta v(x) + k_2^2 n(x) v(x) = h_1(x), \quad in\ \Omega_2, \tag{4.1}$$

$$\Delta w(x) + k_3^2 w(x) = 0, \quad in\ \Omega_3, \tag{4.2}$$

$$\frac{\partial v(x)}{\partial \nu} + i\eta_1(x) v(x) = g_3(x), \quad on\ S_1, \tag{4.3}$$

$$v(x) - w(x) = p_3(x), \qquad \frac{\partial v(x)}{\partial \nu} - \lambda_2 \frac{\partial w(x)}{\partial \nu} = q_3(x), \quad on\ S_2. \tag{4.4}$$

Moreover, there exists a constant $C > 0$ such that

$$\|v(x)\|_{\infty,\overline{\Omega_2}} \leq C\bigl(\|h_1(x)\|_{L^2(\Omega_2)} + \|g_3(x)\|_{\infty,S_1} + \|p_3(x)\|_{\infty,S_2} + \|q_3(x)\|_{\infty,S_2} \bigr). \tag{4.5}$$

Proof First, we prove the uniqueness of solutions, that is $v(x) = 0$ in Ω_2, $w(x) = 0$ in Ω_3 if $h_1(x) = 0$ in Ω_2, $g_3(x) = 0$ on S_1, $p_3(x) = q_3(x) = 0$ on S_2. From equations (4.1)-(4.2), boundary conditions (4.3)-(4.4), noting the assumption that k_2^2 is not a Neumann eigenvalue of $\Delta v(x) + k_2^2 n(x)v(x) = 0$ in Ω_2, and using Green's first theorem over Ω_2 and Ω_3, we can see that

$$\begin{aligned}
0 &= \int_{\Omega_2} \bigl\{\bigl[\Delta v(x) + k_2^2 n(x)v(x)\bigr]\overline{v}(x)\bigr\}\,dx \\
&= \int_{\Omega_2} \bigl[-\bigl|\nabla v(x)\bigr|^2 + k_2^2 n(x)\bigl|v(x)\bigr|^2\bigr]\,dx + \int_{S_1} \overline{v}(x)\frac{\partial v(x)}{\partial \nu}\,ds - \int_{S_2} \overline{v}(x)\frac{\partial v(x)}{\partial \nu}\,ds \\
&= \int_{\Omega_2} \bigl[-\bigl|\nabla v(x)\bigr|^2 + k_2^2 n(x)\bigl|v(x)\bigr|^2\bigr]\,dx \\
&\quad - i\int_{S_1} \eta_1(x)\bigl|v(x)\bigr|^2\,ds - \lambda_2\int_{S_2} \overline{w}(x)\frac{\partial w(x)}{\partial \nu}\,ds \\
&= \int_{\Omega_2} \bigl[-\bigl|\nabla v(x)\bigr|^2 + k_2^2 n(x)\bigl|v(x)\bigr|^2\bigr]\,dx - i\int_{S_1} \eta_1(x)\bigl|v(x)\bigr|^2\,ds \\
&\quad - \lambda_2\int_{\Omega_3} \bigl[\bigl|\nabla w(x)\bigr|^2 + \Delta w(x)\cdot\overline{w}(x)\bigr]\,dx \\
&= \int_{\Omega_2} \bigl[-\bigl|\nabla v(x)\bigr|^2 + k_2^2 n(x)\bigl|v(x)\bigr|^2\bigr]\,dx - i\int_{S_1} \eta_1(x)\bigl|v(x)\bigr|^2\,ds \\
&\quad - \lambda_2\int_{\Omega_3} \bigl[\bigl|\nabla w(x)\bigr|^2 - k_3^2\bigl|w(x)\bigr|^2\bigr]\,dx.
\end{aligned} \tag{4.6}$$

By taking the imaginary part of the above equation (4.6), we can see that $v(x) = 0$ on some part Γ_1 of S_1 since $\eta_1(x) \not\equiv 0$, $\eta_1(x) \leq 0$ on S_1, $\lambda_2 > 0$ is a given number and $\Im[n(x)] \geq 0$. From the boundary condition (4.3), we can see that $\frac{\partial v(x)}{\partial \nu} = -i\eta_1(x)v(x) = 0$ on some part Γ_1 of S_1. Therefore, by using Holmgren's uniqueness theorem [21], we can see that $v(x) = 0$ in Ω_2. By using the transmission boundary conditions (4.4), we have $w(x) = 0$ in Ω_3.

Then we prove the existence of solutions. To achieve this, we introduce the volume potential

$$\bigl(V^\star h_1\bigr)(x) \triangleq \int_{\Omega_2} \Phi_2(x,y)h_1(y)\,dy, \quad x \in \Omega_2, \tag{4.7}$$

from Theorem 8.2 [19], we know this is a bounded operator $V^\star : L^2(\Omega_2) \to H^2(\Omega_2)$. Now we seek a solution in the form

$$\begin{aligned}
v(x) = &-\bigl(V^\star h_1\bigr)(x) + \bigl[\widetilde{S}_{1,2}(\varphi_3 + \varphi_4)\bigr](x) + \lambda_2\bigl[\widetilde{K}_{2,2}(\phi_3 + \phi_4)\bigr](x) \\
&+ \bigl[\widetilde{S}_{2,2}(\psi_1 + \psi_2)\bigr](x) + \bigl(V^\star v\bigr)(x), \quad \text{in } \Omega_2,
\end{aligned} \tag{4.8}$$

$$w(x) = \bigl[\widetilde{K}_{2,3}(\phi_3 + \phi_4)\bigr](x) + \bigl[\widetilde{S}_{2,3}(\psi_1 + \psi_2)\bigr](x), \quad \text{in } \Omega_3, \tag{4.9}$$

with six densities $\varphi_3(x) \in C(S_1)$, $\varphi_4(x) \in H^{\frac{1}{2}}(S_1)$, $\phi_3(x) \in C(S_2)$, $\phi_4(x) \in H^{\frac{1}{2}}(S_2)$, $\psi_1(x) \in C(S_2)$, $\psi_2(x) \in H^{\frac{1}{2}}(S_2)$. From the jump conditions, we can see that the potentials $v(x)$, $w(x)$ given by (4.8) and (4.9) solve the boundary value problem (4.1)-(4.4) if the six mentioned densities satisfy the following system of integral equations:

$$
\begin{aligned}
&\frac{1}{2}\varphi_3(x) + \left[\left(K'_{1,2} + i\eta_1 S_{1,2}\right)\varphi_3\right](x) + \lambda_2\left[(T_{2,2} + i\eta_1 K_{2,2})(\phi_3 + \phi_4)\right](x) \\
&\quad + \left[\left(K'_{2,2} + i\eta_1 S_{2,2}\right)(\psi_1 + \psi_2)\right](x) = g_3(x), \quad \text{on } S_1, && (4.10)\\
&\frac{1}{2}\varphi_4(x) + \left[\left(K'_{1,2} + i\eta_1 S_{1,2}\right)\varphi_4\right](x) = \frac{\partial(V^\star h_1 - V^\star v)(x)}{\partial \nu} + i\eta_1(x)\left(V^\star h_1 - V^\star v\right)(x), \\
&\quad \text{on } S_1, && (4.11)\\
&\frac{\lambda_2 + 1}{2}\phi_3(x) + \left[S_{1,2}(\varphi_3 + \varphi_4)\right](x) + \left[(\lambda_2 K_{2,2} - K_{2,3})\phi_3\right](x) + \left[(S_{2,2} - S_{2,3})\psi_1\right](x) = p_3(x), \\
&\quad \text{on } S_2, && (4.12)\\
&\frac{\lambda_2 + 1}{2}\phi_4(x) + \left[(\lambda_2 K_{2,2} - K_{2,3})\phi_4\right](x) + \left[(S_{2,2} - S_{2,3})\psi_2\right](x) = \left(V^\star h_1 - V^\star v\right)(x), \\
&\quad \text{on } S_2, && (4.13)\\
&-\frac{\lambda_2 + 1}{2}\psi_1(x) + \left[K'_{1,2}(\varphi_3 + \varphi_4)\right](x) + \lambda_2\left[(T_{2,2} - T_{2,3})\phi_3\right](x) \\
&\quad + \left[\left(K'_{2,2} - \lambda_2 K'_{2,3}\right)\psi_1\right](x) = q_3(x), \quad \text{on } S_2, && (4.14)\\
&-\frac{\lambda_2 + 1}{2}\psi_2(x) + \left[\left(K'_{2,2} - \lambda_2 K'_{2,3}\right)\psi_2\right](x) + \lambda_2\left[(T_{2,2} - T_{2,3})\phi_4\right](x) \\
&\quad = \frac{\partial(V^\star h_1 - -V^\star v)(x)}{\partial \nu}, \quad \text{on } S_2. && (4.15)
\end{aligned}
$$

Next, we look for a solution $(v(x), w(x), \varphi_3(x), \varphi_4(x), \phi_3(x), \phi_4(x), \psi_1(x), \psi_2(x)) \in W$ to the above system of eight integral equations, where W is a weighted product space defined as follows:

$$
W \stackrel{\Delta}{=} C(\overline{\Omega_2}) \times C(\overline{\Omega_3}) \times C(S_1) \times H^{\frac{1}{2}}(S_1) \times C(S_2) \times H^{\frac{1}{2}}(S_2) \times C(S_2) \times H^{\frac{1}{2}}(S_2). \tag{4.16}
$$

By using the uniqueness of solutions to the problem and standard arguments, we can easily see that this system has at most one solution in the weighted product space W. Thus, by using the Riesz-Fredholm theory, we can easily obtain the existence of solutions to the boundary value problem with the estimate:

$$
\begin{aligned}
\left\|v(x)\right\|_{\infty,\overline{\Omega_2}} &\le C\Bigg(\left\|\left(V^\star h_1\right)(x)\right\|_{\infty,\overline{\Omega_2}} + \left\| \frac{\partial(V^\star h_1)(x)}{\partial \nu} + i\eta_1(x)\left(V^\star h_1\right)(x) \right\|_{H^{\frac{1}{2}}(S_1 \cup S_2)} \\
&\qquad + \left\|g_3(x)\right\|_{\infty,S_1} + \left\|p_3(x)\right\|_{\infty,S_2} + \left\|q_3(x)\right\|_{\infty,S_2} \Bigg) \\
&\le C\left(\left\|h_1(x)\right\|_{L^2(\Omega_2)} + \left\|g_3(x)\right\|_{\infty,S_1} + \left\|p_3(x)\right\|_{\infty,S_2} + \left\|q_3(x)\right\|_{\infty,S_2}\right),
\end{aligned} \tag{4.17}
$$

for some constant $C > 0$. It finishes the proof of the lemma. □

Lemma 7 *Assume that* $h_2(x) \in L^2(\Omega_3)$, $q_4(x) \in C(S_2)$, $\eta_2(x) \in C(S_2)$ *with* $\eta_2(x) \not\equiv 0$ *and* $\eta_2(x) \le 0$ *on* S_2, *then the following boundary value problem has a unique solution* $w(x) \in$

$C^2(\Omega_3) \cap C(\overline{\Omega_3})$:

$$\Delta w(x) + k_3^2 w(x) = h_2(x), \quad in\ \Omega_3, \tag{4.18}$$

$$\frac{\partial w(x)}{\partial \nu} + i\eta_2(x)w(x) = q_4(x), \quad on\ S_2. \tag{4.19}$$

Furthermore, there exists a constant $C > 0$ such that

$$\|w(x)\|_{\infty,\overline{\Omega_3}} \le C\big(\|h_2(x)\|_{L^2(\Omega_3)} + \|q_4(x)\|_{\infty,S_2}\big). \tag{4.20}$$

Proof The proof is analogous to part of the proof of Lemma 4.4 in [13] and, hence, is omitted. □

So we can obtain our first result as follows.

Theorem 1 *Assume that $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, let $S_1, \widetilde{S}_1$ be two penetrable interfaces and let $\Omega_2, \widetilde{\Omega}_2$ be two penetrable obstacles for the corresponding scattering problem. If the far-field patterns $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S})$ of the scattered fields for the same incident plane wave $u^i(x) = e^{ik_1 x\cdot d}$ coincide at a fixed frequency for every incident direction $d \in \mathbb{S}$ and the observation direction $\widehat{x} \in \mathbb{S}$, then $S_1 = \widetilde{S}_1$, $S_2 = \widetilde{S}_2$.*

Proof Using Lemmas 3, 4, 5, 6, and 7, and arguing analogously to part of the proof of Theorem 3.2 in [9], we can easily prove the uniqueness result. □

4.2 Unique determination of the refractive index $n(x)$ of the inhomogeneous penetrable obstacle Ω_2

In this subsection, we will prove a uniqueness theorem for reconstructing the refractive index $n(x)$ of the inhomogeneous penetrable obstacle Ω_2 from the far-field pattern $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S})$ for incident plane waves $u^i(x) = e^{ik_1 x\cdot d}$. To do this, we need the following two lemmas: one is the completeness result, the other is the orthogonality relation.

Lemma 8 *Assume that k_2^2 is not a Neumann eigenvalue of $\Delta v(x) + k_2^2 n(x)v(x) = 0$ in Ω_2, then the set $\{\frac{\partial v(x,d)}{\partial \nu} \mid d \in \mathbb{S}\}$ of normal derivatives of the fields $v(x, d)$ are complete in $L^2(S_1)$, where the fields $v(x, d)$ correspond to incident plane waves $u^i(x) = e^{ik_1 x\cdot d}$, $d \in \mathbb{S}$ is the incident direction.*

Proof Denote Ω by the complement of Ω_1, that is, $\Omega \triangleq \mathbb{R}^3 \backslash \overline{\Omega_1}$. Choose a large ball B_R centered at the origin such that $\overline{\Omega} \subset B_R$ and such that k_1^2 is not a Dirichlet eigenvalue of $\Delta u(x) + k_1^2 u(x) = 0$ in B_R, then, from Theorem 5.5 in [6], we know that the restriction to ∂B_R of the set of plane waves $u^i(x) = e^{ik_1 x\cdot d}$ are complete in $L^2(\partial B_R)$. Therefore, we just need to prove that the operator $II : L^2(\partial B_R) \to L^2(S_1)$ has a dense range $II\varphi(x) = \frac{\partial v(x)}{\partial \nu}$, where $v(x)$ solves the boundary value problem (1.1)-(1.6) and $\varphi(x)$ is the boundary data of the following interior Dirichlet problem:

$$\Delta u^i(x) + k_1^2 u^i(x) = 0 \quad \text{in } B_R, \qquad u^i(x) = \varphi(x) \quad \text{on } \partial B_R. \tag{4.21}$$

By using Green's formulas, we can see that the L^2-adjoint II' of II is given by

$$II'\phi(x) \triangleq \overline{\left\{ \frac{\partial \widehat{u}(x)}{\partial \nu} - \frac{\partial \widetilde{u}(x)}{\partial \nu} \right\}}\Big|_{\partial B_R}, \quad \phi(x) \in L^2(S_1), \tag{4.22}$$

where $(\widehat{u}(x), \widehat{v}(x))$ solves

$$\Delta\widehat{u}(x) + k_1^2\widehat{v}(x) = 0, \quad \text{in } \Omega_1, \tag{4.23}$$

$$\Delta\widehat{v}(x) + k_2^2 n(x)\widehat{v}(x) = 0, \quad \text{in } \Omega_2, \tag{4.24}$$

$$\widehat{u}(x) - \widehat{v}(x) = \phi(x), \qquad \frac{\partial\widehat{u}(x)}{\partial\nu} - \lambda_1\frac{\partial\widehat{v}(x)}{\partial\nu} = 0, \quad \text{on } S_1, \tag{4.25}$$

$$\lim_{r\to\infty} r\left(\frac{\partial\widehat{u}(x)}{\partial r} - ik_1\widehat{u}(x)\right) = 0, \tag{4.26}$$

and $\widetilde{u}(x)$ is a solution of the following interior Dirichlet problem:

$$\Delta\widetilde{u}(x) + k_1^2\widetilde{u}(x) = 0 \quad \text{in } B_R, \qquad \widetilde{u}(x) = \widehat{u}(x) \quad \text{on } \partial B_R. \tag{4.27}$$

We only have to prove that II' is injective. Let $II'\phi(x) = 0$, then we know that $\frac{\partial\widehat{u}(x)}{\partial\nu} = \frac{\partial\widetilde{u}(x)}{\partial\nu}$ and $\widehat{u}(x) = \widetilde{u}(x)$ on ∂B_R. Define

$$\widetilde{v}(x) = \begin{cases} \widehat{u}(x), & \text{in } \mathbb{R}^3\backslash\overline{B_R}, \\ \widetilde{u}(x), & \text{in } B_R. \end{cases} \tag{4.28}$$

Then, from [8], we know that $\widetilde{v}(x)$ is an entire solution to the Helmholtz equation $\Delta\widetilde{v}(x) + k_1^2\widetilde{v}(x) = 0$ in $\mathbb{R}^3$ satisfying the radiation condition (4.26), so it must vanish identically in $\mathbb{R}^3$. Therefore, $\widehat{u}(x) = 0$ in $\mathbb{R}^3\backslash\overline{B_R}$. By using the unique continuation principle, we can see that $\widehat{u}(x) = 0$ in Ω_1. From the transmission conditions (4.25), Holmgren's uniqueness theorem and by noting the assumption that k_2^2 is not a Neumann eigenvalue of $\Delta v(x) + k_2^2 n(x)v(x) = 0$ in Ω_2, we can see that $\widehat{v}(x) = 0$ in Ω_2 and, hence, $\frac{\partial\widehat{v}(x)}{\partial\nu} = \widehat{v}(x) = 0$ on S_1. It finishes the proof of the lemma. □

Lemma 9 *Assume that the far-field patterns $u^\infty(\widehat{x}, d)$ $(\widehat{x}, d \in \mathbb{S})$ for the refractive indices $n(x)$ and $\widetilde{n}(x)$ coincide, then, for any solution $v(x) \in C^2(\overline{\Omega_2}) \cap C(\Omega_2)$ of $\Delta v(x) + k_2^2 n(x)v(x) = 0$ in Ω_2 and any solution $\widetilde{v}(x) \in C^2(\overline{\Omega_2}) \cap C(\Omega_2)$ of $\Delta\widetilde{v}(x) + k_2^2\widetilde{n}(x)\widetilde{v}(x) = 0$ in Ω_2, we have the following orthogonality relation*:

$$\int_{\Omega_2} \big[n(x) - \widetilde{n}(x)\big]v(x)\widetilde{v}(x)\,dx = 0. \tag{4.29}$$

Proof First, we prove (4.29) for the special case $v(x) = v(x,d), d \in \mathbb{S}$. By using Rellich's lemma and Holmgren's uniqueness theorem, we can see that $v(x,d) = \widetilde{v}(x,d)$ and $\frac{\partial v(x,d)}{\partial\nu} = \frac{\widetilde{v}(x,d)}{\partial\nu}$ on S_1. By using Green's first theorem and the equations satisfied by $v(x,d)$ and $\widetilde{v}(x,d)$, we can see that

$$\begin{aligned}
&\int_{\Omega_2} \big[n(x) - \widetilde{n}(x)\big]v(x,d)\widetilde{v}(x)\,dx \\
&\quad = -\frac{1}{k_2^2}\int_{\Omega_2} \big[\Delta v(x,d) + k_2^2\widetilde{n}(x)v(x,d)\big]\widetilde{v}(x)\,dx \\
&\quad = -\frac{1}{k_2^2}\int_{\Omega_2} \big\{\Delta\big[v(x,d) - \widetilde{v}(x,d)\big] + k_2^2\widetilde{n}(x)\big[v(x,d) - \widetilde{v}(x,d)\big]\big\}\widetilde{v}(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{k_2^2}\int_{\Omega_2}\big[v(x,d)-\widetilde{v}(x,d)\big]\big[\Delta\widetilde{v}(x)+k_2^2\widetilde{n}(x)\widetilde{v}(x)\big]\,dx \\
&= 0.
\end{aligned} \tag{4.30}
$$

Then we have to prove that any general case $v(x)$ can be approximated by functions $v(x,d)$ in $L^2(\Omega_2)$, where $d\in\mathbb{S}$. If not, that is, the set $\{v(x,d)|\, d\in\mathbb{S}\}$ is not dense in $L^2(\Omega_2)$ sense in $\{\acute{v}(x)\in C^2(\overline{\Omega_2})|\Delta\acute{v}(x)+k_2^2 n(x)\acute{v}(x)=0 \text{ in } \Omega_2\}$. Then, from the Hahn-Banach theorem, we see that there exists an $f_3(x)\in L^2(\Omega_2)$ such that

$$
\int_{\Omega_2} f_3(x)v(x,d)\,dx = 0, \tag{4.31}
$$

for $v(x,d)$ with all $d\in\mathbb{S}$, but for some $\acute{v}(x)\in\{\acute{v}(x)\in C^2(\overline{\Omega_2})|\Delta\acute{v}(x)+k_2^2n(x)\acute{v}(x)=0 \text{ in } \Omega_2\}$,

$$
\int_{\Omega_2} f_3(x)\acute{v}(x)\,dx \neq 0. \tag{4.32}
$$

Assume that $\grave{v}(x)\in H^2(\Omega_2)$ is a solution to the following interior Neumann problem:

$$
\Delta\grave{v}(x)+k_2^2n(x)\grave{v}(x)=f_3(x) \quad \text{in } \Omega_2, \qquad \frac{\partial\grave{v}(x)}{\partial\nu}=0 \quad \text{on } S_1. \tag{4.33}
$$

From (4.31) and (4.33), and by using Green's first theorem, we can see that

$$
\begin{aligned}
0 &= \int_{\Omega_2} f_3(x)v(x,d)\,dx = \int_{\Omega_2}\big[\Delta\grave{v}(x)+k_2^2n(x)\grave{v}(x)\big]v(x,d)\,dx \\
&= \int_{S_1}\grave{v}(x)\frac{\partial v(x,d)}{\partial\nu}\,ds.
\end{aligned} \tag{4.34}
$$

From Lemma 8, we know that $\grave{v}(x)=0$ on S_1. Therefore, by using Green's first theorem, we can see that

$$
\begin{aligned}
\int_{\Omega_2} f_3(x)\acute{v}(x)\,dx &= \int_{\Omega_2}\big[\Delta\grave{v}(x)+k_2^2n(x)\grave{v}(x)\big]\acute{v}(x)\,dx \\
&= \int_{\Omega_2}\big[\Delta\acute{v}(x)+k_2^2n(x)\acute{v}(x)\big]\grave{v}(x)\,dx = 0,
\end{aligned} \tag{4.35}
$$

which contradicts (4.32). It finishes the proof of the lemma. □

So we can obtain our second result as follows.

Theorem 2 *Assume that the penetrable interfaces S_i $(i=1,2)$ are known and k_2^2 is not a Neumann eigenvalue of $\Delta v(x)+k_2^2n(x)v(x)=0$ in Ω_2, then the refractive index $n(x)$ of the inhomogeneous penetrable obstacle Ω_2 can be uniquely determined from the far-field pattern $u^\infty(\widehat{x},d)$ $(\widehat{x},d\in\mathbb{S})$ for incident plane waves $u^i(x)=e^{ik_1x\cdot d}$.*

Proof By using the completeness result of Lemma 8 and the orthogonality relation of Lemma 9, we can easily prove the uniqueness result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

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