Jeong et al. *Boundary Value Problems* (2017) 2017:42
DOI 10.1186/s13661-017-0773-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global nonexistence of solutions for a quasilinear wave equation with acoustic boundary conditions

Jin-Mun Jeong[1], Jong-Yeoul Park[2] and Yong Han Kang[3*]

*Correspondence:
yonghann@cu.ac.kr
[3]Institute of Liberal Education, Catholic University of Daegu, Gyeongsan, 712-702, South Korea
Full list of author information is available at the end of the article

Abstract

We consider the quasilinear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) - \operatorname{div}(|\nabla u_t|^{\beta-2}\nabla u_t) + a|u_t|^{m-2}u_t = b|u|^{p-2}u$$

$a, b > 0$, associated with initial and Dirichlet boundary conditions at one part and acoustic boundary conditions at another part, respectively. We prove, under suitable conditions on α, β, m, p and for negative initial energy, a global nonexistence of solutions.

MSC: 35B40; 35B44; 35L72

Keywords: quasilinear wave equation; blow-up; acoustic boundary

1 Introduction

In this paper, we consider the following quasilinear wave equation with acoustic boundary conditions:

$$\begin{aligned} &u_{tt} - \Delta u_t - \operatorname{div}\big(|\nabla u|^{\alpha-2}\nabla u\big) - \operatorname{div}\big(|\nabla u_t|^{\beta-2}\nabla u_t\big) \\ &\quad + a|u_t|^{m-2}u_t = b|u|^{p-2}u \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$\frac{\partial u_t}{\partial \nu} + |\nabla u|^{\alpha-2}\frac{\partial u}{\partial \nu} + |\nabla u_t|^{\beta-2}\frac{\partial u_t}{\partial \nu} = h(x)y_t \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.3}$$

$$u_t + f(x)y_t + q(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.5}$$

$$y(x,0) = y_0(x) \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.6}$$

where $a, b > 0, \alpha, \beta, m, p > 2$, Ω is a regular and bounded domain of $R^n (n \geq 1)$ and $\partial\Omega(= \Gamma) := \Gamma_0 \cup \Gamma_1$. Here Γ_0, Γ_1 are closed and disjoint, and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative. The functions $f, q, h : \Gamma_1 \longrightarrow R^+$ are essentially bounded and $0 < q_0 \leq q(x)$ on Γ_1.

The system (1.1)-(1.6) is a model of a quasilinear wave equation with acoustic boundary conditions. The acoustic boundary conditions were introduced by Morse and Ingard [1] in 1968 and developed by Beale and Rosencrans in [2], where the authors proved the global existence and regularity of the linear problem. Furthermore, Boukhatem and Benabderrahmane [3, 4] studied the existence, blow-up and decay of solutions for viscoelastic wave equations with acoustic boundary conditions. Graber and Said-Houari [5] studied the blow-up solutions for the wave equation with semilinear porous acoustic boundary conditions. Moreover, Wu [6] also considered blow-up solutions for a nonlinear wave equation with porous acoustic boundary conditions. The global nonexistence of solutions for a class of wave equations with nonlinear damping and source terms was proved by Messaoudi and Said-Houari [7–9] (see [10–13] for more details). Recently, Piskin [14] investigated the energy decay and blow-up of solutions for quasilinear hyperbolic equations with nonlinear damping and source terms (see [15–18] for more details).

Motivated by the previous works, in this paper, we study the global nonexistence of solutions for quasilinear wave equations with acoustic boundary conditions. To the best of our knowledge, there are no results of a quasilinear wave equation with acoustic boundary conditions. This work is meaningful. The outline of the paper is the following. In Section 2, we prove the main result.

2 Blow-up results

In order to state and prove our result, we introduce

$$\begin{aligned} Z &= L^\infty\bigl([0,T);W^{1,\alpha}(\Omega)\bigr)\cap W^{1,\infty}\bigl([0,T);L^2(\Omega)\bigr)\\ &\quad \cap W^{1,\beta}\bigl([0,T);W^{1,\beta}(\Omega)\bigr)\cap W^{1,m}\bigl([0,T);L^m(\Omega)\bigr) \end{aligned}$$

for $T > 0$ and the energy functional

$$E(t) = \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{\alpha}\int_\Omega |\nabla u|^\alpha\,dx - \frac{b}{p}\int_\Omega |u|^p\,dx + \frac{1}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \tag{2.1}$$

Theorem 2.1 *Assume that $\alpha,\beta,m,p\ge 2$ such that $\beta<\alpha$, and $max\{m,\alpha\}<p<r_\alpha$, where r_α is the Sobolev critical exponent of $W^{1,\alpha}(\Omega)$. Assume further that*

$$E(0) < 0. \tag{2.2}$$

Then the solution $(u,y)\in Z\times L^2(R^+;L^2(\Gamma_1))$ of (1.1)-(1.6) can not exist for all time.

Remark 2.2 If the solution u of (1.1)-(1.6) is smooth enough, then it blows up in finite time.

Proof We suppose that the solution exists for all time, and we reach a contradiction. For this purpose, we multiply Eq. (1.1) by u_t and, using (1.2)-(1.4), we obtain

$$\begin{aligned} E'(t) &= -\int_\Omega \bigl|\nabla u_t(t)\bigr|^2\,dx - \int_\Omega \bigl|\nabla u_t(t)\bigr|^\beta\,dx\\ &\quad - a\int_\Omega \bigl|u_t(t)\bigr|^m\,dx - \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma \le 0 \end{aligned} \tag{2.3}$$

for any regular solution. Hence we get $E(t)\le E(0)$ $\forall t\ge 0$.

By setting $H(t) = -E(t)$, we deduce

$$0 < H(0) \leq H(t) \leq \frac{b}{p}\int_{\Omega}\left|u(t)\right|^{p}dx, \quad \forall \geq 0. \tag{2.4}$$

Now, we define

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_{\Omega}u(t)u_t(t)\,dx - \frac{\varepsilon}{2}\int_{\Gamma_1}h(x)f(x)y^2(t)\,d\Gamma - \varepsilon\int_{\Gamma_1}h(x)u(t)y(t)\,d\Gamma \tag{2.5}$$

for ε small to be chosen later and

$$0 < \sigma \leq \min\left\{\frac{\alpha-2}{p}, \frac{\alpha-\beta}{p(\beta-1)}, \frac{p-m}{p(m-1)}, \frac{\alpha-2}{2\alpha}\right\}. \tag{2.6}$$

Our goal is to show that $L(t)$ satisfies a differential inequality of the form

$$L'(t) \geq \xi L^{q}(t), \quad q > 1. \tag{2.7}$$

This, of course, will lead to a blow-up in finite time.

By taking a derivative of (2.5), we get

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_{\Omega}u_t^2(t)\,dx + \varepsilon\int_{\Omega}u(t)u_{tt}(t)\,dx \\ &\quad - \varepsilon\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma - \varepsilon\int_{\Gamma_1}h(x)u_t(t)y(t)\,d\Gamma \\ &\quad - \varepsilon\int_{\Gamma_1}h(x)u(t)y_t(t)\,d\Gamma. \end{aligned} \tag{2.8}$$

By using Eqs. (1.1)-(1.4), estimate (2.8) becomes

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_{\Omega}u_t^2(t)\,dx \\ &\quad + \varepsilon\int_{\Omega}u(t)\big[\Delta u_t(t) + \operatorname{div}\big(\left|\nabla u(t)\right|^{\alpha-2}\nabla u(t)\big) + \operatorname{div}\big(\left|\nabla u_t(t)\right|^{\beta-2}\nabla u_t(t)\big) \\ &\quad - a\left|u_t(t)\right|^{m-2}u_t(t) + b\left|u(t)\right|^{p-2}u(t)\big]\,dx - \varepsilon\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma \\ &\quad - \varepsilon\int_{\Gamma_1}h(x)u_t(t)y(t)\,d\Gamma - \varepsilon\int_{\Gamma_1}h(x)u(t)y_t(t)\,d\Gamma \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_{\Omega}u_t^2(t)\,dx - \varepsilon\int_{\Omega}\nabla u_t(t)\nabla u(t)\,dx \\ &\quad - \varepsilon\int_{\Omega}\left|\nabla u(t)\right|^{\alpha}dx - \varepsilon\int_{\Omega}\big(\left|\nabla u_t(t)\right|^{\beta-2}\nabla u_t(t)\big)\nabla u(t)\,dx \\ &\quad - a\varepsilon\int_{\Omega}\left|u_t(t)\right|^{m-2}u_t(t)u(t)\,dx + b\varepsilon\int_{\Omega}\left|u(t)\right|^{p}dx \\ &\quad + \varepsilon\int_{\Gamma_1}\left(\frac{\partial u_t(t)}{\partial\nu} + \left|\nabla u(t)\right|^{\alpha-2}\frac{\partial u(t)}{\partial\nu} + \left|\nabla u_t(t)\right|^{\beta-2}\frac{\partial u_t(t)}{\partial\nu}\right)u(t)\,d\Gamma \\ &\quad - \varepsilon\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma - \varepsilon\int_{\Gamma_1}h(x)u_t(t)y(t)\,d\Gamma - \varepsilon\int_{\Gamma_1}h(x)u(t)y_t(t)\,d\Gamma \end{aligned}$$

$$\begin{aligned}
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2(t)\,dx - \varepsilon\int_\Omega \nabla u_t(t)\nabla u(t)\,dx \\
&\quad - \varepsilon\int_\Omega \left|\nabla u(t)\right|^\alpha dx - \varepsilon\int_\Omega \left(\left|\nabla u_t(t)\right|^{\beta-2}\nabla u_t(t)\right)\nabla u(t)\,dx \\
&\quad - a\varepsilon\int_\Omega \left|u_t(t)\right|^{m-2}u_t(t)u(t)\,dx + b\varepsilon\int_\Omega \left|u(t)\right|^p dx + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma.
\end{aligned} \tag{2.9}$$

Exploiting Hölder's and Young's inequalities, for any $\eta, \mu, \delta > 0$, we obtain

$$\int_\Omega \left|u_t(t)\right|^{m-2}u_t(t)u(t)\,dx \le \frac{\eta^m}{m}\int_\Omega \left|u(t)\right|^m dx + \frac{m-1}{m}\eta^{-\frac{m}{m-1}}\int_\Omega \left|u_t(t)\right|^m dx, \tag{2.10}$$

$$\int_\Omega \nabla u_t(t)\nabla u(t)\,dx \le \frac{1}{4\mu}\int_\Omega \left|\nabla u(t)\right|^2 dx + \mu\int_\Omega \left|\nabla u_t(t)\right|^2 dx, \tag{2.11}$$

$$\int_\Omega \left|\nabla u_t(t)\right|^{\beta-2}\nabla u_t(t)\nabla u(t)\,dx \le \frac{\delta^\beta}{\beta}\int_\Omega \left|\nabla u(t)\right|^\beta dx + \frac{\beta-1}{\beta}\delta^{-\frac{\beta}{\beta-1}}\int_\Omega \left|\nabla u_t(t)\right|^\beta dx. \tag{2.12}$$

A substitution of (2.10)-(2.12) in (2.9) yields

$$\begin{aligned}
L'(t) &\ge (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2(t)\,dx - \frac{\varepsilon}{4\mu}\int_\Omega \left|\nabla u(t)\right|^2 dx \\
&\quad - \varepsilon\mu\int_\Omega \left|\nabla u_t(t)\right|^2 dx - \varepsilon\int_\Omega \left|\nabla u(t)\right|^\alpha dx - \frac{\varepsilon\delta^\beta}{\beta}\int_\Omega \left|\nabla u(t)\right|^\beta dx \\
&\quad - \frac{\varepsilon(\beta-1)}{\beta}\delta^{-\frac{\beta}{\beta-1}}\int_\Omega \left|\nabla u_t(t)\right|^\beta dx - \frac{a\varepsilon\eta^m}{m}\int_\Omega \left|u(t)\right|^m dx \\
&\quad - \frac{a\varepsilon(m-1)}{m}\eta^{-\frac{m}{m-1}}\int_\Omega \left|u_t(t)\right|^m dx + b\varepsilon\int_\Omega \left|u(t)\right|^p dx \\
&\quad + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma.
\end{aligned} \tag{2.13}$$

Therefore, by choosing η, μ, δ so that

$$\begin{aligned}
&\eta^{-\frac{m}{m-1}} = M_1 H^{-\sigma}(t), \\
&\mu = M_2 H^{-\sigma}(t), \\
&\delta^{-\frac{\beta}{\beta-1}} = M_3 H^{-\sigma}(t)
\end{aligned}$$

for M_1, M_2, M_3 to be specified later, and using (2.13), we arrive at

$$\begin{aligned}
L'(t) &\ge (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t^2(t)\,dx - \frac{\varepsilon}{4M_2}H^\sigma(t)\int_\Omega \left|\nabla u(t)\right|^2 dx \\
&\quad - \varepsilon\int_\Omega \left|\nabla u(t)\right|^\alpha dx - \frac{\varepsilon M_3^{-(\beta-1)}}{\beta}H^{\sigma(\beta-1)}(t)\int_\Omega \left|\nabla u(t)\right|^\beta dx \\
&\quad - \frac{a\varepsilon}{m}M_1^{-(m-1)}H^{\sigma(m-1)}(t)\int_\Omega \left|u(t)\right|^m dx + b\varepsilon\int_\Omega \left|u(t)\right|^p dx \\
&\quad - \varepsilon\left[M_2\int_\Omega \left|\nabla u_t(t)\right|^2 dx + \frac{\beta-1}{\beta}M_3\int_\Omega \left|\nabla u_t(t)\right|^\beta dx\right. \\
&\quad \left. + \frac{a(m-1)}{m}M_1\int_\Omega \left|u_t(t)\right|^m dx\right]H^{-\sigma}(t) + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma.
\end{aligned} \tag{2.14}$$

If $M = M_2 + \frac{(\beta-1)M_3}{\beta} + \frac{(m-1)M_1}{m}$, then (2.14) takes the form

$$\begin{aligned} L'(t) \geq{}& (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega u_t^2(t)\,dx - \frac{\varepsilon}{4M_2}H^\sigma(t)\int_\Omega \left|\nabla u(t)\right|^2 dx \\ &- \varepsilon \int_\Omega \left|\nabla u(t)\right|^\alpha dx - \frac{\varepsilon M_3^{-(\beta-1)}}{\beta}H^{\sigma(\beta-1)}(t)\int_\Omega \left|\nabla u(t)\right|^\beta dx \\ &- \frac{a\varepsilon}{m}M_1^{-(m-1)}H^{\sigma(m-1)}(t)\int_\Omega \left|u(t)\right|^m dx + b\varepsilon \int_\Omega \left|u(t)\right|^p dx \\ &+ \varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \varepsilon \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \end{aligned} \tag{2.15}$$

Then we use the embedding $L^p(\Omega) \hookrightarrow L^m(\Omega)$ and (2.4) to get

$$H^{\sigma(m-1)}(t)\int_\Omega \left|u(t)\right|^m dx \leq \left(\frac{b}{p}\right)^{\sigma(m-1)} \left(\int_\Omega \left|u(t)\right|^p dx\right)^{\frac{m+\sigma p(m-1)}{p}}. \tag{2.16}$$

We also exploit the inequality

$$\int_\Omega \left|\nabla u(t)\right|^2 dx \leq c\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{2}{\alpha}},$$

the embedding $W^{1,\alpha}(\Omega) \hookrightarrow H^1(\Omega)$ and (2.4) to obtain

$$H^\sigma(t)\int_\Omega \left|\nabla u(t)\right|^2 dx \leq c\left(\frac{b}{p}\right)^\sigma \left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{p\sigma+2}{\alpha}}. \tag{2.17}$$

Since $\alpha > \beta$, we obtain

$$\int_\Omega \left|\nabla u(t)\right|^\beta dx \leq c\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{\beta}{\alpha}},$$

we derive

$$H^{\sigma(\beta-1)}(t)\int_\Omega \left|\nabla u(t)\right|^\beta dx \leq c\left(\frac{b}{p}\right)^{\sigma(\beta-1)} \left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{p\sigma(\beta-1)+\beta}{\alpha}}, \tag{2.18}$$

where c is a constant depending on Ω only. By using (2.6) and the inequality

$$z^\nu \leq z+1 \leq \left(1+\frac{1}{a}\right)(z+a), \quad \forall z \geq 0, 0 < \nu < 1, a \geq 0, \tag{2.19}$$

we get the following inequalities:

$$\begin{aligned} \left(\int_\Omega \left|u(t)\right|^p dx\right)^{\frac{m+\sigma p(m-1)}{p}} &\leq c\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{m+\sigma p(m-1)}{\alpha}} \\ &\leq d\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(0)\right) \\ &\leq d\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(t)\right), \quad \forall t \geq 0, \end{aligned} \tag{2.20}$$

$$\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{p\sigma+2}{\alpha}} \le d\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(t)\right), \quad \forall t \ge 0, \tag{2.21}$$

and

$$\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{p\sigma(\beta-1)+\beta}{\alpha}} \le d\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(t)\right), \quad \forall t \ge 0, \tag{2.22}$$

where $d = 1 + 1/H(0), a = H(0)$. Inserting (2.16)-(2.18) and (2.20)-(2.22) into (2.15), we deduce

$$\begin{aligned} L'(t) \ge{}& (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) \\ &+ kH(t) + \left(\varepsilon + \frac{k}{2}\right)\int_\Omega u_t^2(t)\,dx \\ &- \frac{\varepsilon c_2}{M_2}\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(t)\right) - \varepsilon\int_\Omega \left|\nabla u(t)\right|^\alpha dx \\ &- \frac{\varepsilon c_3}{M_3^{\beta-1}}\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(t)\right) + \frac{k}{\alpha}\int_\Omega \left|\nabla u(t)\right|^\alpha dx \\ &- \frac{\varepsilon c_1}{M_1^{m-1}}\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx + H(t)\right) + b\left(\varepsilon - \frac{k}{p}\right)\int_\Omega \left|u(t)\right|^p dx \\ &+ \varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \left(\varepsilon + \frac{k}{2}\right)\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma \end{aligned}$$

for some constant k and $c_1 = \frac{acd}{m}(\frac{b}{p})^{\sigma(m-1)}$, $c_2 = \frac{cd}{4}(\frac{b}{p})^\sigma$, $c_3 = \frac{cd}{\beta}(\frac{b}{p})^{\sigma(\beta-1)}$.

Using $k = \varepsilon p$, we arrive at

$$\begin{aligned} L'(t) \ge{}& (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{p+2}{2}\right)\int_\Omega u_t^2(t)\,dx \\ &+ \varepsilon\left(p - \frac{c_2}{M_2} - \frac{c_3}{M_3^{\beta-1}} - \frac{c_1}{M_1^{m-1}}\right)H(t) \\ &+ \varepsilon\left(\frac{p}{\alpha} - \frac{c_2}{M_2} - \frac{c_3}{M_3^{\beta-1}} - \frac{c_1}{M_1^{m-1}} - 1\right)\int_\Omega \left|\nabla u(t)\right|^\alpha dx \\ &+ \varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \varepsilon\left(\frac{p+2}{2}\right)\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma. \end{aligned}$$

At this point, by choosing M_1, M_2, M_3 large enough and using

$$\varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma > 0,$$

we have

$$\begin{aligned} L'(t) \ge{}& (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) \\ &+ r\varepsilon\left(H(t) + \int_\Omega u_t^2(t)\,dx + \int_\Omega \left|\nabla u(t)\right|^\alpha dx + \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right), \end{aligned} \tag{2.23}$$

where r is a positive constant (this is possible since $p > \alpha$).

We choose $0 < \varepsilon < \frac{1-\sigma}{M}$ so that

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1 \, dx - \frac{\varepsilon}{2} \int_{\Gamma_1} h(x) f(x) y_0^2 \, d\Gamma - \varepsilon \int_{\Gamma_1} h(x) u_0 y_0 \, d\Gamma > 0.$$

Then from (2.23) we get

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0,$$

and

$$L'(t) \geq r\varepsilon \left(H(t) + \int_\Omega u_t^2(t) \, dx + \int_\Omega \left| \nabla u(t) \right|^\alpha dx + \int_{\Gamma_1} h(x) q(x) y^2(t) \, d\Gamma \right). \tag{2.24}$$

On the other hand, from (2.5) and $f, h > 0$, we have

$$L(t) \leq H^{1-\sigma}(t) + \varepsilon \int_\Omega u(t) u_t(t) \, dx - \varepsilon \int_{\Gamma_1} h(x) u(t) y(t) \, d\Gamma.$$

Consequently, the above estimate leads to

$$L^{\frac{1}{1-\sigma}}(t) \leq C(\varepsilon, \sigma) \left[H(t) + \left(\int_\Omega u(t) u_t(t) \, dx \right)^{\frac{1}{1-\sigma}} + \left(\int_{\Gamma_1} h(x) u(t) y(t) \, d\Gamma \right)^{\frac{1}{1-\sigma}} \right]. \tag{2.25}$$

From Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega u(t) u_t(t) \, dx &\leq \left(\int_\Omega u_t^2(t) \, dx \right)^{\frac{1}{2}} \left(\int_\Omega u^2(t) \, dx \right)^{\frac{1}{2}} \\ &\leq c \left(\int_\Omega u_t^2(t) \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left| u(t) \right|^\alpha dx \right)^{\frac{1}{\alpha}}, \end{aligned}$$

where c is the positive constant which comes from the embedding $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$. This inequality implies that there exists a positive constant $c_4 > 0$ such that

$$\left(\int_\Omega u(t) u_t(t) \, dx \right)^{\frac{1}{1-\sigma}} \leq c_4 \left(\int_\Omega \left| u(t) \right|^\alpha dx \right)^{\frac{1}{(1-\sigma)\alpha}} \left(\int_\Omega u_t^2(t) \, dx \right)^{\frac{1}{2(1-\sigma)}}.$$

Applying Young's inequality to the right-hand side of the preceding inequality, we have a positive constant, also denoted by $c > 0$, such that

$$\left(\int_\Omega u(t) u_t(t) \, dx \right)^{\frac{1}{1-\sigma}} \leq c \left[\left(\int_\Omega \left| u(t) \right|^\alpha dx \right)^{\frac{\mu}{(1-\sigma)\alpha}} + \left(\int_\Omega u_t^2(t) \, dx \right)^{\frac{\theta}{2(1-\sigma)}} \right]$$

for $\frac{1}{\mu} + \frac{1}{\theta} = 1$. We take $\theta = 2(1-\sigma)$, hence $\mu = 2(1-\sigma)/(1-2\sigma)$, to get

$$\left(\int_\Omega u(t) u_t(t) \, dx \right)^{\frac{1}{1-\sigma}} \leq c \left[\left(\int_\Omega \left| u(t) \right|^\alpha dx \right)^{\frac{2}{(1-2\sigma)\alpha}} + \int_\Omega u_t^2(t) \, dx \right].$$

By Poincare's inequality, we obtain

$$\left(\int_\Omega u(t)u_t(t)\,dx\right)^{\frac{1}{1-\sigma}} \le c\left[\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{2}{(1-2\sigma)\alpha}} + \int_\Omega u_t^2(t)\,dx\right].$$

We use (2.6) and the algebraic inequality (2.19) with $z = \|\nabla u(t)\|_\alpha^\alpha$, $d = 1 + 1/H(0)$, $a = H(0)$, $\nu = 2/\alpha(1-2\sigma)$, condition (2.6) on σ ensures that $0 < \nu < 1$, and it follows that

$$z^\nu \le d\bigl(z + H(0)\bigr) \le d\bigl(z + H(t)\bigr).$$

Therefore, from (2.20), there exists a positive constant, denoted by c_4, such that for all $t \ge 0$,

$$\left(\int_\Omega u(t)u_t(t)\,dx\right)^{\frac{1}{1-\sigma}} \le c_4\bigl[H(t) + \left\|\nabla u(t)\right\|_\alpha^\alpha + \left\|u_t(t)\right\|_2^2\bigr]. \tag{2.26}$$

Furthermore, by the same method, we have

$$\begin{aligned}\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma &= \left|\int_{\Gamma_1} \frac{h(x)q(x)}{q(x)} u(t)y(t)\,d\Gamma\right| \\ &\le \frac{\|h\|_\infty^{\frac{1}{2}}\|q\|_\infty^{\frac{1}{2}}}{q_0}\left(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right)^{\frac{1}{2}}\left(\int_{\Gamma_1} u^2(t)\,d\Gamma\right)^{\frac{1}{2}}.\end{aligned}$$

Using the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Gamma_1)$ and Hölder's inequality, we get

$$\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma \le c_5\frac{\|h\|_\infty^{\frac{1}{2}}\|q\|_\infty^{\frac{1}{2}}}{q_0}\left(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right)^{\frac{1}{2}}\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{1}{\alpha}}.$$

Consequently, there exists a positive constant $c_5 = c_5(\|h\|_\infty, \|q\|_\infty, q_0, \sigma, \alpha)$ such that

$$\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \le c_5\left(\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right)^{\frac{1}{2(1-\sigma)}}\left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{1}{\alpha(1-\sigma)}}.$$

Using Young's inequality exactly as in (2.26), we write

$$\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \le c_6\left[\int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma + \left(\int_\Omega \left|\nabla u(t)\right|^\alpha dx\right)^{\frac{2}{\alpha(1-2\sigma)}}\right],$$

where c_6 is a positive constant depending on c_5 and α. Consequently, applying once again the algebraic inequality (2.19) with $z = \|\nabla u(t)\|_\alpha^\alpha$, $\nu = 2/\alpha(1-2\sigma)$ and making use of (2.6), we obtain by the same method as above

$$\left(\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right)^{\frac{1}{1-\sigma}} \le c_7\left[H(t) + \left\|\nabla u(t)\right\|_\alpha^\alpha + \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right], \tag{2.27}$$

where c_7 is a positive constant. From (2.25), (2.26) and (2.27), we arrive at

$$L^{\frac{1}{1-\sigma}}(t) \le c\left[H(t) + \left\|\nabla u(t)\right\|_\alpha^\alpha + \left\|u_t(t)\right\|_2^2 + \int_{\Gamma_1} h(x)q(x)y^2(t)\,d\Gamma\right], \tag{2.28}$$

where c is a positive constant. Consequently, a combination of (2.24) and (2.28), for some $\xi > 0$, yields

$$L'(t) \geq \xi L^{\frac{1}{1-\sigma}}(t), \quad \forall t \geq 0. \tag{2.29}$$

Integration of (2.29) over $(0, t)$ gives

$$L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\xi\sigma}{1-\sigma}t}, \quad \forall t \geq 0.$$

Hence $L(t)$ blows up in finite time

$$T^* \leq \frac{1-\sigma}{\xi\sigma L^{\frac{\sigma}{1-\sigma}}(0)}.$$

Thus the proof of Theorem 2.1 is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Division of Mathematical Sciences, Pukyong National University, Busan, 608-737, South Korea. [2]Department of Mathematics, Pusan National University, Busan, 609-735, South Korea. [3]Institute of Liberal Education, Catholic University of Daegu, Gyeongsan, 712-702, South Korea.

Acknowledgements
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1A1B03930361).

Received: 7 December 2016 Accepted: 13 February 2017 Published online: 28 March 2017

References

1. Morse, PM, Ingard, KU: Theoretical Acoustics. McGraw-Hill, New York (1968)
2. Beale, JT, Rosencrans, SI: Acoustic boundary conditions. Bull. Am. Math. Soc. **80**, 1276-1278 (1974)
3. Boukhatem, Y, Benabderrahmane, B: Polynomial decay and blow up of solutions for variable coefficients viscoelastic wave equation with acoustic boundary conditions. Acta Math. Sin. Engl. Ser. **32**(2), 153-174 (2016)
4. Boukhatem, Y, Benabderrahmane, B: Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. Nonlinear Anal. **97**, 191-209 (2014)
5. Graber, PJ, Said-Houari, B: On the wave equation with semilinear porous acoustic boundary conditions. J. Differ. Equ. **252**, 4898-4941 (2012)
6. Wu, ST: Non-existence of global solutions for a class of wave equations with nonlinear damping and source terms. Proc. R. Soc. Edinb. A **141**(4), 865-880 (2011)
7. Messaoudi, SA: Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation. J. Math. Anal. Appl. **320**, 902-915 (2006)
8. Messaoudi, SA: Blow up in a non-linearly damping wave equation. Math. Nachr. **231**, 1-7 (2001)
9. Messaoudi, SI, Houari, BS: Global non-existence of solutions of a class of wave equations with non-linear damping and source terms. Math. Methods Appl. Sci. **27**, 1687-1696 (2004)
10. Ball, J: Remarks on blow-up and non-existence theorems for non-linear evolutions equations. Quarterly J. Math. Oxford Series **28**(2), 473-486 (1977)
11. Hao, J, Zhang, Y, Li, S: Global existence and blow-up phenomena for a nonlinear wave equation. Nonlinear Anal. **71**, 4823-4832 (2009)
12. Levine, HA: Some additional remarks on the non-existence of global solutions to non-linear wave equation. SIAM J. Math. Anal. **5**, 138-146 (1974)
13. Yang, Z: Blow up of solutions for a class of non-linear evolution equations with non-linear damping and source terms. Math. Methods Appl. Sci. **25**, 825-833 (2002)
14. Piskin, E: On the decay and blow up of solutions for a quasilinear hyperbolic equations with nonlinear damping and source terms. Bound. Value Probl. **2015**(127), 1 (2015)
15. Ang, DD, Dinh, APN: Strong solutions of quasilinear wave equation with non-linear damping. SIAM J. Math. Anal. **19**, 337-347 (1988)
16. Kalantarov, VK, Ladyzhenskaya, OA: The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type. J. Sov. Math. **10**, 53-70 (1978)

17. Levine, HA, Serrin, J: A global non-existence theorem for quasilinear evolution equation with dissipation. Archives for Rational Mechanics and Analysis **137**, 341-361 (1997)
18. Levine, HA, Park, SR: Global existence and global non-existence of solutions of the Cauchy problem for a non-linearly damped wave equation. J. Math. Anal. Appl. **228**, 181-205 (1998)