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Interior regularity criterion for incompressible Ericksen-Leslie system

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Abstract

An interior regularity criterion of suitable weak solutions is formulated for the Ericksen-Leslie system of liquid crystals. Such a criterion is point-wise, with respect to some appropriate norm of velocity *u* and the gradient of *d*, and it can be viewed as a sort of simply sufficient condition on the local regularity of suitable weak solutions.

MSC: 35Q35; 76D03

Keywords: interior regularity; suitable weak solution; liquid crystal

1 Introduction and main results

In this paper, we investigate the local regularity of weak solutions to the following 3D incompressible Ericksen-Leslie liquid crystal system:

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d), \tag{1.1a}$$

$$\nabla \cdot u = 0, \tag{1.1b}$$

$$\partial_t d + (u \cdot \nabla)d = \Delta d - f(d), \tag{1.1c}$$

with the initial boundary conditions

$$\begin{aligned} &(u,d)(x,t)|_{t=0} = \big(u_0(x), d_0(x)\big), \qquad \nabla \cdot u_0 = 0, \quad x \in \Omega, \\ &(u,d)(x,t)|_{x\in\partial\Omega} = \big(0, d_0(x)\big), \qquad u_0(x) \in H_0^1(\Omega), \qquad d_0(x) \in H_0^2(\Omega), \end{aligned} \tag{1.2}$$

where u, d, P denote the velocity of the fluid, the uniaxial molecular direction, and the pressure, respectively, the i,jth element of $\nabla d \odot \nabla d$ is $\partial_i d^k \partial_j d^k$, $d_0(x)$ is a unit vector, $\Omega \subset \mathbb{R}^3$ is a smooth domain. Additionally, $f(d) = \nabla F(d)$, and $F(d) = \frac{1}{\zeta^2}(|d|^2 - 1)^2, \zeta$ is a small number, formally speaking, as $\zeta \to 0, d$ tends to a unit vector.

The dynamic flows of liquid crystals have been successfully described by the Ericksen-Leslie theory [1–4]. System (1.1a)-(1.1c) is a coupled system of the Navier-Stokes equations with a parabolic system. It is Leray [5] and Hopf [6] that established the global existence of weak solutions to the 3D Navier-Stokes; however, the regularity of the weak solutions is still an open problem. Since the regularity of weak solutions to the 3D Navier-Stokes equations is hard to get, some related conditions or criteria for the regularity of the weak

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solutions are considered, such as the well-known Serrin type criterion [7] and the Beale-Kato-Majda type criterion [8]. Furthermore, based on the suitable weak solutions, some point-wise sufficient regularity criteria were imposed in [9–12].

The global existence of suitable weak solutions to system (1.1a)-(1.1c) was established in [13, 14] by Lin and Liu; however, noticing that system (1.1a)-(1.1c) contains the 3D Navier-Stokes equations as a subsystem, the uniqueness and regularity of these weak solutions are not known. In this paper, we would extend some point-wise sufficient conditions, which guarantee the local regularity of weak solutions for 3D Navier-Stokes equations, to the Ericksen-Leslie system (1.1a)-(1.1c). We would like to mention that when $f(d)$ in system (1.1a)-(1.1c) is replaced by $-|\nabla d|^2 d$, the global existence of weak solutions to the resulting system in three dimensions has only been known under the additional assumption that $d_3 \geq 0$ or small initial data (see [15, 16]). Without these conditions, the general existence of weak solutions is still open. However, the Serrin type criterion and the Beale-Kato-Majda type criterion still hold true even for a weak solution (if it exists) (see [17, 18]).

The suitable weak solution established in [14] can be stated as below.

Definition 1.1 (Suitable weak solutions in $\Omega \times (0,T) \subset \mathbb{R}^3 \times (0,\infty)$) A pair (u,d) is called a suitable weak solution to system (1.1a)-(1.1c) and (1.2) in an open set $\mathcal{O} \subset \mathbb{R}^3 \times (0,\infty)$ (we set $\mathcal{O}_t = \mathcal{O} \cap (\mathbb{R}^3 \times \{t\})$), if it satisfies the following properties:

- (u,d) is a weak solution in the sense of distribution;
- $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), d \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, or generally, there exist constants E_1, E_2, such that

$$\int_{\mathcal{O}_t} \big[|u|^2 + |\nabla d|^2 + F(d)\big]\,\mathrm{d}x < E_1,$$
$$\int\int_{\mathcal{O}} \big[|\nabla u|^2 + \big|\Delta d - f(d)\big|^2 + F(d)\big]\,\mathrm{d}x\,\mathrm{d}t < E_2;$$

- for any $\varphi \in C_c^\infty(\mathcal{O})$, more specifically, for any $\varphi \in C_c^\infty(B(x_0,R) \times (t_0 - R^2, t_0))$, the following generalized energy inequality holds

$$\begin{aligned}
&\int_{B(x_0,R)} \big(|u|^2 + |\nabla d|^2\big)\varphi\,\mathrm{d}x + 2\int_{t_0-R^2}^{t}\int_{B(x_0,R)} \big(|\nabla u|^2 + \big|\nabla^2 d\big|^2\big)\varphi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad \leq \int_{t_0-R^2}^{t}\int_{B(x_0,R)} \big\{\big(|u|^2 + |\nabla d|^2\big)(\varphi_t + \Delta\varphi) + \big(|u|^2 + |\nabla d|^2 + 2P\big)u\cdot\nabla\varphi\big\}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad + 2\int_{t_0-R^2}^{t}\int_{B(x_0,R)} \big((u\cdot\nabla)d\nabla d\nabla\varphi - \nabla f(d) : \nabla d\varphi\big)\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (1.3)
\end{aligned}$$

In the following, we can take $Q((x_0,t_0),R) \equiv B(x_0,R) \times (t_0 - R^2, t_0)$, $B(x_0,R) \equiv \{y \in \mathbb{R}^3 | |y - x_0| < R\}, z_0 \equiv (x_0,t_0)$ for simplicity.

We now state our main result of this paper.

Theorem 1.2 *Let* (u,d) *be a suitable weak solution to liquid crystal system* (1.1a)-(1.1c) *in* $Q(z_0,R)$. *The real numbers* $l \geq 1$ *and* $s \geq 1$ *satisfy*

$$\frac{1}{2} \geq \frac{3}{s} + \frac{2}{l} - \frac{3}{2} > \max\left\{\frac{1}{2l}, \frac{1}{2} - \frac{1}{s}, \frac{1}{s} - \frac{1}{6}\right\}.$$

Then there is a positive number $\varepsilon = \varepsilon(s,l)$, such that if

$$M^{s,l}(z_0,R) = \frac{1}{R^\kappa}\int_{t_0-R^2}^{t_0}\left(\int_{B(x_0,R)} |u|^s + |\nabla d|^s \,\mathrm{d}x\right)^{\frac{l}{s}} \mathrm{d}t < \varepsilon, \quad \kappa = \frac{3l}{s} + 2 - l,$$

then z_0 is a regular point of $(u,\nabla d)$, i.e. $(u,\nabla d)$ is Hölder continuous in $Q(z_0,r)$, for some $r \in (0,R]$.

Throughout this paper, we use c to denote a generic positive constant which can be different from line to line.

2 Preliminaries

As the preparation for proving Theorem 1.2, we first give two auxiliary lemmas.

Lemma 2.1 *We have*

$$D(z_0,r;p) \le c\left[\frac{r}{\rho}D(z_0,\rho;p) + \left(\frac{\rho}{r}\right)^2 C(z_0,\rho;u,\nabla d)\right], \tag{2.1}$$

where

$$C(z_0,r;u,\nabla d) = \frac{1}{r^2}\int_{Q(z_0,r)} \left(|u|^3 + |\nabla d|^3\right)\mathrm{d}z, \qquad D(z_0,r;p) = \frac{1}{r^2}\int_{Q(z_0,r)} |p|^{\frac{3}{2}}\,\mathrm{d}z.$$

Proof Step 1. For (1.1a), we choose the test function $w = \chi\nabla q$, for any $\chi \in C_c^\infty((t_0 - \rho^2, t_0))$, $q \in C_c^\infty(B(x_0,\rho))$, then it yields

$$\int_{Q(z_0,\rho)} -u\cdot\partial_t\chi\nabla q - (u\otimes u + \nabla d \odot \nabla d) : \chi\nabla^2 q - u\cdot\chi\nabla\Delta q\,\mathrm{d}z = \int_{Q(z_0,\rho)} p\chi\,\Delta q\,\mathrm{d}z.$$

It follows from $\nabla\cdot u = 0$ that

$$-\int_{Q(z_0,\rho)} p\chi\,\Delta q\,\mathrm{d}z = \int_{Q(z_0,\rho)} \chi(u\otimes u + \nabla d\odot\nabla d) : \nabla^2 q\,\mathrm{d}z.$$

Therefore, for a.e. $t \in (t_0 - \rho^2, t_0)$, we have

$$-\int_{B(x_0,\rho)} p\Delta q\,\mathrm{d}x = \int_{B(x_0,\rho)} (u\otimes u + \nabla d\odot\nabla d) : \nabla^2 q\,\mathrm{d}x, \quad \forall q \in C_c^\infty\big(B(x_0,\rho)\big). \tag{2.2}$$

Step 2. Approximate p with p_1 by confining q in $W^{2,3}(B(x_0,\rho))$.

Set $p_1 \in L^{\frac{3}{2}}(Q(z_0,\rho))$ such that, for a.e. $t \in (t_0 - \rho^2, t_0)$,

$$-\int_{B(x_0,\rho)} p_1\Delta q\,\mathrm{d}x = \int_{B(x_0,\rho)} (u\otimes u + \nabla d\odot\nabla d) : \nabla^2 q\,\mathrm{d}x, \tag{2.3}$$

for any $q(\cdot,t) \in W^{2,3}(B(x_0,\rho))$, and $q(\cdot,t) = 0$ on $\partial B(x_0,\rho)$. The existence of p_1 is established due to the Lax-Milgram theorem with appropriate approximating process on u and d (see [11]).

Next, choose $q_0(\cdot,t) \in W^{2,3}(B(x_0,\rho))$, such that, for a.e. $t \in (t_0-\rho^2, t_0)$,

$$\Delta q_0(\cdot,t) = -\left|p_1(\cdot,t)\right|^{\frac{1}{2}} \operatorname{sgn} p_1(\cdot,t), \quad \text{in } B(x_0,\rho), \qquad q_0(\cdot,t) = 0, \quad \text{on } \partial B(x_0,\rho).$$

Then, by the Calderon-Zygmund inequality, it yields

$$\left(\int_{B(x_0,\rho)} \left|\nabla^2 q_0(\cdot,t)\right|^3 \mathrm{d}x\right)^{\frac{1}{3}} \le c\left(\int_{B(x_0,\rho)} \left|p_1(\cdot,t)\right|^{\frac{3}{2}} \mathrm{d}x\right)^{\frac{1}{3}}, \quad \text{a.e. } t \in \left(t_0-\rho^2, t_0\right).$$

Therefore, it follows from (2.3) and the Hölder inequality that

$$\begin{aligned}\int_{B(x_0,\rho)} \left|p_1(\cdot,t)\right|^{\frac{3}{2}} \mathrm{d}x &\le c\left(\int_{B(x_0,\rho)} |u|^3 + |\nabla d|^3 \,\mathrm{d}x\right)^{\frac{2}{3}} \left(\int_{B(x_0,\rho)} \left|\nabla^2 q\right|^3 \mathrm{d}x\right)^{\frac{1}{3}} \\ &\le c\left(\int_{B(x_0,\rho)} |u|^3 + |\nabla d|^3 \,\mathrm{d}x\right)^{\frac{2}{3}} \left(\int_{B(x_0,\rho)} |p_1|^{\frac{3}{2}} \mathrm{d}x\right)^{\frac{1}{3}},\end{aligned}$$

which yields $\int_{Q(z_0,\rho)} |p_1(\cdot,t)|^{\frac{3}{2}} \,\mathrm{d}z \le c\rho^2 C(z_0,\rho;u,\nabla d)$.

Step 3. Estimates for the remainder $p-p_1$.

For a.e. $t \in (t_0-\rho^2,t_0)$, let $p_2 = p-p_1$, then from (2.2)-(2.3) one infers that

$$\Delta p_2(\cdot,t) = 0, \quad \text{in } B(x_0,\rho).$$

By the harmonic property, one can get

$$\frac{1}{r^3}\int_{Q(z_0,r)} |p_2|^{\frac{3}{2}} \,\mathrm{d}z \le \frac{c}{\rho^3}\int_{Q(z_0,\rho)} |p_2|^{\frac{3}{2}} \,\mathrm{d}z, \quad \forall r < \rho,$$

while

$$\int_{Q(z_0,\rho)} |p_2|^{\frac{3}{2}} \,\mathrm{d}z \le \int_{Q(z_0,\rho)} \left(|p|^{\frac{3}{2}} + |p_1|^{\frac{3}{2}}\right) \mathrm{d}z \le c\rho^2\left(D(z_0,\rho;p) + C(z_0,\rho;u,\nabla d)\right).$$

Step 4. Estimates for p.

We have

$$\begin{aligned}D(z_0,r;p) &\le c\left(\frac{1}{r^2}\int_{Q(z_0,r)} |p_1|^{\frac{3}{2}} \,\mathrm{d}z + \frac{r}{\rho^3}\int_{Q(z_0,\rho)} |p_2|^{\frac{3}{2}} \,\mathrm{d}z\right) \\ &\le c\left(\frac{\rho^2}{r^2}\frac{1}{\rho^2}\int_{Q(z_0,r)} |p_1|^{\frac{3}{2}} \,\mathrm{d}z + \frac{r}{\rho}\frac{1}{\rho^2}\int_{Q(z_0,\rho)} |p_2|^{\frac{3}{2}} \,\mathrm{d}z\right) \\ &\le c\left[\frac{\rho^2}{r^2}C(z_0,\rho;u,\nabla d) + \frac{r}{\rho}\left(D(z_0,\rho;p) + C(z_0,\rho;u,\nabla d)\right)\right] \\ &\le c\left[\frac{r}{\rho}D(z_0,\rho;p) + \left(\frac{\rho}{r}\right)^2 C(z_0,\rho;u,\nabla d)\right].\end{aligned}$$

□

We denote

$$A(\rho) = \operatorname*{ess\,sup}_{t_0-\rho^2<t<t_0} \frac{1}{\rho}\int_{B(x_0,\rho)} \left(\left|u(t)\right|^2 + \left|\nabla d(t)\right|^2\right) \mathrm{d}x,$$

$$E(\rho) = \frac{1}{\rho}\int_{Q(z_0,\rho)} \left(|\nabla u|^2 + \left|\nabla^2 d\right|^2\right) \mathrm{d}z, \qquad H(\rho) = \frac{1}{\rho^3}\int_{Q(z_0,\rho)} \left(|u|^2 + |\nabla d|^2\right) \mathrm{d}z.$$

Lemma 2.2 *Under the assumptions of Theorem* 1.2, *we have*

$$C(\rho) \le c\epsilon^{\frac{1}{q}}\big(E(\rho) + A(\rho) + 1\big),$$

where $q = 2l(\frac{3}{s} + \frac{2}{l} - \frac{3}{2})$, *and* $q' = \frac{q}{q-1}$.

Proof With the help of the Hölder and Sobolev embedding inequalities, one gets

$$\begin{aligned}
\int_{B(x_0,\rho)} |v|^3 \,\mathrm{d}x &= \int_{B(x_0,\rho)} |v|^{\lambda s+2\mu+6\gamma} \,\mathrm{d}x \\
&\le \left(\int_{B(x_0,\rho)} |v|^2 \,\mathrm{d}x\right)^{\mu} \left(\int_{B(x_0,\rho)} |v|^s \,\mathrm{d}x\right)^{\lambda} \left(\int_{B(x_0,\rho)} |v|^6 \,\mathrm{d}x\right)^{\gamma} \\
&\le \frac{c}{2}\rho^{\mu}\left(\operatorname*{ess\,sup}_{t_0-\rho^2<t<t_0} \frac{1}{\rho}\int_{B(x_0,\rho)} |v|^2 \,\mathrm{d}x\right)^{\mu} \left(\int_{B(x_0,\rho)} |v|^s \,\mathrm{d}x\right)^{\lambda} \\
&\quad \times \left(\int_{B(x_0,\rho)} |\nabla v|^2 + \frac{1}{\rho^2}|v|^2 \,\mathrm{d}x\right)^{3\gamma},
\end{aligned}$$

where $\lambda s + 2\mu + 6\gamma = 3, \lambda + \mu + \gamma = 1$. Substituting v by u and ∇d, respectively, then one can get the summation

$$\begin{aligned}
\int_{B(x_0,\rho)} |u|^3 + |\nabla d|^3 \,\mathrm{d}x &\le c\rho^{\mu}A^{\mu}(\rho)\left(\int_{B(x_0,\rho)} \left(|u|^s + |\nabla d|^s\right) \mathrm{d}x\right)^{\lambda} \\
&\quad \times \left(\int_{B(x_0,\rho)} \left(|\nabla u|^2 + \left|\nabla^2 d\right|^2\right) + \frac{1}{\rho^2}\left(|u|^2 + |\nabla d|^2\right) \mathrm{d}x\right)^{3\gamma}.
\end{aligned}$$

Therefore, by choosing appropriate parameters $\lambda = \frac{1}{2s(\frac{3}{s}+\frac{2}{l}-\frac{3}{2})}$, $\mu = \frac{\frac{3}{s}+\frac{3}{l}-2}{2(\frac{3}{s}+\frac{2}{l}-\frac{3}{2})}$, $\gamma = \frac{\frac{2}{s}+\frac{1}{l}-1}{2(\frac{3}{s}+\frac{2}{l}-\frac{3}{2})}$, and integrating from $t_0 - \rho^2$ to t_0 with the variable t, it follows from the Hölder and Young inequalities that

$$\begin{aligned}
C(\rho) &\le c\rho^{\mu-2}A^{\mu}(\rho)\left(\int_{Q(z_0,\rho)} \left(|\nabla u|^2 + \left|\nabla^2 d\right|^2\right) + \frac{1}{\rho^2}\left(|u|^2 + |\nabla d|^2\right) \mathrm{d}z\right)^{\frac{1}{q'}} \\
&\quad \times \left[\int_{t_0-\rho^2}^{t_0} \left(\int_{B(x_0,\rho)} \left(|u|^s + |\nabla d|^s\right) \mathrm{d}x\right)^{\frac{l}{s}} \mathrm{d}t\right]^{\frac{1}{q}} \\
&\le c\rho^{\mu-2}A^{\mu}(\rho)\rho^{\frac{1}{q'}}\big(E(\rho) + H(\rho)\big)^{\frac{1}{q'}}\big(\rho^{\kappa}M^{s,l}(\rho)\big)^{\frac{1}{q}} \\
&\le cA^{\mu}(\rho)\big(E(\rho) + H(\rho)\big)^{\frac{1}{q'}}\big(M^{s,l}(\rho)\big)^{\frac{1}{q}} \\
&\le c\epsilon^{\frac{1}{q}}A^{\mu}(\rho)\big(E(\rho) + H(\rho)\big)^{\frac{1}{q'}} \\
&\le c\epsilon^{\frac{1}{q}}\big(A^{\mu q}(\rho) + E(\rho) + H(\rho)\big) \\
&\le c\epsilon^{\frac{1}{q}}\big(E(\rho) + A(\rho) + 1\big),
\end{aligned}$$

where $\kappa = \frac{3l}{s} + 2 - l$ as in Theorem 1.2, and in the last step, we used the fact that $\mu q \leq 1, H(\rho) \leq A(\rho)$. □

3 Proof of Theorem 1.2

Due to the induction argument as Proposition 2.6 in [10] or Lemma 2.2 in [19] (the parabolic version of the Campanato criterion), to get the desired consequence, it suffices to prove $C(\theta^k) + D(\theta^k) < \epsilon_0$ for some small ϵ_0. Here θ is a small number, which will be chosen later.

From the generalized energy inequality, it is easy to check that, for $\rho \in (0, R]$,

$$A\left(\frac{\rho}{2}\right) + E\left(\frac{\rho}{2}\right) \leq c\left[C^{\frac{2}{3}}(\rho) + C(\rho) + D(\rho)\right].$$

Denoting $G(\rho) = A(\rho) + E(\rho) + D(\rho)$, due to Lemmas 2.1-2.2, and the fact that $C(2\theta\rho) \leq \frac{1}{4\theta^2}C(\rho)$, we can get

$$\begin{aligned} G(\theta\rho) &\leq c\left[C^{\frac{2}{3}}(2\theta\rho) + C(2\theta\rho) + D(2\theta\rho) + \theta D(\rho) + \frac{1}{\theta^2}C(\rho)\right] \\ &\leq c\left[\frac{1}{\theta^{\frac{4}{3}}}C^{\frac{2}{3}}(\rho) + \frac{1}{\theta^2}C(\rho) + \theta D(\rho)\right] \\ &\leq c\left[\frac{\epsilon^{\frac{2}{3q}}}{\theta^{\frac{4}{3}}}\big(G(\rho)+1\big)^{\frac{2}{3}} + \frac{\epsilon^{\frac{1}{q}}}{\theta^2}\big(G(\rho)+1\big) + \theta G(\rho)\right] \\ &\leq c\left[\left(\theta + \frac{\epsilon^{\frac{1}{2q}}}{\theta^2}\right)G(\rho) + \frac{\epsilon^{\frac{1}{2q}}}{\theta^2}\right], \end{aligned}$$

where in the last step we have used $\frac{\epsilon^{\frac{2}{3q}}}{\theta^{\frac{4}{3}}}(G(\rho)+1)^{\frac{2}{3}} \leq c[\epsilon^{\frac{1}{q}} + \frac{\epsilon^{\frac{1}{2q}}}{\theta^2}(G(\rho)+1)]$. Now choosing θ and ϵ such that $c\theta < \frac{1}{4}$ and $c\frac{\epsilon^{\frac{1}{2q}}}{\theta^2} < \frac{1}{4}$, then it yields $G(\theta\rho) \leq \frac{1}{2}G(\rho) + c\frac{\epsilon^{\frac{1}{2q}}}{\theta^2}$. Iterating the above process, we obtain $G(\theta^k\rho) \leq \frac{1}{2^k}G(\rho) + c\frac{\epsilon^{\frac{1}{2q}}}{\theta^2}$, therefore,

$$D\big(\theta^k\rho\big) \leq \frac{1}{2^k}G(\rho) + c\frac{\epsilon^{\frac{1}{2q}}}{\theta^2}. \tag{3.1}$$

For $C(\theta^k\rho)$, by Lemma 2.2, we have

$$C\big(\theta^k\rho\big) \leq c\epsilon^{\frac{1}{q}}\left[G\big(\theta^k\rho\big) + 1\right] \leq c\epsilon^{\frac{1}{q}}\left[\frac{1}{2^k}G(\rho) + \frac{\epsilon^{\frac{1}{2q}}}{\theta^2} + 1\right] \leq c\left[\frac{1}{2^k}G(\rho) + \frac{\epsilon^{\frac{1}{2q}}}{\theta^2}\right], \tag{3.2}$$

where in the last step we use the fact that $\epsilon^{\frac{1}{q}} \leq \frac{\epsilon^{\frac{1}{2q}}}{\theta^2}$ for ϵ small enough. With these inequalities in hand, for fixed ρ and ϵ_0, we can choose k_0 large enough such that $c\frac{1}{2^{k_0}}G(\rho) < \frac{\epsilon_0}{4}$, and choose ϵ small enough, such that $c\frac{\epsilon^{\frac{1}{2q}}}{\theta^2} < \frac{\epsilon_0}{4}$. With these prerequisites and (3.1)-(3.2), it follows that $D(\theta^k\rho) + C(\theta^k\rho) < \epsilon_0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally in writing this paper. They both read and approved the final manuscript.

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