Li et al. *Boundary Value Problems* (2017) 2017:30
DOI 10.1186/s13661-017-0761-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

New periodic solutions with a prescribed energy for a class of Hamiltonian systems

Fengying Li[1], Ying Lv[2*] and Shiqing Zhang[3]

*Correspondence:
ly0904@swu.edu.cn
[2]Department of Mathematics and Statistics, Southwest University, Chongqing, 400715, China
Full list of author information is available at the end of the article

Abstract
We consider a class of second order Hamiltonian systems with a C^2 potential function. The existence of new periodic solutions with a prescribed energy is established by the use of constrained variational methods.

MSC: 34C15; 34C25; 58F

Keywords: second order Hamiltonian systems; C^2 periodic solutions; constrained variational minimizing methods

1 Introduction

In this paper, we examine the existence of periodic solutions for second order Hamiltonian systems

$$\ddot{q} + V'(q) = 0, \tag{1.1}$$

$$\frac{1}{2}|\dot{q}|^2 + V(q) = h, \tag{1.2}$$

with a fixed energy. The first major result in this direction we would like to highlight can be derived from the work of Benci [1], Gluck-Ziller [2], and Hayashi [3], which is based on the earlier work of Seifert [4] in 1948 and following the highly influential papers of Rabinowitz [5, 6] in 1978 and 1979. Utilizing the Jacobi metric and a very involved interplay between geodesic methods and algebraic topology, the following general theorem is established.

Theorem 1.1 *Suppose $V \in C^1(\mathbb{R}^n, \mathbb{R})$. If the potential well*

$$\{x \in \mathbb{R}^n : V(x) \le h\}$$

is bounded and non-empty, then the system (1.1)-(1.2) *has a periodic solution with energy h. Furthermore, if*

$$V'(x) \neq 0, \quad \forall x \in \{x \in \mathbb{R}^n : V(x) = h\},$$

then the system (1.1)-(1.2) *has a non-constant periodic solution with energy h.*

For the existence of multiple periodic solutions for (1.1)-(1.2) with compact energy surfaces, we can refer the reader to Groessen [7] and Long [8] and the references therein.

For the weakly attractive potential V defined on an open subset Ω of $\mathbb{R}^n$, Ambrosetti and Coti Zelati [9] (Theorem 16.7) proved the following.

Theorem 1.2 *Suppose $V \in C^2(\Omega, \mathbb{R})$ satisfies*

($V10$) $3\langle V'(x), x\rangle + \langle V''(x)x, x\rangle \neq 0$, $\forall x \in \Omega$;
($V11$) $\langle V'(x), x\rangle > 0$, $\forall x \in \Omega$;
($V12$) $\exists \alpha \in (0,2)$, *such that* $\langle V'(x), x\rangle \geq -\alpha V(x)$, $\forall x \in \Omega$;
($V13$) $\exists \beta \in (0,2)$ *and* $r > 0$ *such that* $\langle V'(x), x\rangle \leq -\beta V(x)$, $\forall 0 < |x| < r$;
($V14$) $G_\infty \geq 0$; *where* $G_\infty = \lim_{|x|\to\infty} \inf G(x)$, $G(x) = [V(x) + \frac{1}{2}\langle V'(x), x\rangle]$.

Then $\forall h < 0$, the system (1.1)-(1.2) (*referred to as* (P_h)) *has at least one non-constant weak periodic solution with the given energy h.*

Using a simpler constrained variational minimizing method, we obtain the following result.

Theorem 1.3 *Suppose $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and $h \in \mathbb{R}$ satisfy*

(V_1) $V(-q) = V(q)$;
(V_2) $\langle V'(q), q\rangle > 0$, $\forall q \neq 0$;
(V_3) $3\langle V'(q), q\rangle + \langle V''(q)q, q\rangle > 0$, $\forall q \neq 0$;
(V_4) $\exists \mu_1 > 0$, $\mu_2 \geq 0$, *such that* $\langle V'(q), q\rangle \geq \mu_1 V(q) - \mu_2$;
(V_5) $\lim_{|q|\to\infty} \sup[V(q) + \frac{1}{2}\langle V'(q), q\rangle] \leq A$;
(V_6) $\frac{\mu_2}{\mu_1} < h < A$.

Then the system (1.1)-(1.2) *has at least one non-constant periodic solution with the given energy h.*

Remark 1.4 Comparing Theorem 16.7 of Ambrosetti and Coti Zelati [9] with our Theorem 1.3, we notice that our condition (V_2) corresponds to their ($V11$), our condition (V_3) corresponds to their ($V10$), our condition (V_4) corresponds to their ($V12$) and ($V13$), our conditions (V_5) and (V_6) correspond to their ($V14$). Since the potential in Theorem 16.7 of Ambrosetti and Coti Zelati has a singularity, but the potential in Theorem 1.3 has no singularity, the two theorems are essentially different.

Remark 1.5 Take for $V(x)$ the following C^∞ function:

$$V(x) = e^{\frac{-1}{|x|}}, \quad \forall x \neq 0;$$
$$V(0) = 0.$$

Then $V(x)$ satisfies (V_1)-(V_5) in Theorem 1.3 if we take $\mu_1 = \mu_2 > 0$ and $A = 1$, but (V_6) does not hold.

Proof of Theorem 1.3 We verify (V_1)-(V_5) by calculation:
(1) It is obvious for (V_1).

(2) For (V_2) and (V_3), we notice that

$$\begin{aligned}
&\langle V'(x), x\rangle = \frac{1}{|x|} e^{\frac{-1}{|x|}} > 0, \quad \forall x \neq 0, \\
&\langle V''(x)x, x\rangle = e^{\frac{-1}{|x|}} \left(\frac{-2}{|x|} + \frac{1}{|x|^2}\right), \\
&3\langle V'(x), x\rangle + \langle V''(x)x, x\rangle = e^{\frac{-1}{|x|}} \left(\frac{1}{|x|} + \frac{1}{|x|^2}\right) > 0, \quad \forall x \neq 0.
\end{aligned}$$

(3) For (V_4), we set

$$w(x) = \left(\frac{1}{|x|} - \mu_1\right) e^{\frac{-1}{|x|}}; \quad x \neq 0, w(0) = 0.$$

We will prove $w(x) > -\mu_1$; in fact,

$$w'(x) = \left[\frac{1}{|x|} - (1 + \mu_1)\right] \frac{x}{|x|^3} e^{\frac{-1}{|x|}}; \quad x \neq 0, w'(0) = 0.$$

From $w'(x) = 0$, we have $x = -\frac{1}{1+\mu_1}$ or 0 or $\frac{1}{1+\mu_1}$.

It is easy to see that $w(x)$ is strictly increasing on $(-\infty, -\frac{1}{1+\mu_1}]$ and $[0, \frac{1}{1+\mu_1}]$, but strictly decreasing on $[\frac{-1}{1+\mu_1}, 0]$ and $[\frac{1}{1+\mu_1}, +\infty)$. We notice that

$$\lim_{|x|\to+\infty} w(x) = -\mu_1$$

and

$$w(0) = 0.$$

So

$$w(x) > -\mu_1.$$

When we take $\mu_2 = \mu_1 > 0$, (V_4) holds.

(4) For (V_5), we have

$$\begin{aligned}
&V(x) + \frac{1}{2}\langle V'(x), x\rangle = e^{\frac{-1}{|x|}} \left(1 + \frac{1}{2}\frac{1}{|x|}\right) < 1, \quad \forall x \neq 0; \\
&V(0) + \frac{1}{2}\langle V'(0), 0\rangle = 0.
\end{aligned}$$

□

Corollary 1.6 *Given $a > 0$, $n \in \mathbb{N}$, define $V(x) = a|x|^{2n} + e^{\frac{-1}{|x|}}$, $x \neq 0$; $V(0) = 0$. Then, for $h > 1$, the system* (1.1)-(1.2) *has at least one non-constant periodic solution with the given energy h.*

Remark 1.7 The potential $V(x) = e^{\frac{-1}{|x|}}$, $\forall x \neq 0$; $V(0) = 0$ in Remark 1.5 is noteworthy since the potential function is non-convex and bounded which satisfies neither of the conditions of Theorems 1.1, Offin's geometrical conditions [10], nor Berg-Pasquotto-Vandervorst's complex topological assumptions [11]. For this potential, the potential well

$\{x \in \mathbb{R}^n : V(x) \leq h\}$ is a bounded set if $h < 1$, but for $h \geq 1$ it is $\mathbb{R}^n$ - an unbounded set. We also notice that the symmetrical condition on the potential simplified our Theorem 1.2 and its proof. It would be interesting to obtain non-constant periodic solutions when the symmetrical condition is deleted.

2 A few lemmas

Let

$$H^1 = W^{1,2}(\mathbb{R}_{\text{per}}, \mathbb{R}^n) = \{u : \mathbb{R} \to \mathbb{R}^n, u(t+1) = u(t), u \in L^2[0,1], \dot{u} \in L^2[0,1]\}$$

denotes the periodic functional space of period 1. Then the standard H^1 norm is

$$\|u\| = \|u\|_{H^1} = \left(\int_0^1 |\dot{u}|^2\,dt\right)^{1/2} + \left(\int_0^1 |u|^2\,dt\right)^{1/2}.$$

Lemma 2.1 ([12]) *For $u \in H^1$, define*

$$g(u) = \int_0^1 \left[V(u) + \frac{1}{2}\langle V'(u), u\rangle\right] dt,$$
$$M = \{u \in H^1 : g(u) = h\}.$$

For $u, v \in H^1$ and $s \in \mathbb{R}$, let

$$\phi(s) = g(u + sv).$$

Then

$$\phi'(0) = \langle g'(u), v\rangle = \frac{1}{2}\int_0^1 \{3\langle V'(u), v\rangle + \langle V''(u)v, u\rangle\}\,dt$$

and

$$\langle g'(u), u\rangle = \frac{1}{2}\int_0^1 \{3\langle V'(u), u\rangle + \langle V''(u)u, u\rangle\}\,dt;$$

therefore, if (V_3) holds, then on M, $g'(u) \neq 0$, which implies M is a C^1 manifold with codimension 1 in H^1.

Let

$$f(u) = \frac{1}{4}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \langle V'(u), u\rangle\,dt \tag{2.1}$$

and $\widetilde{u} \in M$ such that $f'(\widetilde{u}) = 0$ and $f(\widetilde{u}) > 0$. Set

$$\frac{1}{T^2} = \frac{\int_0^1 \langle V'(\widetilde{u}), \widetilde{u}\rangle\,dt}{\int_0^1 |\dot{\widetilde{u}}|^2\,dt}.$$

If (V_2) holds, then $\widetilde{q}(t) = \widetilde{u}(t/T)$ is a non-constant T-periodic solution for (1.1)-(1.2).

When the potential is even, then by Palais' symmetrical principle [13] and Lemma 2.1 we have the following.

Lemma 2.2 ([12]) *Let*

$$F = \left\{ u \in M : u\left(t + \frac{1}{2}\right) = -u(t) \right\} \tag{2.2}$$

and suppose (V_1)-(V_3) *hold. If* $\widetilde{u} \in F$ *is such that* $f'(\widetilde{u}) = 0$ *and* $f(\widetilde{u}) > 0$, *then* $\widetilde{q}(t) = \widetilde{u}(\frac{t}{T})$ *is a non-constant* T*-periodic solution for* (1.1)-(1.2); *in addition, we have*

$$\forall u \in F, \quad \int_0^1 u(t)\,dt = 0.$$

Wirtinger's inequality [14] *implies*

$$\int_0^1 |\dot{u}|^2\,dt \geq (2\pi)^2 \int_0^1 |u|^2,$$

from which it follows that $(\int_0^1 |\dot{u}|^2\,dt)^{1/2}$ *is an equivalent norm for the space* H^1.

Lemma 2.3 *Let* X *be a Banach space and* $F \subset X$ *a weakly closed subset. Suppose* Φ *defined on* F *is Gateaux-differentiable, weakly lower semi-continuous and bounded from below on* F. *Suppose further that* Φ *satisfies the following* $(WPS)_{\inf \Phi, F}$ *condition*:

- *If* $\{x_n\} \subset F$ *such that* $\Phi(x_n) \to c$ *and* $\|\Phi'(x_n)\| \to 0$, *then* $\{x_n\}$ *has a weakly convergent subsequence.*

Then Φ *attains its infimum on* F.

Proof By Ekeland's variational principle [15, 16], we know that there is a sequence $\{x_n\} \subset F$ satisfying

$$\Phi(x_n) \to \inf \Phi \quad \text{and} \quad \left\|\Phi'(x_n)\right\| \to 0.$$

Since Φ satisfies the $(WPS)_{\inf \Phi, F}$ condition, $\{x_n\}$ has a weakly convergent subsequence which as a weak limit x. Because $F \subset X$ is a weakly closed subset, we have $x \in F$. Finally, by the weakly lower semi-continuous assumption on Φ, we conclude that Φ attains its infimum on F. □

3 The proof of Theorem 1.3

We prove Theorem 1.3 by the following sequence of lemmas. In the following, f and F are defined as in (2.1) and (2.2), respectively.

Lemma 3.1 *If* (V_1)-(V_6) *hold, then, for any given* $c > 0$, f *satisfies the* $(PS)_{c,F}$ *condition*; *that is, if* $\{u_n\} \subset F$ *satisfies*

$$f(u_n) \to c > 0 \quad \textit{and} \quad f|'_F(u_n) \to 0, \tag{3.1}$$

then $\{u_n\}$ *has a strongly convergent subsequence.*

Proof We first prove that under our assumptions the constrained set $F \neq \emptyset$. For any given $u \in H^1$ satisfying $u(t) \neq 0$, $\forall t \in [0,1]$ and for $a > 0$, let

$$g_u(a) = g(au) = \int_0^1 \left[V(au) + \frac{1}{2}\langle V'(au), au\rangle \right] dt. \tag{3.2}$$

By the assumption (V_3), we have

$$\frac{d}{da} g_u(a) > 0 \tag{3.3}$$

and so g_u is strictly increasing. Since $V \in C^2$, we know that, for any given $a > 0$,

$$\left[V\bigl(au(t)\bigr) + \frac{1}{2}\bigl\langle V'\bigl(au(t)\bigr), au(t)\bigr\rangle \right]$$

is uniformly continuous on $[0,1]$.

Hence by (V_5), we have

$$\lim_{a\to+\infty} g_u(a) \leq \int_0^1 \lim_{a\to+\infty} \sup \left[V(au) + \frac{1}{2}\langle V'(au), au\rangle \right] dt \leq A. \tag{3.4}$$

By (V_4), we notice that

$$g_u(0) = V(0) \leq \frac{\mu_2}{\mu_1}. \tag{3.5}$$

Since $\frac{\mu_2}{\mu_1} < h < A$, we see that the equation $g_u(a) = h$ has a unique solution $a(u)$ with $a(u)u \in M$.

By $f(u_n) \to c$, we have

$$\frac{1}{4}\int_0^1 \left|\dot{u}_n(t)\right|^2 dt \cdot \int_0^1 \bigl\langle V'(u_n), u_n\bigr\rangle dt \to c, \tag{3.6}$$

and by (V_4) we see that

$$h = \int_0^1 \left[V(u_n) + \frac{1}{2}\bigl\langle V'(u_n), u_n\bigr\rangle \right] dt \leq \left(\frac{1}{\mu_1} + \frac{1}{2} \right) \int_0^1 \bigl\langle V'(u_n), u_n\bigr\rangle dt + \frac{\mu_2}{\mu_1}. \tag{3.7}$$

By (3.6) and (3.7), we have

$$\int_0^1 \bigl\langle V'(u_n), u_n\bigr\rangle dt \geq \frac{h - \frac{\mu_2}{\mu_1}}{\frac{1}{2} + \frac{1}{\mu_1}}. \tag{3.8}$$

Condition (V_6) provides $h > \frac{\mu_2}{\mu_1}$. Then (3.6) and (3.8) imply $\int_0^1 |\dot{u}_n(t)|^2\, dt$ is bounded and $\|u_n\| = \|\dot{u}_n\|_{L^2}$ is bounded.

We know that H^1 is a reflexive Banach space, so $\{u_n\}$ has a weakly convergent subsequence; furthermore, by the embedding theorem the weakly convergent subsequence also uniformly converges to some $u \in H^1$. The standard argument can show that $\{u_n\}$ has a subsequence which converges under the H^1 norm. We omit the details of this standard demonstration. □

Lemma 3.2 *$f(u)$ is weakly lower semi-continuous on F.*

Proof For any $u_n \subset F$ with $u_n \rightharpoonup u$, by Sobolev's embedding theorem we have the uniform convergence

$$\left|u_n(t) - u(t)\right|_\infty \to 0.$$

Since $V \in C^1(\mathbb{R}^n, \mathbb{R})$, we have

$$\left|V\big(u_n(t)\big) - V\big(u(t)\big)\right|_\infty \to 0.$$

By the weakly lower semi-continuity of the norm, we see that

$$\liminf \left[\int_0^1 |\dot{u}_n|^2\,dt\right]^{\frac{1}{2}} \geq \left(\int_0^1 |\dot{u}|^2\,dt\right)^{\frac{1}{2}},$$

and so

$$\liminf \left(\int_0^1 |\dot{u}_n|^2\,dt\right) \geq \int_0^1 |\dot{u}|^2\,dt.$$

Then

$$\begin{aligned}\liminf f(u_n) &= \liminf \frac{1}{4}\int_0^1 |\dot{u}_n|^2\,dt \int_0^1 \langle V'(u_n), u_n\rangle\,dt\\ &\geq \frac{1}{4}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \langle V'(u), u\rangle\,dt = f(u).\end{aligned}$$

□

Lemma 3.3 *F is a weakly closed subset in H^1.*

Proof This follows easily from Sobolev's embedding theorem and $V \in C^1(\mathbb{R}^n, \mathbb{R})$. □

Lemma 3.4 *The functional $f(u)$ has a positive lower bound on F.*

Proof By the definitions of $f(u)$, F, and the assumption (V_2), we have

$$f(u) = \frac{1}{4}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \langle V'(u), u\rangle\,dt \geq 0, \quad \forall u \in F.$$

We claim further that

$$\inf f(u) > 0;$$

otherwise, (V_2) implies $u(t) = const$, and by the symmetrical property $u(t + 1/2) = -u(t)$ we have $u(t) = 0$, $\forall t \in \mathbb{R}$. But assumptions (V_4) and (V_6) imply

$$V(0) \leq \frac{\mu_2}{\mu_1} < h,$$

which contradicts the definition of F since $V(0) = h$ if we have $0 \in F$. Now by Lemmas 3.1-3.4 and Lemma 2.3, we see that $f(u)$ attains the infimum on F and we know that the minimizer is non-constant. □

Competing interests
The authors declare that no competing interests exist.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

Author details
[1]School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, 61130, China. [2]Department of Mathematics and Statistics, Southwest University, Chongqing, 400715, China. [3]Department of Mathematics, Sichuan University, Chengdu, 610064, China.

Acknowledgements
The authors sincerely thank the editor and the referees for their many valuable comments and suggestions. Shiqing Zhang and Fengying Li were partially supported by NSFC (11671278). Ying Lv was partially supported by NSFC (11601438).

Received: 30 November 2016 Accepted: 22 February 2017 Published online: 09 March 2017

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