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Stability estimate and regularization for a radially symmetric inverse heat conduction problem

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Abstract
This paper investigates a radially symmetric inverse heat conduction problem, which determines the internal surface temperature distribution of the hollow sphere from measured data at the fixed location inside it. This is an inverse and ill-posed problem. A conditional stability estimate is given on its solution by using Hölder's inequality. A wavelet regularization method is proposed to recover the stability of solution, and the technique is based on the dual least squares method and Shannon wavelet. A quite sharp error estimate between the approximate solution and the exact ones is obtained by choosing a suitable regularization parameter.

MSC: 65M30; 35R30; 35R25

Keywords: inverse heat conduction; ill-posed problems; stability estimate; regularization; error estimate

1 Introduction

A physical model considered here is a hollow sphere, and R and r_0 denote its external and internal radius, respectively. Let the hollow sphere be adiabatic at its external surface, and a thermocouple is installed inside the hollow sphere at the radius $r = r_1$, $r_0 < r_1 < R$, as illustrated in Figure 1. Assuming a spherically symmetric temperature distribution of the model, the correspondingly mathematical model can be described as the following radially symmetric heat conduction problem:

$$\begin{aligned} &u_t = u_{rr} + \frac{2}{r}u_r, \quad r_0 < r < R, t > 0, \\ &u(r,0) = 0, \quad r_0 \le r \le R, \\ &u(r_1,t) = g(t), \quad t \ge 0, \\ &u_r(R,t) = 0, \quad t \ge 0, \end{aligned} \tag{1.1}$$

where r denotes the radial coordinate, $g(t)$ is the temperature history at one fixed radius r_1, $r_0 < r_1 < R$. We want to recover the temperature distribution $u(r,\cdot)$ ($r_0 \le r < r_1$) based on the measured data of $g(\cdot)$. This is an inverse heat conduction problem.

The inverse heat conduction problem (IHCP) has numerous important applications in various sciences and engineering [1]. For example, determination of thermal fields at sur-

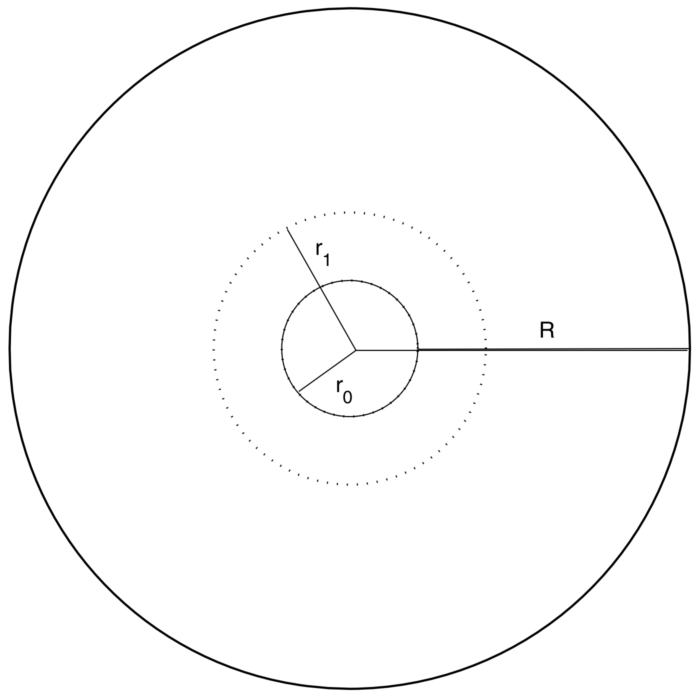

Figure 1 Physical model.

faces without access, obtaining the force applied to a complex structure from knowledge of the response and transfer function which describes the system, or the diagnosis of a disease by computerized tomography [2]. In all cases, the boundary conditions of these problems are inaccessible to measurements or not known. Usually sensors are installed beneath the surface and the unknown boundary conditions of these problems are estimated.

The solutions of inverse heat conduction problems (IHCPs) are very challenging, because IHCPs are severely ill-posed in the Hadamard sense that the solution (if it exists) does not depend continuously on the given data, i.e., a small measurement error in the given data can cause an enormous error in the solution [3, 4]. Therefore, an appropriate regularization method needs to be applied. These methods include the filtering method [5], the spectral method [6], the mollification method [7, 8], the boundary element method [9], the fundamental solution method [10], the wavelet and wavelet-Galerkin method [11–14], the Fourier method [15], the differential-difference method [16], the global time method [17], the modified Tikhonov method [18], the iterative method [19] etc. However, the results already published in the works on IHCP are mainly devoted to the heat equation with constant coefficient. The works presented for heat equation with variable coefficient are still limited. A few works have been developed for the inverse problems on heat equation with variable coefficient [20–23]. In [20], Fu used a simplified Tikhonov and a Fourier regularization methods to deal with an IHCP on heat equation with variable coefficient and provided two kinds of convergence rates. Grabski et al. [21] applied the method of fundamental solutions for identifying a time-dependent perfusion coefficient in the bioheat equation. A non-iterative inverse determination of temperature-dependent thermal conductivity was solved by Mierzwiczak and Kolodziej [22]. Chang and Chang [23] investigated the determination of spatially- and temperature- dependent thermal conductivity by a semi-discretization method. In this work, we will use a wavelet method to deal with IHCP (1.1) ($r_0 \le r < r_1$) with variable coefficient and to obtain a quite sharp error estimate between the approximate solution and the exact solution.

The wavelet method has become a powerful method for solving partial differential equations (PDEs). And the method has been applied to direct problems as well as to various types of inverse problems such as the IHCP [24], the Cauchy problem of Laplace equation [25, 26], the backward heat conduction problem [27], the inverse source identification problems [28, 29] and the Cauchy problem for the modified Helmholtz equation [30, 31]. It is worth mentioning that Feng and Ning [32] used a Meyer wavelet regularization method for solving numerical analytic continuation and presented the Hölder-type stability estimates. In this paper, we solve the radially symmetric inverse heat conduction problem (1.1) in the interval $[r_0, r_1)$ by determining the temperature distribution using a wavelet dual least squares method generated by the family of Shannon wavelets.

When we deal with problem (1.1) in $L^2(\mathbb{R})$ with respect to variable t, we extend all functions of variable t appearing in the paper to be zero for $t < 0$. Since the measurement data of $g(t)$ contain noises, the solutions have to be sought from the data function $g^\delta(t) \in L^2(\mathbb{R})$, which satisfy

$$\|g - g^\delta\| \le \delta, \tag{1.2}$$

where the constant $\delta > 0$ denotes a bound on the measurement error, and $\|\cdot\|$ represents the $L^2(\mathbb{R})$ norm. It is also assumed that there exists an a priori bound for function $u(r_0, t)$

$$\|u(r_0, \cdot)\|_{H^p} \le E, \quad p \ge 0, \tag{1.3}$$

where $\|u(r_0, \cdot)\|_{H^p}$ denotes the norm in the Sobolev space $H^p(\mathbb{R})$ defined by

$$\|u(r_0, \cdot)\|_{H^p} := \left(\int_{-\infty}^{\infty} (1 + \xi^2)^p \left|\hat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Using the Fourier transform with respect to the variable t, problem (1.1) can be formulated in a frequency space as follows:

$$\begin{cases} i\xi\,\hat{u}(r,\xi) = \frac{\partial^2 \hat{u}(r,\xi)}{\partial r^2} + \frac{2}{r}\frac{\partial \hat{u}(r,\xi)}{\partial r}, & r \in (r_0, R], \xi \in \mathbb{R}, \\ \hat{u}(r_1, \xi) = \hat{g}(\xi), & \xi \in \mathbb{R}, \\ \hat{u}_r(R, \xi) = 0, & \xi \in \mathbb{R}. \end{cases} \tag{1.4}$$

We can get a formal solution for problem (1.4), refer to [33],

$$\hat{u}(r,\xi) = (r_1/r)\varphi(r,\xi)e^{(r_1 - r)\sqrt{i\xi}}\hat{g}(\xi), \quad r \in [r_0, R), \xi \in \mathbb{R}, \tag{1.5}$$

where

$$\varphi(r,\xi) = \frac{(\sqrt{i\xi}R + 1)e^{2r\sqrt{i\xi}} + (\sqrt{i\xi}R - 1)e^{2R\sqrt{i\xi}}}{(\sqrt{i\xi}R + 1)e^{2r_1\sqrt{i\xi}} + (\sqrt{i\xi}R - 1)e^{2R\sqrt{i\xi}}}.$$

According to Lemma 2.3 in [33], function $\varphi(r,\xi)$ satisfies

$$c_1 \leq \left|\varphi(r,\xi)\right| \leq c_2, \quad r \in [r_0, r_1), \xi \in \mathbb{R}, \tag{1.6}$$

where c_1 and c_2 are positive constants. Due to $|(r_1/r)\varphi(r,\xi)e^{(r_1-r)\sqrt{i\xi}}|$ increases rapidly with exponential order as $|\xi| \to \infty$, the Fourier transform of the exact data $g(t)$ must decay rapidly at high frequencies for $r_1 > r$. But such a decay is not likely to occur in $g^\delta(t)$. So, a small measurement error in the given data $g^\delta(t)$ in high frequency components can completely destroy the solution of problem (1.1) for $r \in [r_0, r_1)$.

For problem (1.1), we define an operator $A_r : u(r,\cdot) \longmapsto g(\cdot)$ in the space $X = L^2(\mathbb{R})$. Then problem (1.1) can be rewritten as

$$A_r u(r,t) = g(t), \quad \forall u(r,\cdot) \in X, r_0 \leq r < r_1. \tag{1.7}$$

According to expression (1.5), there holds

$$\widehat{A_r u}(r,\xi) = \hat{g}(\xi) = (r/r_1)e^{(r-r_1)\sqrt{i\xi}}\varphi^{-1}(r,\xi)\hat{u}(r,\xi), \quad r \in [r_0, r_1). \tag{1.8}$$

Then we have $\widehat{A_r u(r,\xi)} := \widehat{A_r}\hat{u}(r,\xi)$ and a multiplication operator $\widehat{A_r} : L^2(\mathbb{R}) \longmapsto L^2(\mathbb{R})$ given by

$$\widehat{A_r}\hat{u}(r,\xi) = (r/r_1)e^{(r-r_1)\sqrt{i\xi}}\varphi^{-1}(r,\xi)\hat{u}(r,\xi). \tag{1.9}$$

Therefore, we have the following lemma.

Lemma 1.1 *If A_r^* is the adjoint of A_r, then A_r^* corresponds to the following problem*:

$$\begin{cases} -\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}, & r_0 < r \leq R, t \geq 0, \\ U(r,0) = 0, & r_0 \leq r \leq R, \\ U(r_1,t) = g(t), & t \geq 0, \\ U_r(R,t) = 0, & t \geq 0, \end{cases} \tag{1.10}$$

and

$$\widehat{A_r^*} = (r/r_1)e^{(r-r_1)\overline{\sqrt{i\xi}}}\overline{\varphi^{-1}(r,\xi)}. \tag{1.11}$$

Proof Using the following relations and expression (1.9)

$$\langle A_r u, v\rangle = \langle \widehat{A_r}\hat{u}, \hat{v}\rangle = \left\langle \hat{u}, \widehat{A_r}^* \hat{v}\right\rangle = \left\langle u, A_r^* v\right\rangle = \left\langle \hat{u}, \widehat{A_r^*}\hat{v}\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product, we can obtain the adjoint operator A_r^* of A_r in the frequency domain

$$\widehat{A_r^*} = \widehat{A_r}^* = (r/r_1)e^{(r-r_1)\overline{\sqrt{i\xi}}}\overline{\varphi^{-1}(r,\xi)}.$$

Applying the Fourier transform with respect to the variable t, we can rewrite problem (1.10) in the following form (in the frequency space):

$$\begin{cases} -i\xi \hat{U}(r,\xi) = \frac{\partial^2 \hat{U}(r,\xi)}{\partial r^2} + \frac{2}{r}\frac{\partial \hat{U}(r,\xi)}{\partial r}, & r \in (r_0, R], \xi \in \mathbb{R}, \\ \hat{U}(r_1,\xi) = \hat{g}(\xi), & \xi \in \mathbb{R}, \\ |\hat{U}_r(R,\xi)| = 0, & \xi \in \mathbb{R}. \end{cases} \tag{1.12}$$

Taking the conjugate operator for problem (1.4), we know that $\hat{U}(r,\xi) = \overline{\hat{u}(r,\xi)}$. So, combining with (1.5), we get

$$\hat{U}(r,\xi) = \overline{\hat{u}(r,\xi)} = (r_1/r)e^{(r_1-r)\overline{\sqrt{i\xi}}}\overline{\varphi(r,\xi)}\hat{g}(\xi) \tag{1.13}$$

and

$$\hat{g}(\xi) = (r/r_1)e^{(r-r_1)\overline{\sqrt{i\xi}}}\overline{\varphi^{-1}(r,\xi)}\hat{U}(r,\xi) = \widehat{A_r^*}\hat{U}(r,\xi) = \widehat{A_r^*U}. \tag{1.14}$$

□

The outline of the paper is as follows. In Section 2, using Hölder's inequality, we prove the conditional stability for IHCP (1.1) in the interval $[r_0, r_1)$. The relevant properties of Shannon wavelets are summarized in Section 3. The last section presents error estimates via wavelet dual least squares method approximation.

2 A conditional stability estimate

In this section, we give a conditional stability in the following theorem.

Theorem 2.1 *Let $u(r,t)$ be the exact solution of problem* (1.1) *given by* (1.5) *and the a priori bound* (1.3) *hold. Then, for a fixed $r \in (r_0, r_1)$, we have the following estimate*:

$$\|u(r,\cdot)\| \le c_2(c_1)^{\frac{r-r_1}{r_1-r_0}}(r_1/r_0)^{\frac{r-r_0}{r_1-r_0}}\|u(r_0,\cdot)\|^{\frac{r_1-r}{r_1-r_0}}\|g\|^{\frac{r-r_0}{r_1-r_0}}, \tag{2.1}$$

where c_1 and c_2 are constants given by (1.6).

Proof Using Parseval's formula, expression (1.5) and Hölder's inequality, we have

$$\begin{aligned}
\|u(r,\cdot)\|^2 &= \|\hat{u}(r,\cdot)\|^2 = \int_{-\infty}^{\infty}\left|(r_1/r)\varphi(r,\xi)e^{(r_1-r)\sqrt{i\xi}}\hat{g}(\xi)\right|^2 d\xi \\
&= \int_{-\infty}^{\infty}\left[\left|(r_1/r)\varphi(r,\xi)\right|^{\frac{2(r_1-r_0)}{r_1-r}}\left|e^{(r_1-r_0)\sqrt{i\xi}}\hat{g}(\xi)\right|^2\right]^{\frac{r_1-r}{r_1-r_0}}\left[\left|\hat{g}(\xi)\right|^2\right]^{\frac{r-r_0}{r_1-r_0}} d\xi \\
&\le \left(\int_{-\infty}^{\infty}\left|(r_1/r)\varphi(r,\xi)\right|^{\frac{2(r_1-r_0)}{r_1-r}}\left|e^{(r_1-r_0)\sqrt{i\xi}}\hat{g}(\xi)\right|^2 d\xi\right)^{\frac{r_1-r}{r_1-r_0}}\left(\int_{-\infty}^{\infty}\left|\hat{g}(\xi)\right|^2 d\xi\right)^{\frac{r-r_0}{r_1-r_0}} \\
&= \left(\int_{-\infty}^{\infty}\left|(r_1/r)\varphi(r,\xi)\right|^{\frac{2(r_1-r_0)}{r_1-r}}\left|(r_1/r_0)\varphi(r_0,\xi)\right|^{-2}\left|\hat{u}(r_0,\xi)\right|^2 d\xi\right)^{\frac{r_1-r}{r_1-r_0}}\|g\|^{\frac{2(r-r_0)}{r_1-r_0}} \\
&\le \sup_{\xi\in\mathbb{R}}\left[\left|(r_1/r)\varphi(r,\xi)\right|^2\left|(r_1/r_0)\varphi(r_0,\xi)\right|^{\frac{2(r-r_1)}{r_1-r_0}}\right]\left\|\hat{u}(r_0,\cdot)\right\|^{\frac{2(r_1-r)}{r_1-r_0}}\|g\|^{\frac{2(r-r_0)}{r_1-r_0}}.
\end{aligned}$$

From inequalities (1.6), we get

$$\left|(r_1/r)\varphi(r,\xi)\right|^2\left|(r_1/r_0)\varphi(r_0,\xi)\right|^{\frac{2(r-r_1)}{r_1-r_0}} \le \left|(r_1/r)c_2\right|^2\left|(r_1/r_0)c_1\right|^{\frac{2(r-r_1)}{r_1-r_0}}.$$

Then there holds

$$\left\|u(r,\cdot)\right\|^2 \le c_2^2(c_1)^{\frac{2(r-r_1)}{r_1-r_0}}(r_1/r_0)^{\frac{2(r-r_0)}{r_1-r_0}}\left\|u(r_0,\cdot)\right\|^{\frac{2(r_1-r)}{r_1-r_0}}\|g\|^{\frac{2(r-r_0)}{r_1-r_0}}.$$

The conclusion of Theorem 2.1 is proved. □

Remark 2.2 If $u_1(r,t)$ and $u_2(r,t)$ are the solutions of problem (1.1) with the exact data $g_1(t)$ and $g_2(t)$, respectively, then for a fixed $r \in (r_0,r_1)$, there holds

$$\left\|u_1(r,\cdot)-u_2(r,\cdot)\right\| \le C\left\|u_1(r_0,\cdot)-u_2(r_0,\cdot)\right\|^{\frac{r_1-r}{r_1-r_0}}\left\|g_1(\cdot)-g_1(\cdot)\right\|^{\frac{r-r_0}{r_1-r_0}}, \tag{2.2}$$

where $C = c_2(c_1)^{\frac{r-r_1}{r_1-r_0}}(r_1/r_0)^{\frac{r-r_0}{r_1-r_0}}$. It is obvious that if $\|g_1(\cdot)-g_1(\cdot)\| \to 0$, then $\|u_1(r,\cdot)-u_2(r,\cdot)\|$ for $r_0 < r \le r_1$.

In the next section, the relevant properties of Shannon wavelets are summarized.

3 The Shannon wavelets

Suppose that ϕ and ψ are the Shannon scaling and wavelet functions whose Fourier transforms are given by

$$\hat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le \pi, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

and

$$\hat{\psi}(\xi) = \begin{cases} e^{-i\frac{\xi}{2}}, & \pi \le |\xi| \le 2\pi, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Let $\phi_{j,k}(t) := 2^{\frac{j}{2}}\phi(2^jt-k)$, $\psi_{j,k}(t) := 2^{\frac{j}{2}}\psi(2^jt-k)$, $j,k \in \mathbb{Z}$, $\Psi_{-1,k} := \phi_{0,k}$ and $\Psi_{l,k} := \psi_{l,k}$ for $l \ge 0$, the index set

$$\tilde{I} = \left\{\{j,k\} : j,k \in \mathbb{Z}\right\} \subset \mathbb{Z}^2,$$
$$\tilde{I}_J = \left\{\{j,k\} : j = -1,0,\ldots,J-1; k \in \mathbb{Z}\right\} \subset \mathbb{Z}^2.$$

Then the subspaces V_J can be defined

$$V_J = \overline{\operatorname{span}\{\Psi_\lambda\}_{\lambda\in\tilde{I}_J}} \tag{3.3}$$

and an orthogonal projection $P_J : L^2(\mathbb{R}) \longmapsto V_J$:

$$P_J\varphi = \sum_{\lambda\in\tilde{I}_J}\langle\varphi,\Psi_\lambda\rangle\Psi_\lambda, \quad \forall\varphi \in L^2(\mathbb{R}). \tag{3.4}$$

We have, for any $k \in \mathbb{Z}$,

$$\operatorname{supp}(\hat{\psi}_{j,k}) = \{\xi : \pi 2^j \leq |\xi| \leq \pi 2^{j+1}\}, \tag{3.5}$$

$$\operatorname{supp}(\hat{\phi}_{j,k}) = \{\xi : |\xi| \leq \pi 2^j\}. \tag{3.6}$$

Expression (3.4) shows that P_J can be considered as a low pass filter.

4 Regularization and error estimates

In this section, the wavelet dual least squares method will be described and error estimates be given by Theorems 4.1-4.3.

4.1 Dual least squares method

We now introduce the dual least squares method for approximation of the solutions of problem (1.1). For the operator equation $Au = g$, a general projection method is generated by two subspace families $\{V_j\}$ and $\{Y_j\}$ of X. Then the approximate solution $u_j \in V_j$ is defined to be the solution of the following problem:

$$\langle Au_j, y\rangle = \langle g, y\rangle, \quad \forall y \in Y_j. \tag{4.1}$$

If $V_j \subset R(A^*)$ and subspaces Y_j are chosen in such a way that

$$A^* Y_j = V_j,$$

then there is a special case of projection method known as the dual least squares method. Suppose that $\{\psi_\lambda\}_{\lambda \in \tilde{I}_j}$ is an orthogonal basis of V_j and y_λ is the solution of the equation

$$A^* y_\lambda = k_\lambda \psi_\lambda, \quad \|y_\lambda\| = 1. \tag{4.2}$$

Then we can obtain the approximate solution

$$u_j = \sum_{\lambda \in \tilde{I}_j} \langle g, y_\lambda\rangle \frac{1}{k_\lambda} \psi_\lambda. \tag{4.3}$$

According to (4.3), we easily conclude $u_J = P_J u$. In order to give an error estimate for the regularized solution, we need a sequence of subspaces Y_j approximating the space X and contained in the range of A^*. From $A^* Y_j = V_j$, the subspaces Y_j are spanned by w_λ, $\lambda \in \tilde{I}_J$, where

$$A^* w_\lambda = \Psi_\lambda \quad \text{and} \quad k_\lambda = \|w_\lambda\|^{-1}, \qquad y_\lambda = \frac{w_\lambda}{\|w_\lambda\|} = k_\lambda w_\lambda. \tag{4.4}$$

We know that w_λ is a solution of the following problem (see Lemma 1.1):

$$\begin{cases} -\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}, & r_0 < r \leq R, t \geq 0, \\ U(r,0) = 0, & r_0 \leq r \leq R, \\ U(r_1,t) = \Psi_{j,k}(t), & t \geq 0, \\ U_r(R,t) = 0, & t \geq 0. \end{cases} \tag{4.5}$$

Because supp $\hat{\psi}_{j,k}$ is compact, the solution exists for any $t \in (0,\infty)$. Analogously, the solution of the adjoint equation is unique. So, for given Ψ_λ, w_λ can be uniquely determined according to (4.5). From (1.13), there holds

$$\hat{w}_\lambda = \frac{r_1}{r} e^{(r_1-r)\overline{\sqrt{i\xi}}}\overline{\varphi(r,\xi)}\hat{\Psi}_\lambda(\xi), \quad \lambda = \{j,k\}, \tag{4.6}$$

combining with (4.4), we have

$$\hat{y}_\lambda = \frac{r_1}{r} e^{(r_1-r)\overline{\sqrt{i\xi}}}\overline{\varphi(r,\xi)}k_\lambda\hat{\Psi}_\lambda(\xi), \quad \lambda = \{j,k\}. \tag{4.7}$$

Thus we have the approximate solution for noisy data g^δ given by

$$P_J u^\delta(r,t) = u_J^\delta = \sum_{\lambda\in\tilde{I}_J}\langle u^\delta, \Psi_\lambda\rangle\Psi_\lambda = \sum_{\lambda\in\tilde{I}_J}\langle g^\delta, y_\lambda\rangle\frac{1}{k_\lambda}\Psi_\lambda. \tag{4.8}$$

4.2 Error estimates

We estimate firstly the errors $\|P_J u - P_J u^\delta\|$ and $\|u - P_J u\|$ by Theorems 4.1 and 4.2, respectively.

Theorem 4.1 (Stability) *Let $P_J u(r,t)$ given by* (4.3) *and $P_J u^\delta(r,t)$ given by* (4.8) *be the regularized approximate solutions to $u(r,t)$ for the data g and g^δ, respectively. If the measured data $g^\delta(t)$ satisfy condition* (1.2), *then for any fixed $r \in [r_0, r_1)$, we have*

$$\|P_J u - P_J u^\delta\| \le (c_2 r_1/r_0) e^{(r_1-r)\sqrt{\frac{1}{2}\pi 2^J}}\delta. \tag{4.9}$$

Proof Due to (4.3), (4.8) and (4.7), for any fixed $r \in [r_0, r_1)$, there holds

$$\begin{aligned}
\|P_J u(r,\cdot) - P_J u^\delta(r,\cdot)\| &= \left\|\sum_{\lambda\in\tilde{I}_J}\langle g - g^\delta, y_\lambda\rangle\frac{1}{k_\lambda}\Psi_\lambda\right\| \\
&= \left\|\sum_{\lambda\in\tilde{I}_J}\langle \hat{g} - \widehat{g^\delta}, \hat{y}_\lambda\rangle\frac{1}{k_\lambda}\hat{\Psi}_\lambda\right\| \\
&= \left\|\sum_{\lambda\in\tilde{I}_J}\left\langle \hat{g} - \widehat{g^\delta}, \frac{r_1}{r}e^{(r_1-r)\overline{\sqrt{i\xi}}}\overline{\varphi(r,\xi)}k_\lambda\hat{\Psi}_\lambda\right\rangle\frac{1}{k_\lambda}\hat{\Psi}_\lambda\right\| \\
&\le \sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J}\left|(r_1/r)e^{(r_1-r)\overline{\sqrt{i\xi}}}\overline{\varphi(r,\xi)}\right| \cdot \left\|\sum_{\lambda\in\tilde{I}_J}\langle \hat{g} - \widehat{g^\delta}, \hat{\Psi}_\lambda\rangle\hat{\Psi}_\lambda\right\| \\
&\le \sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J}\left|(r_1/r)e^{(r_1-r)\sqrt{i\xi}}\varphi(r,\xi)\right| \cdot \left\|\hat{P}_J\left(\hat{g} - \widehat{g^\delta}\right)\right\|,
\end{aligned}$$

combining with inequality (1.6) and condition (1.2),

$$\begin{aligned}
\|P_J u(r,\cdot) - P_J u^\delta(r,\cdot)\| &\le (c_2 r_1/r_0)\sup_{\pi 2^{J-1}\le|\xi|\le\pi 2^J} e^{(r_1-r)\sqrt{|\xi|/2}}\delta \\
&\le (c_2 r_1/r_0)e^{(r_1-r)\sqrt{\frac{1}{2}\pi 2^J}}\delta.
\end{aligned}$$

□

Theorem 4.2 (Convergence) *If $u(r,t)$ is the solution of problem* (1.1) *satisfying the a priori condition* (1.3), *then for any fixed $r \in [r_0, r_1)$, we have*

$$\left\| u(r,\cdot) - P_J u(r,\cdot) \right\| \le (c_2/c_1)\left(2^{J+1}\right)^{-p} e^{(r_0-r)\sqrt{\frac{1}{2}\pi 2^J}} E. \tag{4.10}$$

Proof According to (3.4), we get

$$\begin{aligned} u(r,\cdot) &= \sum_{\lambda} \langle u(r,\cdot), \Psi_\lambda \rangle \Psi_\lambda, \\ P_J u(r,\cdot) &= \sum_{\lambda \in \tilde{I}_J} \langle u(r,\cdot), \Psi_\lambda \rangle \Psi_\lambda. \end{aligned}$$

By using Parseval's relation and (1.5), (1.6), (1.3), there holds

$$\begin{aligned} &\left\| u(r,\cdot) - P_J u(r,\cdot) \right\| \\ &\quad = \left\| \hat{u}(r,\cdot) - \widehat{P_J u}(r,\cdot) \right\| = \left\| \sum_{\lambda \in \tilde{I}} \langle \hat{u}, \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda - \sum_{\lambda \in \tilde{I}_J} \langle \hat{u}, \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda \right\| \\ &\quad = \left\| \sum_{\lambda \in \tilde{I}_{j \ge J+1}} \langle \hat{u}, \hat{\Psi}_\lambda \rangle \hat{\Psi}_\lambda \right\| = \left\| \sum_{\lambda \in \tilde{I}_{j \ge J+1}} \left\langle (r_1/r)\varphi(r,\cdot) e^{(r_1-r)\sqrt{i(\cdot)}} \hat{g}(\cdot), \hat{\Psi}_\lambda \right\rangle \hat{\Psi}_\lambda \right\| \\ &\quad = \left\| \sum_{\lambda \in \tilde{I}_{j \ge J+1}} \left\langle (r_0/r)\varphi(r,\cdot)\varphi^{-1}(r_0,\cdot) e^{(r_0-r)\sqrt{i(\cdot)}} \hat{u}(r_0,\cdot), \hat{\Psi}_\lambda \right\rangle \hat{\Psi}_\lambda \right\| \\ &\quad \le \sup_{\pi 2^J \le |\xi| \le \pi 2^{J+1}} \left| \frac{r_0 \varphi(r,\xi)}{r \varphi(r_0,\xi)} e^{(r_0-r)\sqrt{i\xi}} |\xi|^{-p} \right| \left\| \sum_{\lambda \in \tilde{I}_{j \ge J+1}} \left\langle \left(1 + (\cdot)^2\right)^{p/2} \hat{u}(r_0,\cdot), \hat{\Psi}_\lambda \right\rangle \hat{\Psi}_\lambda \right\| \\ &\quad \le \sup_{\pi 2^J \le |\xi| \le \pi 2^{J+1}} (c_2/c_1)|\xi|^{-p} e^{(r_0-r)\sqrt{|\xi|/2}} E \le (c_2/c_1)\left(2^{J+1}\right)^{-p} e^{(r_0-r)\sqrt{\frac{1}{2}\pi 2^J}} E. \end{aligned}$$

We have proved estimate (4.10). □

Theorem 4.3 *Let $u(r,t)$ be the exact solution of* (1.1) *and $P_J u^\delta$ given by* (4.8) *be the regularized approximate solution to $u(r,t)$. If the measured data $g^\delta(t)$ satisfies condition* (1.2) *and the a priori condition* (1.3) *is valid when we select*

$$J = \log_2 \left[\frac{2}{\pi} \left(\frac{1}{r_1 - r_0} \ln \left(\frac{E}{\delta} \left(\ln \frac{E}{\delta} \right)^{-2p} \right) \right)^2 \right], \tag{4.11}$$

then for any fixed $r \in [r_0, r_1)$, we have

$$\begin{aligned} &\left\| u(r,\cdot) - P_J u^\delta(r,\cdot) \right\| \\ &\quad \le E^{1-\frac{r-r_0}{r_1-r_0}} \delta^{\frac{r-r_0}{r_1-r_0}} \left(\ln \frac{E}{\delta} \right)^{-2p(1-\frac{r-r_0}{r_1-r_0})} \left(C + o(1) \right) \quad \textit{for } \delta \to 0, \end{aligned} \tag{4.12}$$

where $C = (c_2/c_1)(r_1 - r_0)^{2p} + (c_2 r_1/r_0)$.

Proof From Theorem 4.1 , Theorem 4.2 and the choice rule (4.11) of J, we can get

$$\begin{aligned}
&\bigl\|u(r,\cdot)-P_J u^\delta(r,\cdot)\bigr\| \\
&\quad\le (c_2/c_1)\bigl(2^{J+1}\bigr)^{-p} e^{(r_0-r)\sqrt{\frac{1}{2}\pi 2^J}} E + (c_2 r_1/r_0) e^{(r_1-r)\sqrt{\frac{1}{2}\pi 2^J}} \delta \\
&\quad\le (c_2/c_1) E (r_1-r_0)^{2p} \left(\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)\right)^{-2p} \left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)^{\frac{r_0-r}{r_1-r_0}} \\
&\qquad + (c_2 r_1/r_0)\delta \left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)^{\frac{r_1-r}{r_1-r_0}} \\
&\quad\le E^{1-\frac{r-r_0}{r_1-r_0}} \delta^{\frac{r-r_0}{r_1-r_0}} \left(\ln\frac{E}{\delta}\right)^{-2p(1-\frac{r-r_0}{r_1-r_0})} \left\{\frac{c_2((r_1-r_0)\ln\frac{E}{\delta})^{2p}}{c_1(\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p}))^{2p}} + \frac{c_2 r_1}{r_0}\right\}.
\end{aligned}$$

Note that

$$\frac{\ln\frac{E}{\delta}}{\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p})} = \frac{\ln\frac{E}{\delta}}{\ln\frac{E}{\delta} - 2p\ln(\ln\frac{E}{\delta})} \to 1 \quad \text{for } \delta\to 0.$$

We have obtained estimate (4.12). □

Remark 4.4

(i) If $p=0$ and $r_0<r<r_1$, estimate (4.12) becomes

$$\bigl\|u(r,\cdot)-P_J u^\delta(r,\cdot)\bigr\| \le \bigl((c_2/c_1)+(c_2 r_1/r_0)\bigr) E^{1-\frac{r-r_0}{r_1-r_0}} \delta^{\frac{r-r_0}{r_1-r_0}}, \tag{4.13}$$

which is a Hölder stability estimate.

(ii) If $p>0$, estimate (4.12) is a logarithmical-Hölder stability estimate, especially at $r=r_0$, it becomes

$$\bigl\|u(r_0,\cdot)-u^\delta(r_0,\cdot)\bigr\| \le E\left(\ln\frac{E}{\delta}\right)^{-2p} \bigl(C+o(1)\bigr) \to 0 \quad \text{for } \delta\to 0, \tag{4.14}$$

which is a logarithmical stability estimate.

5 Conclusion

In this paper the radially symmetric inverse heat conduction problem is considered. A conditional stability result is established by utilizing the a priori bound. We obtain a regularized solution by a wavelet dual least squares method and the error estimate of logarithmic Hölder type between the approximate solution and the exact ones by choosing a suitable regularization parameter.

Abbreviation

IHCP, inverse heat conduction problem.

Competing interests

The author declares that she has no competing interests.

Author's contributions

The paper was realized by the author. The author read and approved the final manuscript.

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