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The existence and concentration of ground-state solutions for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents

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Abstract

We are concerned with ground-state solutions for the following Kirchhoff type equation with critical nonlinearity:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x)u = \lambda W(x)|u|^{p-2}u + |u|^4 u & \text{in } \mathbb{R}^3, \\ u > 0, \qquad u \in H^1(\mathbb{R}^3), \end{cases}$$

where ε is a small positive parameter, $a, b > 0$, $\lambda > 0$, $2 < p \leq 4$, V and W are two potentials. Under proper assumptions, we prove that, for $\varepsilon > 0$ sufficiently small, the above problem has a positive ground-state solution u_ε by using a monotonicity trick and a new version of global compactness lemma. Moreover, we use another global compactness method due to Gui (Commun. Partial Differ. Equ. 21:787-820, 1996) to show that u_ε is concentrated around a set which is related to the set where the potential $V(x)$ attains its global minima or the set where the potential $W(x)$ attains its global maxima as $\varepsilon \to 0$.

MSC: Primary 35J20; 35J60; 35J92

Keywords: existence; concentration; Kirchhoff type equation; critical growth

1 Introduction

In this paper, we study the following Kirchhoff type equation with critical nonlinearity:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x)u = \lambda W(x)|u|^{p-2}u + |u|^4 u & \text{in } \mathbb{R}^3, \\ u > 0, \qquad u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where ε is a small positive parameter, $a, b > 0$, $\lambda > 0$, $2 < p \leq 4$.

Problem (1.1) is a variant type of the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain. Such problems are often referred to as nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2)\Delta u$, which implies that equation (1.2) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. On the other hand, problem (1.2) is related to the stationary analog of the equation

$$\begin{cases} u_{tt} - (a + b\int_\Omega |\nabla_x u|^2)\Delta_x u = f(x,u) \quad (x \in \Omega, t > 0), \\ u(\cdot,t)|_{\partial\Omega} = 0 \quad (t \geq 0), \end{cases} \tag{1.3}$$

proposed by Kirchhoff in [2] as the existence of the classical D'Alembert wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.3), u denotes the displacement, $f(x,u)$ the external force and b the initial tension, while a is related to the intrinsic properties of the string (such as Young's modulus). We have to point out that nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself (for example, the population density). After the pioneer work of Lions [3], where a functional analysis approach was proposed, the Kirchhoff type equations began to arouse the attention of researchers.

In [4], Alves, Corrêa and Ma used the mountain pass theorem to get the existence result of the following Kirchhoff type problem:

$$\begin{cases} M(\int_\Omega |\nabla u|^2)\Delta u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, M is a positive function, and f is of subcritical growth.

In [5], Arosio and Panizzi proved the well-posedness (existence, uniqueness and continuous dependence of the local solution upon the initial data) of the Cauchy-Dirichlet type problem related to (1.3) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space.

In [6], Perera and Zhang studied (1.2) under the conditions $N = 1,2,3$, f is a Carathéodory function on $\Omega \times R$ and satisfies $\lim_{t\to 0} \frac{f(x,t)}{at} = \lambda$, $\lim_{|t|\to\infty} \frac{f(x,t)}{bt^3} = \mu$ uniformly for $x \in \Omega$. They used the Yang index and critical group to obtain a nontrivial solution of (1.2).

In [7], He and Zou considered and obtained infinitely many solutions of (1.2) by using a local minimum method and the fountain theorem.

In [8], Chen *et al.* considered the following Kirchhoff type equation:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $1 < q < 2 < p < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1,2$), the weight function $f, g \in C(\bar{\Omega})$ satisfies $\max\{f, 0\} \not\equiv 0$ and $\max\{g, 0\} \not\equiv 0$. By using the Nehari manifold and fibering map methods, multiple positive solutions were obtained under proper assumptions.

Recently, in [9], Li and Ye studied

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u + V(x)u = |u|^{p-2}u, & x\in\mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \qquad u>0, \qquad x\in\mathbb{R}^3, & 3<p<6, \end{cases} \tag{1.4}$$

and the potential V satisfies

(V_1) $V(x)\in C(\mathbb{R}^3,\mathbb{R})$ is weakly differentiable and satisfies $(\nabla V(x),x)\in L^{\frac{3}{2}}(\mathbb{R}^3)\cup L^\infty(\mathbb{R}^3)$ and $V(x)-(\nabla V(x),x)\geq 0$ a.e. $x\in\mathbb{R}^3$.

(V_2) $V(x)\leq \liminf_{|y|\to+\infty}V(y)<+\infty$ and the inequality is strict in a subset of positive Lebesgue measure.

(V_3) $\inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|\nabla u|^2+V(x)u^2}{\int_{\mathbb{R}^3}u^2}>0$.

They proved that (1.4) has a positive ground-state solution. For more results, we can refer to [7, 10–13] and the references therein.

We note that problem (1.4) with $b=0$ is motivated by the search for standing wave solutions for the nonlinear Schrödinger equation, which is one of the main subjects in nonlinear analysis. Different approaches have been taken to deal with this problem under various hypotheses on the potentials and the nonlinearities (see [14, 15] and so on).

Our motivation to study (1.1) mainly comes from the results of perturbed Schrödinger equations, *i.e.*

$$-\varepsilon^2\Delta u + V(x)u = |u|^{q-2}u, \quad x\in\mathbb{R}^N, \tag{1.5}$$

where $2<q<2^*$, $N\geq 1$.

Many mathematicians proved the existence, concentration and multiplicity of solutions for (1.5), we refer to [1, 16–18].

Under the condition

(V_4) $V_\infty=\liminf_{|x|\to\infty}V(x)>V_0=\inf_{x\in\mathbb{R}^N}V(x)>0$

on $V(x)$, He and Zou in [19] studied (1.1) with the nonlinearity replaced by $f(u)$, where $f\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ and satisfies

(AR) $\exists\mu>4$ such that

$$0<\mu\int_0^u f(s)\,ds\leq f(u)u \quad \text{for all } u\geq 0,$$

$\lim_{s\to 0}\frac{f(s)}{s^3}=0$, $\lim_{|s|\to\infty}\frac{f(s)}{|s|^q}=0$ for some $3<q<5$ and $\frac{f(s)}{s^3}$ is strictly increasing for $s>0$. They obtained the existence, concentration and multiplicity of solutions for (1.1) by the same arguments as in [16–18]. In [20], Wang *et al.* extended the result of [19] with the case that the nonlinearity is of critical growth.

2 Main results

Before stating our theorem, we first give some notations. Set

$$\tau := \min_{\mathbb{R}^3} V, \qquad \mathcal{V} := \left\{x\in\mathbb{R}^3 : V(x)=\tau\right\}, \qquad \tau_\infty := \lim_{|x|\to\infty} V(x),$$

$$\kappa := \max_{\mathbb{R}^3} W, \qquad \mathcal{W} := \left\{x\in\mathbb{R}^3 : W(x)=\kappa\right\}, \qquad \kappa_\infty := \lim_{|x|\to\infty} W(x).$$

We will use the following hypotheses on the potentials:

(P_1) V and W are bounded locally Hölder continuous functions with $\tau > 0$ and $\inf_{\mathbb{R}^3} W > 0$.

(P_2) Either (i) $\tau < \tau_\infty$ and there exist $R > 0$, $x_v \in \mathcal{V}$ such that $W(x_v) \geq W(x)$ for all $|x| \geq R$, or (ii) $\kappa > \kappa_\infty$ and there exist $R > 0$, $x_w \in \mathcal{W}$ such that $V(x_w) \leq V(x)$ for all $|x| \geq R$.

(P_3) V and W are weakly differentiable and satisfy

$$\big(\nabla V(x), x\big) \in L^{r_1}\big(\mathbb{R}^3\big) \quad \text{for some } r_1 \in \left[\frac{3}{2}, \infty\right]$$

and

$$\big(\nabla W(x), x\big) \in L^{r_2}\big(\mathbb{R}^3\big) \quad \text{for some } r_2 \in \left[\frac{6}{6-p}, \infty\right]$$

with

$$(q-2)V(x) - 2\big(\nabla V(x), x\big) \geq 0, \qquad (p-q)W(x) + 2\big(\nabla W(x), x\big) \geq 0, \quad \text{a.e. } \mathbb{R}^3$$

for some $2 < q < p$, where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^3$.

Note that the idea of introducing condition (P_2) is actually due to Ding. In [21], Ding and Liu studied the existence and concentration of semiclassical solutions for Schrödinger equations with magnetic fields under the condition (P_2). It seems that, under the conditions (P_1), (P_2), the existence and concentration behavior of positive solutions to (1.1) have not ever been studied. So in this paper we shall fill this gap. Precisely, we will find a family of positive ground-state solutions for (1.1) with some properties, such as concentration and exponential decay.

Observe that, in case (P_2)-(i), we can assume that $W(x_v) = \max_{x \in \mathcal{V}} W(x)$ and set

$$\mathcal{A}_v := \big\{x \in \mathcal{V} : W(x) = W(x_v)\big\} \cup \big\{x \notin \mathcal{V} : W(x) > W(x_v)\big\},$$

in case (P_2)-(ii), we can assume that $V(x_w) = \min_{x \in \mathcal{W}} V(x)$ and set

$$\mathcal{A}_w := \big\{x \in \mathcal{W} : V(x) = V(x_w)\big\} \cup \big\{x \notin \mathcal{W} : V(x) < V(x_w)\big\}.$$

Obviously, $\mathcal{A}_v$ and $\mathcal{A}_w$ are bounded. Moreover, $\mathcal{A}_v = \mathcal{A}_w = \mathcal{V} \cap \mathcal{W}$ if $\mathcal{V} \cap \mathcal{W} \neq \emptyset$. In particular, $\mathcal{A}_v = \mathcal{V}$ if W is a constant and $\mathcal{A}_w = \mathcal{W}$ if V is a constant.

Our main results are as follows.

Theorem 2.1 *Let* (P_1), (P_3) *holds.* (A) *Suppose* (P_2)-(i) *holds.*

(a_1) *There exist* $\lambda^* > 0$ *and* $\varepsilon^* > 0$ *such that, for each* $\lambda \in [\lambda^*, \infty)$ *and* $\varepsilon \in (0, \varepsilon^*)$, (1.1) *possesses a positive ground-state solution* $u_\varepsilon \in H^1(\mathbb{R}^3)$. *If additionally, V and W are uniformly continuous functions on* $\mathbb{R}^3$, *then* u_ε *satisfies*:

(a_2) *there exists a maximum point* x_ε *of* u_ε *with*

$$\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathcal{A}_v) = 0,$$

(a_3) $\exists C_1, C_2 > 0$,

$$u_\varepsilon(x) \le C_1 \exp\left(-\frac{C_2}{\varepsilon}|x - x_\varepsilon|\right).$$

(B) *Suppose* (P_2)-(ii) *holds, then all the conclusions of* (A) (*with* $\mathcal{A}_v$ *replaced by* $\mathcal{A}_w$) *remain true.*

The proof is based on the variational method. The main difficulties in proving Theorem 2.1 lie in the fact that the nonlinearity $\lambda W(x)|u|^{p-2}u + |u|^4 u$ $(2 < p \le 4)$ does not satisfy the (AR) condition, which prevents us from obtaining a bounded (PS) sequence and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^3$ and the nonlinearity with the critical Sobolev growth. As we will see later, the competing effect of $\lambda W(x)|u|^{p-2}u + |u|^4 u$ $(2 < p \le 4)$ and the lack of compactness of the embedding prevent us from using the variational method in a standard way.

To overcome these difficulties, inspired by [22], we use a proposition due to Jeanjean (Proposition 2.2 below) to construct a special bounded (PS) sequence and we recover the compactness by using a version of global compactness lemma (Lemma 3.4 below).

To complete this section, we sketch our proof.

We will work with the following equation, equivalent to (1.1):

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(\varepsilon x)u = \lambda W(\varepsilon x)|u|^{p-2}u + |u|^4 u & \text{in } \mathbb{R}^3, \\ u > 0, \qquad u \in H^1(\mathbb{R}^3), \end{cases} \tag{2.1}$$

with the energy functional

$$\begin{aligned} I_\varepsilon(u) = {} & \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ & - \frac{\lambda}{p}\int_{\mathbb{R}^3} W(\varepsilon x)\left(u^+\right)^p - \frac{1}{6}\int_{\mathbb{R}^3} \left(u^+\right)^6, \quad u \in H^1\left(\mathbb{R}^3\right). \end{aligned}$$

We can easily check that I_ε possesses the mountain-pass geometry. But it is difficult to get the boundedness of any (PS) sequence for $2 < p \le 4$. To overcome this difficulty, in the spirit of [9, 13], we use the following proposition due to Jeanjean [22].

Proposition 2.2 (Theorem 1.1 of [22]) *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval, we consider a family $\{\Phi_\mu\}_{\mu\in J}$ of C^1-functional on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \ge 0$, $\forall u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\mu\left(\gamma(t)\right) > \max\left\{\Phi_\mu(v_1), \Phi_\mu(v_2)\right\}, \quad \forall \mu \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in J$, there is a bounded $(PS)_{c_\mu}$ sequence for Φ_μ, that is, there is a sequence $\{u_n(\mu)\} \subset X$ such that

(i) *$\{u_n(\mu)\}$ is bounded in X,*

(ii) $\Phi_\mu(u_n(\mu)) \to c_\mu$,

(iii) *$\Phi'_\mu(u_n(\mu)) \to 0$ in X^{-1}, where X^{-1} is the dual space of X.*

Applying Proposition 2.2 to the following functional:

$$\begin{aligned} I_{\varepsilon,\mu}(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \mu\left[\frac{\lambda}{p}\int_{\mathbb{R}^3} W(\varepsilon x)(u^+)^p + \frac{1}{6}\int_{\mathbb{R}^3} (u^+)^6\right], \quad u \in H^1(\mathbb{R}^3), \mu \in [1-\delta_0, 1], \end{aligned}$$

then, for a.e. $\mu \in [1-\delta_0, 1]$, $\varepsilon > 0$ small but fixed, there exists a bounded $(\mathrm{PS})_{c_{\varepsilon,\mu}}$ sequence $\{u_n\}$ for $I_{\varepsilon,\mu}$ in $H^1(\mathbb{R}^3)$, where $c_{\varepsilon,\mu}$, δ_0 are given below.

In order to prove that $I_{c_{\varepsilon,\mu}}$ satisfies the $(\mathrm{PS})_{c_{\varepsilon,\mu}}$ condition, inspired by [9], we will establish a version of global compactness lemma (Lemma 3.4 below).

At last, we note that the concentration result in Theorem 2.1 is obtained by using a similar method which is related to Proposition 2.2 in [1].

3 Proof of Theorem 2.1

The equation

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + u = \lambda|u|^{p-2}u + |u|^4u & \text{in } \mathbb{R}^3, \\ u > 0, \quad u \in H^1(\mathbb{R}^3), \end{cases} \tag{3.1}$$

is the limiting equation of (1.1). In view of [23], we have the following.

Proposition 3.1 *Equation* (3.1) *has a positive ground-state solution $\tilde{u} \in H^1(\mathbb{R}^3)$ with $c < \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{\frac{3}{2}}$, where c is the least energy level of* (3.1).

Equation (1.1) can be rewritten as

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(\varepsilon x)u = \lambda W(\varepsilon x)|u|^{p-2}u + |u|^4u & \text{in } \mathbb{R}^3, \\ u > 0, \quad u \in H^1(\mathbb{R}^3), \end{cases} \tag{3.2}$$

and the corresponding energy functional is

$$\begin{aligned} I_\varepsilon(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \frac{\lambda}{p}\int_{\mathbb{R}^3} W(\varepsilon x)(u^+)^p - \frac{1}{6}\int_{\mathbb{R}^3} (u^+)^6, \quad u \in H^1(\mathbb{R}^3). \end{aligned}$$

Since V is bounded and $\tau := \min_{\mathbb{R}^3} V > 0$,

$$\|u\|_\varepsilon := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(\varepsilon x)u^2 \right)^{\frac{1}{2}}$$

is an equivalent norm in $H^1(\mathbb{R}^3)$.

By Proposition 3.1, for any $x_0 \in \mathbb{R}^3$, let w_μ be a positive ground-state solution to the equation

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x_0)u = \mu[\lambda W(x_0)|u|^{p-2}u + |u|^4 u] & \text{in } \mathbb{R}^3, \\ u > 0, \qquad u \in H^1(\mathbb{R}^3), \qquad 0 < \mu \le 1, \end{cases}$$

with the energy functional

$$\begin{aligned} I_{V(x_0),W(x_0),\mu}(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x_0)u^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \frac{1}{p}\mu\lambda \int_{\mathbb{R}^3} W(x_0)\left(u^+\right)^p - \frac{1}{6}\mu\int_{\mathbb{R}^3} \left(u^+\right)^6, \quad u \in H^1\left(\mathbb{R}^3\right), 0 < \mu \le 1. \end{aligned}$$

Denote the mountain-pass level of $I_{V(x_0),W(x_0),\mu}$ by $c_{V(x_0),W(x_0),\mu}$. From [23], we see that

$$\begin{aligned} c_{V(x_0),W(x_0),\mu} &:= \inf_{\gamma \in \Gamma_{V(x_0),W(x_0),\mu}} \max_{t \in [0,1]} I_{V(x_0),W(x_0),\mu}\left(\gamma(t)\right) \\ &= \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_{V(x_0),W(x_0),\mu}(u_t) = \inf_{u \in \mathcal{M}_{V(x_0),W(x_0),\mu}} I_{V(x_0),W(x_0),\mu}(u) > 0, \end{aligned}$$

where

$$\begin{aligned} \Gamma_{V(x_0),W(x_0),\mu} &:= \left\{\gamma \in C\left([0,1], H^1\left(\mathbb{R}^3\right)\right) : \gamma(0) = 0, I_{V(x_0),W(x_0),\mu}\left(\gamma(1)\right) < 0\right\}, \\ \mathcal{M}_{V(x_0),W(x_0),\mu} &:= \left\{u \in H^1\left(\mathbb{R}^3\right)\setminus\{0\} : G_{V(x_0),W(x_0),\mu}(u) = 0\right\} \end{aligned}$$

and

$$\begin{aligned} G_{V(x_0),W(x_0),\mu}(u) &= 2a\int_{\mathbb{R}^3} |\nabla u|^2 + 4\int_{\mathbb{R}^3} V(x_0)u^2 + 2b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \mu\left[\frac{p+6}{p}\lambda\int_{\mathbb{R}^3} W(x_0)\left(u^+\right)^p + 2\int_{\mathbb{R}^3} \left(u^+\right)^6\right]. \end{aligned}$$

We have the following lemma.

Lemma 3.2 *For any $\{\mu_n\}$ with $\mu_n \to 1^-$, up to a subsequence, $\exists\{y_n\} \subset \mathbb{R}^3$ such that $\{w_{\mu_n}(x + y_n)\}$ is convergent in $H^1(\mathbb{R}^3)$.*

Proof Since

$$\begin{aligned} c_{V(x_0),W(x_0),\frac{1}{2}} &\ge c_{V(x_0),W(x_0),\mu_n} \\ &= I_{V(x_0),W(x_0),\mu_n}(w_{\mu_n}) - \frac{1}{p+6}G_{V(x_0),W(x_0),\mu_n}(w_{\mu_n}) \end{aligned}$$

$$\begin{aligned}&= \frac{p+2}{2(p+6)} a \int_{\mathbb{R}^3} |\nabla w_{\mu_n}|^2 + \frac{p-2}{2(p+6)} \int_{\mathbb{R}^3} V(x_0) w_{\mu_n}^2 \\ &\quad + \frac{p-2}{4(p+6)} b \left(\int_{\mathbb{R}^3} |\nabla w_{\mu_n}|^2 \right)^2 + \frac{6-p}{6(p+6)} \mu_n \int_{\mathbb{R}^3} w_{\mu_n}^6,\end{aligned}$$

$\{w_{\mu_n}\}$ is bounded in $H^1(\mathbb{R}^3)$.

By the vanishing theorem, $\exists \{y_n\} \subset \mathbb{R}^3$ and set $\tilde{w}_{\mu_n}(x) := w_{\mu_n}(x + y_n)$, we may assume that $\exists \tilde{w} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\begin{cases} \tilde{w}_{\mu_n} \rightharpoonup \tilde{w} & \text{in } H^1(\mathbb{R}^3), \\ \tilde{w}_{\mu_n} \to \tilde{w} & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } 1 \le s < 6, \\ \tilde{w}_{\mu_n} \to \tilde{w} & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Moreover, $\tilde{w}$ satisfies

$$-\left(a + bA^2\right) \Delta \tilde{w} + V(x_0) \tilde{w} = \lambda W(x_0) \left(\tilde{w}^+\right)^{p-1} + \left(\tilde{w}^+\right)^5,$$

where $A^2 = \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla \tilde{w}_{\mu_n}|^2$ and $\int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 \le A^2$.

Next, we claim that

$$\lim_{n\to\infty} c_{V(x_0),W(x_0),\mu_n} = c_{V(x_0),W(x_0),1}. \tag{3.3}$$

Indeed, $\exists t_n > 0$ such that $(w_1)_{t_n} \in \mathcal{M}_{V(x_0),W(x_0),\mu_n}$, then $\frac{dI_{V(x_0),W(x_0),\mu_n}((w_1)_t)}{dt}|_{t=t_n} = 0$ shows that $\{t_n\}$ is bounded. Hence, we have

$$\begin{aligned} c_{V(x_0),W(x_0),1} &\le c_{V(x_0),W(x_0),\mu_n} \le I_{V(x_0),W(x_0),\mu_n}\left((w_1)_{t_n}\right) \\ &= I_{V(x_0),W(x_0),1}\left((w_1)_{t_n}\right) + \frac{1}{p}(1-\mu_n)\lambda \int_{\mathbb{R}^3} W(x_0)(w_1)_{t_n}^p + \frac{1}{6}(1-\mu_n) \int_{\mathbb{R}^3} (w_1)_{t_n}^6 \\ &\le I_{V(x_0),W(x_0),1}(w_1) + o(1) = c_{V(x_0),W(x_0),1} + o(1), \end{aligned}$$

(3.3) holds.

Since $\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla \tilde{w}_{\mu_n}|^2 \ge \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2$, we check that $G_{V(x_0),W(x_0),1}(\tilde{w}) \le 0$, then by (3.3), we get $\tilde{w}_{\mu_n} \to \tilde{w}$ in $H^1(\mathbb{R}^3)$. □

By Lemma 3.2, $\tilde{w}_{\mu_n}^6$, $\tilde{w}_{\mu_n}^p$, $\tilde{w}_{\mu_n}^2$ are uniformly integrable near ∞. Since $\{\mu_n\}$ is arbitrary, then $\exists \delta_0 > 0$ small but fixed, $\{y_\mu\} \subset \mathbb{R}^3$ for all $\mu \in [1-\delta_0, 1]$,

$$\tilde{w}_\mu^6, \tilde{w}_\mu^p, \tilde{w}_\mu^2 \text{ are uniformly integrable near } \infty, \tag{3.4}$$

where $\tilde{w}_\mu(x) := w_\mu(x + y_\mu)$.

Next, we will show that $\exists \bar{C} > 0$ which is independent of $\mu \in [1-\delta_0, 1]$ such that

$$\int_{\mathbb{R}^3} w_\mu^p + \int_{\mathbb{R}^3} w_\mu^6 \ge \bar{C}. \tag{3.5}$$

Indeed, assuming the contrary, $\exists \{\mu_j\} \subset [1-\delta_0, 1]$ with $\mu_j \to 1^-$ such that

$$\int_{\mathbb{R}^3} w_{\mu_j}^p + \int_{\mathbb{R}^3} w_{\mu_j}^6 \to 0 \quad \text{as } j \to \infty.$$

In view of the definition of w_{μ_j},

$$a\int_{\mathbb{R}^3}|\nabla w_{\mu_j}|^2+\int_{\mathbb{R}^3}V(x_0)w_{\mu_j}^2+b\left(\int_{\mathbb{R}^3}|\nabla w_{\mu_j}|^2\right)^2=\lambda\mu_j\int_{\mathbb{R}^3}W(x_0)w_{\mu_j}^p+\mu_j\int_{\mathbb{R}^3}w_{\mu_j}^6,$$

then $\|w_{\mu_j}\|_{H^1(\mathbb{R}^3)}\to 0$ as $j\to\infty$, which contradicts $c_{V(x_0),W(x_0),1}>0$ by (3.3).

Consider the following functional:

$$\begin{aligned}I_{\varepsilon,\mu}(u)&=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\\&\quad-\mu\left[\frac{\lambda}{p}\int_{\mathbb{R}^3}W(\varepsilon x)(u^+)^p+\frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\right],\quad u\in H^1(\mathbb{R}^3),\mu\in[1-\delta_0,1].\end{aligned}$$

Denote

$$A(u):=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2$$

and

$$B(u):=\left[\frac{\lambda}{p}\int_{\mathbb{R}^3}W(\varepsilon x)(u^+)^p+\frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\right],$$

we will show that $A(u)$ and $B(u)$ satisfy the conditions of Proposition 2.2 for $\varepsilon>0$ small.

For any $u\in H^1(\mathbb{R}^3)$,

$$\frac{\lambda}{p}\int_{\mathbb{R}^3}W(\varepsilon x)(u^+)^p+\frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6\ge 0$$

and

$$\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\to+\infty\quad\text{as }\|u\|_{H^1(\mathbb{R}^3)}\to\infty.$$

Set $W_{\varepsilon,\mu,t}(x):=t\eta(\sqrt{\varepsilon}\frac{x}{t^2}-\frac{x_0}{\sqrt{\varepsilon}t^2})\tilde{w}_\mu(\frac{x}{t^2}-\frac{x_0}{\varepsilon t^2})$, where η is a smooth cut-off function with $0\le\eta\le 1$, $\eta=1$ on $B_1(0)$, $\eta=0$ on $\mathbb{R}^3\backslash B_2(0)$, $|\nabla\eta|\le C$.

Since $\delta_0>0$ is small, we may assume that $1-\delta_0>\frac{1}{2}$, then $I_{\varepsilon,\mu}(u)\le I_{\|V\|_{L^\infty},\inf W,\frac{1}{2}}(u)$ and

$$\begin{aligned}&I_{\|V\|_{L^\infty},\inf W,\frac{1}{2}}(W_{\varepsilon,\mu,t})\\&\quad=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,\mu,t}|^2+\frac{1}{2}\|V\|_{L^\infty}\int_{\mathbb{R}^3}W_{\varepsilon,\mu,t}^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,\mu,t}|^2\right)^2\\&\qquad-\frac{\lambda}{2p}\inf W\int_{\mathbb{R}^3}W_{\varepsilon,\mu,t}^p-\frac{1}{12}\int_{\mathbb{R}^3}W_{\varepsilon,\mu,t}^6\\&\quad\overset{x'=\frac{x}{t^2}-\frac{x_0}{\varepsilon t^2}}{=}\frac{a}{2}t^4\int_{\mathbb{R}^3}\left|\nabla\eta(\sqrt{\varepsilon}x')\sqrt{\varepsilon}\tilde{w}_\mu(x')+\eta(\sqrt{\varepsilon}x')\nabla\tilde{w}_\mu(x')\right|^2\\&\qquad+\frac{1}{2}\|V\|_{L^\infty}t^8\int_{\mathbb{R}^3}\eta^2(\sqrt{\varepsilon}x')\tilde{w}_\mu^2(x')\\&\qquad+\frac{b}{4}t^8\left(\int_{\mathbb{R}^3}\left|\nabla\eta(\sqrt{\varepsilon}x')\sqrt{\varepsilon}\tilde{w}_\mu(x')+\eta(\sqrt{\varepsilon}x')\nabla\tilde{w}_\mu(x')\right|^2\right)^2\end{aligned}$$

$$\begin{aligned}
&-\frac{\lambda}{2p}\inf_{\mathbb{R}^3} W t^{p+6}\int_{\mathbb{R}^3}\eta^p\left(\sqrt{\varepsilon}x'\right)\tilde{w}_\mu^p\left(x'\right)\\
&-\frac{1}{12}t^{12}\int_{\mathbb{R}^3}\eta^6\left(\sqrt{\varepsilon}x'\right)\tilde{w}_\mu^6\left(x'\right)\\
\leq{}& Ct^4\left(\int_{\mathbb{R}^3}\tilde{w}_\mu^2+\int_{\mathbb{R}^3}|\nabla\tilde{w}_\mu|^2\right)+Ct^8\left(\left(\int_{\mathbb{R}^3}\tilde{w}_\mu^2\right)^2+\left(\int_{\mathbb{R}^3}|\nabla\tilde{w}_\mu|^2\right)^2\right)+Ct^8\int_{\mathbb{R}^3}\tilde{w}_\mu^2\\
&-\frac{\lambda}{2p}\inf_{\mathbb{R}^3} W t^{p+6}\int_{B_{1/\sqrt{\varepsilon}}(0)}\tilde{w}_\mu^p-\frac{1}{12}t^{12}\int_{B_{1/\sqrt{\varepsilon}}(0)}\tilde{w}_\mu^6\to-\infty
\end{aligned}$$

as $t\to+\infty$ uniformly for all $\varepsilon>0$ small and $\mu\in[1-\delta_0,1]$, where we have used (3.4) and (3.5). Taking $t_0>0$ large, we get

$$I_{\varepsilon,\mu}(W_{\varepsilon,\mu,t_0})\leq I_{\|V\|_{L^\infty},\inf W,\frac{1}{2}}(W_{\varepsilon,\mu,t_0})<-2$$

for all $\mu\in[1-\delta_0,1]$, $\varepsilon>0$ small.

Using the Sobolev embedding theorem, we have

$$\begin{aligned}
I_{\varepsilon,\mu}(u)&\geq\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{\tau}{2}\int_{\mathbb{R}^3}u^2-\frac{\kappa}{p}\lambda\int_{\mathbb{R}^3}\left(u^+\right)^p-\frac{1}{6}\int_{\mathbb{R}^3}\left(u^+\right)^6\\
&\geq C\|u\|_{H^1(\mathbb{R}^3)}^2-C\lambda\|u\|_{H^1(\mathbb{R}^3)}^p-C\|u\|_{H^1(\mathbb{R}^3)}^6>0
\end{aligned}$$

for all $u\in H^1(\mathbb{R}^3)$ with $\|u\|_{H^1(\mathbb{R}^3)}$ small since $p>2$.

Hence, we can define

$$c_{\varepsilon,\mu}:=\inf_{\gamma\in\Gamma_\mu}\max_{t\in[0,1]}I_{\varepsilon,\mu}\left(\gamma(t)\right)>\max\left\{I_{\varepsilon,\mu}(0),I_{\varepsilon,\mu}(W_{\varepsilon,\mu,t_0})\right\}$$

for all $\mu\in[1-\delta_0,1]$, $\varepsilon>0$ small, where

$$\Gamma_\mu:=\left\{\gamma\in C\left([0,1],H^1\left(\mathbb{R}^3\right)\right):\gamma(0)=0,\gamma(1)=W_{\varepsilon,\mu,t_0}\right\}.$$

Lemma 3.3 *For any $x_0\in\mathbb{R}^3$, $\lim_{\varepsilon\to0}c_{\varepsilon,\mu}\leq c_{V(x_0),W(x_0),\mu}$ uniformly for all $\mu\in[1-\delta_0,1]$.*

Proof Define $W_{\varepsilon,\mu,0}:=\lim_{t\to0}W_{\varepsilon,\mu,t}$ in $H^1(\mathbb{R}^3)$ sense, then $W_{\varepsilon,\mu,0}=0$. Thus, setting $\gamma_\mu(s):=W_{\varepsilon,\mu,st_0}$ $(0\leq s\leq1)$, we have $\gamma_\mu\in\Gamma_\mu$, then

$$c_{\varepsilon,\mu}\leq\max_{s\in[0,1]}I_{\varepsilon,\mu}\left(\gamma_\mu(s)\right)=\max_{t\in[0,t_0]}I_{\varepsilon,\mu}(W_{\varepsilon,\mu,t})$$

and we just need to verify that

$$\lim_{\varepsilon\to0}\max_{t\in[0,t_0]}I_{\varepsilon,\mu}(W_{\varepsilon,\mu,t})\leq c_{V(x_0),W(x_0),\mu}\tag{3.6}$$

uniformly for $\mu\in[1-\delta_0,1]$.

Indeed,

$$\begin{aligned}
&\max_{t\in[0,t_0]}I_{\varepsilon,\mu}(W_{\varepsilon,\mu,t})\\
&\overset{x'=\frac{x}{t^2}-\frac{x_0}{\varepsilon t^2}}{=}\max_{t\in[0,t_0]}\frac{a}{2}t^4\int_{\mathbb{R}^3}\left|\nabla\eta\left(\sqrt{\varepsilon}x'\right)\sqrt{\varepsilon}\tilde{w}_\mu\left(x'\right)+\eta\left(\sqrt{\varepsilon}x'\right)\nabla\tilde{w}_\mu\left(x'\right)\right|^2
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{2}t^8\int_{\mathbb{R}^3}V(\varepsilon t^2x'+x_0)\eta^2(\sqrt{\varepsilon}x')\tilde{w}_\mu^2(x')\\
&+\frac{b}{4}t^8\left(\int_{\mathbb{R}^3}\left|\nabla\eta(\sqrt{\varepsilon}x')\sqrt{\varepsilon}\tilde{w}_\mu(x')+\eta(\sqrt{\varepsilon}x')\nabla\tilde{w}_\mu(x')\right|^2\right)^2\\
&-\frac{\lambda}{p}\mu t^{p+6}\int_{\mathbb{R}^3}W(\varepsilon t^2x'+x_0)\eta^p(\sqrt{\varepsilon}x')\tilde{w}_\mu^p(x')-\frac{1}{6}\mu t^{12}\int_{\mathbb{R}^3}\eta^6(\sqrt{\varepsilon}x')\tilde{w}_\mu^6(x')\\
\le{}& o(1)+\max_{t\in[0,t_0]}\frac{a}{2}t^4\int_{\mathbb{R}^3}|\nabla\tilde{w}_\mu|^2+\frac{1}{2}t^8\int_{\mathbb{R}^3}V(x_0)\tilde{w}_\mu^2+\frac{b}{4}t^8\left(\int_{\mathbb{R}^3}|\nabla\tilde{w}_\mu|^2\right)^2\\
&-\frac{\lambda}{p}\mu t^{p+6}\int_{\mathbb{R}^3}W(x_0)\tilde{w}_\mu^p-\frac{1}{6}\mu t^{12}\int_{\mathbb{R}^3}\tilde{w}_\mu^6\\
\le{}& o(1)+\sup_{t\in[0,+\infty)}I_{V(x_0),W(x_0),\mu}\left((\tilde{w}_\mu)_t\right)\\
={}& o(1)+c_{V(x_0),W(x_0),\mu},
\end{aligned}$$

where we have used (3.4). Notice that $o(1)\to 0$ as $\varepsilon\to 0$ uniformly for $\mu\in[1-\delta_0,1]$, then (3.6) holds, the lemma is proved. □

Suppose that (P_1)-(i) holds, assume that $x_\nu\in\mathcal{V}$ such that

$$W(x_\nu):=\max_{x\in\mathcal{V}}W(x).$$

By (P_2)-(i), $\tau<\tau_\infty$ and $W(x_\nu)\ge\kappa_\infty$, then $c_{\tau,W(x_\nu),\mu}<c_{\tau_\infty,\kappa_\infty,\mu}$, and combining with Lemma 3.3, we have

$$c_{\varepsilon,\mu}<c_{\tau_\infty,\kappa_\infty,\mu}\tag{3.7}$$

for all $\mu\in[1-\delta_0,1]$ and $\varepsilon>0$ small. Similarly, if (P_2)-(ii) holds, (3.7) is still true for all $\mu\in[1-\delta_0,1]$ and $\varepsilon>0$ small.

Lemma 3.4 *Suppose that (P_1), (P_2), (P_3) hold and $p\in(3,4]$. Fix $\varepsilon>0$, for every $\mu\in[1-\delta_0,1]$, let $\{u_n\}\subset H^1(\mathbb{R}^3)$ be a bounded $(PS)_c$ sequence for $I_{\varepsilon,\mu}$ with $0<c<\frac{1}{4}ab\frac{S^3}{\mu}+\frac{1}{24}b^3\frac{S^6}{\mu^2}+\frac{1}{24}(b^2\frac{S^4}{\mu^{4/3}}+4a\frac{S}{\mu^{1/3}})^{\frac{3}{2}}$, then there exists a $u\in H^1(\mathbb{R}^3)$, a number $k\in\mathbb{N}\cup\{0\}$, k functions $w_1,\dots,w_k$ of $H^1(\mathbb{R}^3)$ and k sequences of points $\{y_n^j\}\subset\mathbb{R}^3$, $1\le j\le k$ and $A\in\mathbb{R}$, such that*
(i) *$u_n\rightharpoonup u$ in $H^1(\mathbb{R}^3)$ with $J'_{\varepsilon,\mu}(u)=0$ and $\int_{\mathbb{R}^3}|\nabla u_n|^2\to A^2$ as $n\to\infty$;*
(ii) *$|y_n^j|\to+\infty$, $|y_n^i-y_n^j|\to+\infty$ as $n\to\infty$ if $i\ne j$;*
(iii) *$w_j\ne 0$ and $J'_{\tau_\infty,\kappa_\infty,\mu}(w_j)=0$;*
(iv) *$\|u_n-u-\sum_{j=1}^k w_j(\cdot-y_n^j)\|_{H^1(\mathbb{R}^3)}\to 0$ as $n\to\infty$;*
(v) *$I_{\varepsilon,\mu}(u_n)+\frac{b}{4}A^4=J_{\varepsilon,\mu}(u)+\sum_{j=1}^k J_{\tau_\infty,\kappa_\infty,\mu}(w_j)+o(1)$;*
(vi) *$A^2=\int_{\mathbb{R}^3}|\nabla u|^2+\sum_{j=1}^k\int_{\mathbb{R}^3}|\nabla w_j|^2$,*

where

$$\begin{aligned}
J_{\varepsilon,\mu}(u)={}&\frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2\\
&-\frac{\mu}{p}\lambda\int_{\mathbb{R}^3}W(\varepsilon x)(u^+)^p-\frac{1}{6}\mu\int_{\mathbb{R}^3}(u^+)^6,\quad u\in H^1(\mathbb{R}^3),
\end{aligned}$$

and

$$J_{\tau_\infty,\kappa_\infty,\mu}(u) = \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{2}\tau_\infty\int_{\mathbb{R}^3}u^2 - \frac{\mu}{p}\lambda\kappa_\infty\int_{\mathbb{R}^3}\left(u^+\right)^p - \frac{1}{6}\mu\int_{\mathbb{R}^3}\left(u^+\right)^6, \quad u\in H^1\left(\mathbb{R}^3\right).$$

Proof Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, then $\exists u\in H^1(\mathbb{R}^3)$ and $A\in\mathbb{R}$, up to a subsequence, such that as $n\to\infty$,

$$u_n \rightharpoonup u \quad \text{in } H^1\left(\mathbb{R}^3\right), \qquad \int_{\mathbb{R}^3}|\nabla u_n|^2 \to A^2 \quad \text{and} \quad \int_{\mathbb{R}^3}|\nabla u|^2 \le A^2.$$

$I'_{\varepsilon,\mu}(u_n)\to 0$ implies that

$$\left(a+bA^2\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla\varphi + \int_{\mathbb{R}^3}V(\varepsilon x)u\varphi - \mu\lambda\int_{\mathbb{R}^3}W(\varepsilon x)\left(u^+\right)^{p-1}\varphi - \mu\int_{\mathbb{R}^3}\left(u^+\right)^5\varphi = 0, \quad \forall\varphi\in H^1\left(\mathbb{R}^3\right),$$

i.e. $J'_{\varepsilon,\mu}(u) = 0$.

Since

$$\begin{aligned}
&J_{\varepsilon,\mu}(u_n)\\
&\quad= \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u_n^2 - \frac{\mu}{p}\lambda\int_{\mathbb{R}^3}W(\varepsilon x)\left(u_n^+\right)^p - \frac{1}{6}\mu\int_{\mathbb{R}^3}\left(u_n^+\right)^6\\
&\quad= \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u_n^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 - \frac{\mu}{p}\lambda\int_{\mathbb{R}^3}W(\varepsilon x)\left(u_n^+\right)^p\\
&\qquad - \frac{1}{6}\mu\int_{\mathbb{R}^3}\left(u_n^+\right)^6 + \frac{b}{4}A^4 + o(1)\\
&\quad= I_{\varepsilon,\mu}(u_n) + \frac{b}{4}A^4 + o(1)
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle J'_{\varepsilon,\mu}(u_n),\varphi\right\rangle\\
&\quad= \left(a+bA^2\right)\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\varphi + \int_{\mathbb{R}^3}V(\varepsilon x)u_n\varphi - \mu\lambda\int_{\mathbb{R}^3}W(\varepsilon x)\left(u_n^+\right)^{p-1}\varphi - \mu\int_{\mathbb{R}^3}\left(u_n^+\right)^5\varphi\\
&\quad= \left\langle I'_{\varepsilon,\mu}(u_n),\varphi\right\rangle + o(1)\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\varphi\\
&\quad= \left\langle I'_{\varepsilon,\mu}(u_n),\varphi\right\rangle + o(1)\|\varphi\|_{H^1(\mathbb{R}^3)},
\end{aligned}$$

we conclude that as $n\to\infty$,

$$J_{\varepsilon,\mu}(u_n) \to c + \frac{b}{4}A^4 \tag{3.8}$$

and

$$J'_{\varepsilon,\mu}(u_n) \to 0 \quad \text{in } \left(H^1\left(\mathbb{R}^3\right)\right)^{-1}. \tag{3.9}$$

Step 1: Set $u_{n,1} = u_n - u$, by the Brezis-Lieb theorem ([24], Theorem 1),

$$\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 = \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 + o(1), \tag{3.10}$$

$$\int_{\mathbb{R}^3} u_{n,1}^2 = \int_{\mathbb{R}^3} u_n^2 - \int_{\mathbb{R}^3} u^2 + o(1),$$

$$\int_{\mathbb{R}^3} (u_{n,1}^+)^p = \int_{\mathbb{R}^3} (u_n^+)^p - \int_{\mathbb{R}^3} (u^+)^p + o(1),$$

$$\int_{\mathbb{R}^3} (u_{n,1}^+)^6 = \int_{\mathbb{R}^3} (u_n^+)^6 - \int_{\mathbb{R}^3} (u^+)^6 + o(1), \tag{3.11}$$

$$J_{\tau_\infty,\kappa_\infty,\mu}(u_{n,1}) = J_{\varepsilon,\mu}(u_n) - J_{\varepsilon,\mu}(u) + o(1), \tag{3.12}$$

$$J'_{\tau_\infty,\kappa_\infty,\mu}(u_{n,1}) \to 0 \quad \text{in } \big(H^1(\mathbb{R}^3)\big)^{-1}. \tag{3.13}$$

Next, we claim that one of the following conclusions holds for $u_{n,1}$:

(1) $u_{n,1} \to 0$ in $H^1(\mathbb{R}^3)$ or

(2) $\exists r, \beta > 0$ and a sequence $\{y_n^1\} \subset \mathbb{R}^3$ such that

$$\int_{B_r(y_n^1)} u_{n,1}^2 \geq \beta > 0.$$

Indeed, suppose that (2) does not hold, then by the vanishing theorem due to Lion ([25], Lemma 1.1), we have

$$u_{n,1} \to 0 \quad \text{in } L^s\big(\mathbb{R}^3\big) \text{ for } s \in (2,6), \tag{3.14}$$

and combining with $\langle J'_{\tau_\infty,\kappa_\infty,\mu}(u_{n,1}), u_{n,1}\rangle = o(1)$, we get

$$\big(a + bA^2\big) \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 + \tau_\infty \int_{\mathbb{R}^3} u_{n,1}^2 - \mu \int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6 = o(1). \tag{3.15}$$

Now, we have the following equalities:

$$\begin{cases} J_{\varepsilon,\mu}(u) = \frac{a+bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x)u^2 - \frac{\mu}{p}\lambda \int_{\mathbb{R}^3} W(\varepsilon x)(u^+)^p - \frac{1}{6}\mu \int_{\mathbb{R}^3} (u^+)^6, \\ 0 = (a + bA^2) \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 - \mu\lambda \int_{\mathbb{R}^3} W(\varepsilon x)(u^+)^p - \mu \int_{\mathbb{R}^3} (u^+)^6, \\ 0 = \frac{a+bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \frac{1}{2} \int_{\mathbb{R}^3} (DV(\varepsilon x), \varepsilon x)u^2 \\ \quad - \frac{3}{p}\mu\lambda \int_{\mathbb{R}^3} W(\varepsilon x)(u^+)^p - \frac{1}{p}\mu\lambda \int_{\mathbb{R}^3} (DW(\varepsilon x), \varepsilon x)(u^+)^p - \frac{1}{2}\mu \int_{\mathbb{R}^3} (u^+)^6. \end{cases}$$

The first one comes from the definition of $J_{\varepsilon,\mu}$. The second one follows by $\langle J'_{\varepsilon,\mu}(u), u\rangle = 0$. The third one is the Pohozaev identity applying to $J'_{\varepsilon,\mu}(u) = 0$. From these equalities and (P_3), we have

$$\begin{aligned} &J_{\varepsilon,\mu}(u) - \frac{b}{4}A^2 \int_{\mathbb{R}^3} |\nabla u|^2 \\ &\quad = J_{\varepsilon,\mu}(u) - \frac{b}{4}A^2 \int_{\mathbb{R}^3} |\nabla u|^2 \\ &\qquad - \frac{1}{q+6}\bigg[2\big(a + bA^2\big) \int_{\mathbb{R}^3} |\nabla u|^2 + 4 \int_{\mathbb{R}^3} V(\varepsilon x)u^2 + \int_{\mathbb{R}^3} \big(DV(\varepsilon x), \varepsilon x\big)u^2 \end{aligned}$$

$$
\begin{aligned}
&- \frac{p+6}{p}\mu\lambda \int_{\mathbb{R}^3} W(\varepsilon x)\big(u^+\big)^p - \frac{2}{p}\mu\lambda \int_{\mathbb{R}^3} \big(DW(\varepsilon x), \varepsilon x\big)\big(u^+\big)^p - 2\mu \int_{\mathbb{R}^3} \big(u^+\big)^6 \Big] \\
&= \frac{2(q+2)a + (q-2)bA^2}{4(q+6)} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{6-q}{6(q+6)}\mu \int_{\mathbb{R}^3} \big(u^+\big)^6 \\
&\quad + \frac{q-2}{2(q+6)} \int_{\mathbb{R}^3} V(\varepsilon x)u^2 - \frac{1}{(q+6)} \int_{\mathbb{R}^3} \big(DV(\varepsilon x), \varepsilon x\big)u^2 \\
&\quad + \frac{1}{p}\frac{p-q}{q+6}\mu\lambda \int_{\mathbb{R}^3} W(\varepsilon x)\big(u^+\big)^p + \frac{1}{p}\frac{2}{q+6}\mu\lambda \int_{\mathbb{R}^3} \big(DW(\varepsilon x), \varepsilon x\big)\big(u^+\big)^p \geq 0.
\end{aligned} \tag{3.16}
$$

In view of (3.8), (3.10), (3.11), (3.12), (3.14) and (3.16), we have

$$
\begin{aligned}
c &= J_{\tau_\infty,\kappa_\infty,\mu}(u_{n,1}) + J_{\varepsilon,\mu}(u) - \frac{b}{4}A^4 + o(1) \\
&\geq J_{\tau_\infty,\kappa_\infty,\mu}(u_{n,1}) + \frac{b}{4}A^2 \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{b}{4}A^4 + o(1) \\
&= \frac{a+bA^2}{2} \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 + \frac{1}{2}\tau_\infty \int_{\mathbb{R}^3} u_{n,1}^2 - \frac{1}{p}\mu\lambda\kappa_\infty \int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^p - \frac{1}{6}\mu \int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6 \\
&\quad + \frac{b}{4}A^2 \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{b}{4}A^4 + o(1) \\
&= \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 + \frac{1}{2}\tau_\infty \int_{\mathbb{R}^3} u_{n,1}^2 + \frac{b}{4}A^2 \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 - \frac{1}{6}\mu \int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6 + o(1).
\end{aligned} \tag{3.17}
$$

Using the definition of S, we get

$$
\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 \geq S\left(\int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6\right)^{\frac{1}{3}}. \tag{3.18}
$$

In view of (3.15), we assume that

$$
a\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 + \tau_\infty \int_{\mathbb{R}^3} u_{n,1}^2 \to l_1, \tag{3.19}
$$

$$
bA^2 \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 \to l_2, \tag{3.20}
$$

$$
\mu \int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6 \to l_1 + l_2. \tag{3.21}
$$

Equations (3.10), (3.17), (3.18), (3.19), (3.20) and (3.21) yield

$$
\begin{aligned}
l_1 &= \lim_{n\to\infty} a\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 + \tau_\infty \int_{\mathbb{R}^3} u_{n,1}^2 \geq \lim_{n\to\infty} a\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 \\
&\geq \lim_{n\to\infty} aS\left(\int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6\right)^{\frac{1}{3}} = aS\left(\frac{l_1+l_2}{\mu}\right)^{\frac{1}{3}},
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
l_2 &= \lim_{n\to\infty} bA^2 \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 = \lim_{n\to\infty} b\left(\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 + \int_{\mathbb{R}^3} |\nabla u|^2\right) \int_{\mathbb{R}^3} |\nabla u_{n,1}|^2 \\
&\geq \lim_{n\to\infty} b\left(\int_{\mathbb{R}^3} |\nabla u_{n,1}|^2\right)^2 \geq \lim_{n\to\infty} bS^2\left(\int_{\mathbb{R}^3} \big(u_{n,1}^+\big)^6\right)^{\frac{2}{3}} \geq bS^2\left(\frac{l_1+l_2}{\mu}\right)^{\frac{2}{3}},
\end{aligned} \tag{3.23}
$$

and

$$c \geq \frac{1}{2}l_1 + \frac{1}{4}l_2 - \frac{1}{6}(l_1 + l_2) = \frac{1}{3}l_1 + \frac{1}{12}l_2. \tag{3.24}$$

Combining (3.22) and (3.23), we have

$$l_1 + l_2 \geq aS\left(\frac{l_1 + l_2}{\mu}\right)^{\frac{1}{3}} + bS^2\left(\frac{l_1 + l_2}{\mu}\right)^{\frac{2}{3}}.$$

If $l_1 + l_2 \neq 0$, we get

$$(l_1 + l_2)^{\frac{1}{3}} \geq \frac{1}{2}\left[b\left(\frac{S}{\mu^{1/3}}\right)^2 + \sqrt{\frac{b^2S^4}{\mu^{4/3}} + \frac{4aS}{\mu^{1/3}}}\right],$$

then

$$\begin{aligned} c &\geq \frac{1}{3}l_1 + \frac{1}{12}l_2 \geq \frac{1}{3}\frac{aS}{\mu^{1/3}}(l_1 + l_2)^{\frac{1}{3}} + \frac{1}{12}\frac{bS^2}{\mu^{2/3}}(l_1 + l_2)^{\frac{2}{3}} \\ &\geq \frac{1}{4}ab\frac{S^3}{\mu} + \frac{1}{24}b^3\frac{S^6}{\mu^2} + \frac{1}{24}\left(b^2\frac{S^4}{\mu^{4/3}} + 4a\frac{S}{\mu^{1/3}}\right)^{\frac{3}{2}}, \end{aligned}$$

a contradiction. Hence $l_1 + l_2 = 0$, *i.e.*

$$u_{n,1} \to 0 \quad \text{in } H^1(\mathbb{R}^3) \text{ as } n \to \infty,$$

(1) holds.

If (1) holds, the proof is completed for $k = 0$. If (2) holds, denote $w_{n,1}(x) = u_{n,1}(x + y_n^1)$, then

$$\int_{B_r(0)} w_{n,1}^2 \geq \beta > 0.$$

Up to a subsequence, $w_{n,1} \rightharpoonup w_1$ in $H^1(\mathbb{R}^3)$ with $w_1 \neq 0$ and $J'_{\tau_\infty,\kappa_\infty,\mu}(w_1) = 0$. Moreover, $u_{n,1} \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $\{y_n^1\}$ is unbounded.

Step 2: Set $u_{n,2}(x) = u_n(x) - u(x) - w_1(x - y_n^1)$, we can similarly check that

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u_{n,2}|^2 = \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} |\nabla w_1|^2 + o(1), \\ &\int_{\mathbb{R}^3} u_{n,2}^2 = \int_{\mathbb{R}^3} u_n^2 - \int_{\mathbb{R}^3} u^2 - \int_{\mathbb{R}^3} w_1^2 + o(1), \\ &\int_{\mathbb{R}^3} (u_{n,2}^+)^p = \int_{\mathbb{R}^3} (u_n^+)^p - \int_{\mathbb{R}^3} (u^+)^p - \int_{\mathbb{R}^3} (w_1^+)^p + o(1), \\ &\int_{\mathbb{R}^3} (u_{n,2}^+)^6 = \int_{\mathbb{R}^3} (u_n^+)^6 - \int_{\mathbb{R}^3} (u^+)^6 - \int_{\mathbb{R}^3} (w_1^+)^6 + o(1), \\ &J_{\tau_\infty,\kappa_\infty,\mu}(u_{n,2}) = J_{\varepsilon,\mu}(u_n) - J_{\varepsilon,\mu}(u) - J_{\tau_\infty,\kappa_\infty,\mu}(w_1) + o(1), \\ &J'_{\tau_\infty,\kappa_\infty,\mu}(u_{n,2}) \to 0 \quad \text{in } \left(H^1(\mathbb{R}^3)\right)^{-1}. \end{aligned}$$

Similar to *Step 1*, if (1) holds for $u_{n,2}$, then

$$\|u_{n,2}\|_{H^1(\mathbb{R}^3)} = \|u_n - u - w_1(x - y_n^1)\|_{H^1(\mathbb{R}^3)} \to 0 \quad \text{as } n \to \infty,$$

$$c + \frac{b}{4}A^4 + o(1) = J_{\varepsilon,\mu}(u_n) = J_{\varepsilon,\mu}(u) + J_{\tau_\infty,\kappa_\infty,\mu}(w_1) + o(1)$$

and

$$\begin{aligned} A^2 + o(1) &= \int_{\mathbb{R}^3} |\nabla u_n|^2 = \int_{\mathbb{R}^3} |\nabla u_{n,2}|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} |\nabla w_1|^2 + o(1) \\ &= \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} |\nabla w_1|^2 + o(1), \end{aligned}$$

the lemma holds for $k = 1$.

If (2) holds for $u_{n,2}$, *i.e.* $\exists r', \beta' > 0$ and a sequence $\{y_n^2\} \subset \mathbb{R}^3$ such that

$$\int_{B_{r'}(y_n^2)} u_{n,2}^2 \geq \beta' > 0,$$

then

$$\int_{B_{r'}(y_n^2 - y_n^1)} u_{n,2}^2(x + y_n^1) \geq \beta' > 0.$$

$u_{n,2}(x + y_n^1) \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $|y_n^2 - y_n^1| \to +\infty$.

Since $\{y_n^1\}$ is unbounded and $w_1 \in H^1(\mathbb{R}^3)$, we can easily check that

$$w_1(x - y_n^1) \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^3),$$

then

$$u_{n,2}(x) := u_n(x) - u(x) - w_1(x - y_n^1) \rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^3) \text{ as } n \to \infty,$$

which implies that $\{y_n^2\}$ must be unbounded. Denote $w_{n,2}(x) = u_{n,2}(x + y_n^2)$, then

$$\int_{B_{r'}(0)} w_{n,2}^2 \geq \beta' > 0,$$

up to a subsequence, $w_{n,2} \rightharpoonup w_2$ in $H^1(\mathbb{R}^3)$ with $w_2 \not\equiv 0$ and $J'_{\tau_\infty,\kappa_\infty,\mu}(w_2) = 0$ and next proceed by iteration. Since w_k is a nontrivial critical point of $J_{\tau_\infty,\kappa_\infty,\mu}$, $J_{\tau_\infty,\kappa_\infty,\mu}(w_k) \geq c'_{\tau_\infty,\kappa_\infty,\mu}$, where $c'_{\tau_\infty,\kappa_\infty,\mu}$ is the mountain-pass value of the functional $J_{\tau_\infty,\kappa_\infty,\mu}$. Hence the iteration must stop at some finite index k. The proof is completed. □

Proof of Theorem 2.1(A)-(a_1) We divide the proof into three steps.

Step 1: Since $I_{\varepsilon,\mu}$ possesses the geometry of Proposition 2.2 for $\varepsilon > 0$ small with $\mu \in [1 - \delta_0, 1]$, then by Proposition 2.2, for $\varepsilon > 0$ small but fixed, for almost every $\mu \in [1 - \delta_0, 1]$, there exists a bounded $(\mathrm{PS})_{c_{\varepsilon,\mu}}$ sequence $\{u_n\}$ for $I_{\varepsilon,\mu}$. Using the same argument as in the proof of Lemma 3.5 of [23], we can check that

$$c_{\tau_\infty,\kappa_\infty,\mu} < \frac{1}{4}ab\frac{S^3}{\mu} + \frac{1}{24}b^3\frac{S^6}{\mu^2} + \frac{1}{24}\left(b^2\frac{S^4}{\mu^{4/3}} + 4a\frac{S}{\mu^{1/3}}\right)^{\frac{3}{2}}, \quad \mu \in [1 - \delta_0, 1],$$

for $\lambda > 0$ large. Combining with (3.7), we have

$$c_{\varepsilon,\mu} < \frac{1}{4}ab\frac{S^3}{\mu} + \frac{1}{24}b^3\frac{S^6}{\mu^2} + \frac{1}{24}\left(b^2\frac{S^4}{\mu^{4/3}} + 4a\frac{S}{\mu^{1/3}}\right)^{\frac{3}{2}}, \quad \mu \in [1-\delta_0, 1],$$

for $\lambda > 0$ large, $\varepsilon > 0$ small.

In view of Lemma 3.4, there exist a $u_{\varepsilon,\mu} \in H^1(\mathbb{R}^3)$, a number $k \in \mathbb{N} \cup \{0\}$, k functions $w_1, \dots, w_k$ of $H^1(\mathbb{R}^3)$ and k sequences of points $\{y_n^j\} \subset \mathbb{R}^3$, $1 \le j \le k$ and $A_{\varepsilon,\mu} \in \mathbb{R}$, such that

(i) $u_n \rightharpoonup u_{\varepsilon,\mu}$ in $H^1(\mathbb{R}^3)$ with $J'_{\varepsilon,\mu}(u_{\varepsilon,\mu}) = 0$ and $\int_{\mathbb{R}^3} |\nabla u_n|^2 \to A_{\varepsilon,\mu}^2$ as $n \to \infty$;
(ii) $|y_n^j| \to +\infty$, $|y_n^i - y_n^j| \to +\infty$ as $n \to \infty$ if $i \neq j$;
(iii) $w_j \not\equiv 0$ and $J'_{\tau_\infty,\kappa_\infty,\mu}(w_j) = 0$;
(iv) $\|u_n - u_{\varepsilon,\mu} - w_j(\cdot - y_n^j)\|_{H^1(\mathbb{R}^3)} \to 0$ as $n \to \infty$;
(v) $I_{\varepsilon,\mu}(u_n) + \frac{b}{4}A_{\varepsilon,\mu}^4 = J_{\varepsilon,\mu}(u_{\varepsilon,\mu}) + \sum_{j=1}^k J_{\tau_\infty,\kappa_\infty,\mu}(w_j) + o(1)$;
(vi) $A_{\varepsilon,\mu}^2 = \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu}|^2 + \sum_{j=1}^k \int_{\mathbb{R}^3} |\nabla w_j|^2$.

By (3.16), we have

$$J_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \ge \frac{b}{4}A_{\varepsilon,\mu}^2 \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu}|^2. \tag{3.25}$$

Applying Pohozaev's identity to $J'_{\tau_\infty,\kappa_\infty,\mu}(w_j) = 0$, we have

$$\tilde{P}_{\varepsilon,\mu}(w_j) = \frac{a + bA_{\varepsilon,\mu}^2}{2}\int_{\mathbb{R}^3} |\nabla w_j|^2 + \frac{3}{2}\tau_\infty \int_{\mathbb{R}^3} w_j^2 - \frac{3}{p}\mu\lambda\kappa_\infty \int_{\mathbb{R}^3} \left(w_j^+\right)^p - \frac{1}{2}\mu \int_{\mathbb{R}^3} \left(w_j^+\right)^6 = 0,$$

then

$$\begin{aligned} 0 &= \left\langle J'_{\tau_\infty,\kappa_\infty,\mu}(w_j), w_j\right\rangle + 2\tilde{P}_{\varepsilon,\mu}(w_j) \\ &= 2\left(a + bA_{\varepsilon,\mu}^2\right)\int_{\mathbb{R}^3} |\nabla w_j|^2 + 4\tau_\infty \int_{\mathbb{R}^3} w_j^2 - \frac{p+6}{p}\mu\lambda\kappa_\infty \int_{\mathbb{R}^3} \left(w_j^+\right)^p - 2\mu \int_{\mathbb{R}^3} \left(w_j^+\right)^6 \\ &\ge G_{\tau_\infty,\kappa_\infty,\mu}(w_j). \end{aligned} \tag{3.26}$$

Hence, there exists $t_j \in (0,1]$ such that $(w_j)_{t_j} := t_j w_j(t_j^{-2}x) \in \mathcal{M}_{\tau_\infty,\kappa_\infty,\mu}$, we get

$$\begin{aligned} & J_{\tau_\infty,\kappa_\infty,\mu}(w_j) - \frac{b}{4}A_{\varepsilon,\mu}^2 \int_{\mathbb{R}^3} |\nabla w_j|^2 \\ &= J_{\tau_\infty,\kappa_\infty,\mu}(w_j) - \frac{b}{4}A_{\varepsilon,\mu}^2 \int_{\mathbb{R}^3} |\nabla w_j|^2 - \frac{1}{8}\left(\left\langle J'_{\tau_\infty,\kappa_\infty,\mu}(w_j), w_j\right\rangle + 2\tilde{P}_{\varepsilon,\mu}(w_j)\right) \\ &= \frac{a}{4}\int_{\mathbb{R}^3} |\nabla w_j|^2 + \frac{1}{12}\tau_\infty \int_{\mathbb{R}^3} w_j^2 + \frac{p-3}{6p}\mu\lambda\kappa_\infty \int_{\mathbb{R}^3} \left(w_j^+\right)^p + \frac{1}{12}\mu \int_{\mathbb{R}^3} \left(w_j^+\right)^6 \\ &\ge \frac{a}{4}t_j^3 \int_{\mathbb{R}^3} |\nabla w_j|^2 + \frac{1}{12}\tau_\infty t_j^5 \int_{\mathbb{R}^3} w_j^2 + \frac{p-3}{6p}\mu\lambda\kappa_\infty t_j^{p+3} \int_{\mathbb{R}^3} \left(w_j^+\right)^p + \frac{1}{12}\mu t_j^9 \int_{\mathbb{R}^3} \left(w_j^+\right)^6 \\ &= I_{\tau_\infty,\kappa_\infty,\mu}\left((w_j)_{t_j}\right) - \frac{1}{6}G_{\tau_\infty,\kappa_\infty,\mu}\left((w_j)_{t_j}\right) \ge c_{\tau_\infty,\kappa_\infty,\mu}, \end{aligned} \tag{3.27}$$

and combining with (3.25), we have

$$\begin{aligned} c_{\varepsilon,\mu} + \frac{b}{4}A_{\varepsilon,\mu}^4 &= J_{\varepsilon,\mu}(u_{\varepsilon,\mu}) + \sum_{j=1}^{k} J_{\tau_\infty,\kappa_\infty,\mu}(w_j) \\ &\geq \frac{b}{4}A_{\varepsilon,\mu}^2 \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu}|^2 + kc_{\tau_\infty,\kappa_\infty,\mu} + \frac{b}{4}A_{\varepsilon,\mu}^2 \sum_{j=1}^{k} \int_{\mathbb{R}^3} |\nabla w_j|^2 \\ &= \frac{b}{4}A_{\varepsilon,\mu}^4 + kc_{\tau_\infty,\kappa_\infty,\mu}. \end{aligned}$$

If $k \geq 1$, we get $c_{\varepsilon,\mu} \geq c_{\tau_\infty,\kappa_\infty,\mu}$ for $\varepsilon > 0$ small, which contradicts (3.7). Hence $k = 0$, then $u_n \to u_{\varepsilon,\mu}$ in $H^1(\mathbb{R}^3)$ for $\varepsilon > 0$ small and almost every $\mu \in [1-\delta_0, 1]$, *i.e.* for $\varepsilon > 0$ small and almost every $\mu \in [1-\delta_0, 1]$, $I'_{\varepsilon,\mu}(u_{\varepsilon,\mu}) = 0$ and $I_{\varepsilon,\mu}(u_{\varepsilon,\mu}) = c_{\varepsilon,\mu}$.

Step 2: Fix $\varepsilon > 0$ small, choose a sequence $\{\mu_n\} \subset [1-\delta_0, 1]$ satisfying $\mu_n \to 1$, we get a sequence of nontrivial critical points $\{u_{\varepsilon,\mu_n}\}$ of I_{ε,μ_n} with $I_{\varepsilon,\mu_n}(u_{\varepsilon,\mu_n}) = c_{\varepsilon,\mu_n}$. We have the following equalities:

$$\begin{cases} \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u_{\varepsilon,\mu_n}^2 + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2)^2 \\ \quad - \mu_n\frac{\lambda}{p}\int_{\mathbb{R}^3} W(\varepsilon x)(u_{\varepsilon,\mu_n}^+)^p - \mu_n\frac{1}{6}\int_{\mathbb{R}^3} (u_{\varepsilon,\mu_n}^+)^6 = c_{\varepsilon,\mu_n}, \\ a\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u_{\varepsilon,\mu_n}^2 + b(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2)^2 \\ \quad - \mu_n\lambda\int_{\mathbb{R}^3} W(\varepsilon x)(u_{\varepsilon,\mu_n}^+)^p - \mu_n\int_{\mathbb{R}^3} (u_{\varepsilon,\mu_n}^+)^6 = 0, \\ \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2 + \frac{3}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u_{\varepsilon,\mu_n}^2 + \frac{1}{2}\int_{\mathbb{R}^3} (DV(\varepsilon x), \varepsilon x)u_{\varepsilon,\mu_n}^2 \\ \quad + \frac{b}{2}(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2)^2 - \frac{3}{p}\mu_n\lambda\int_{\mathbb{R}^3} W(\varepsilon x)(u_{\varepsilon,\mu_n}^+)^p \\ \quad - \frac{1}{p}\mu_n\lambda\int_{\mathbb{R}^3} (DW(\varepsilon x), \varepsilon x)(u_{\varepsilon,\mu_n}^+)^p - \frac{1}{2}\mu_n\int_{\mathbb{R}^3} (u_{\varepsilon,\mu_n}^+)^6 = 0. \end{cases}$$

The first one comes from the definition of c_{ε,μ_n}. The second one follows by $\langle I'_{\varepsilon,\mu_n}(u_{\varepsilon,\mu_n}), u_{\varepsilon,\mu_n}\rangle = 0$. The third one is the Pohozaev identity applying to $I'_{\varepsilon,\mu_n}(u_{\varepsilon,\mu_n}) = 0$, then we get

$$\begin{aligned} &\frac{q+2}{2(q+6)}a\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2 + \frac{q-2}{4(q+6)}b\left(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2\right)^2 + \frac{6-q}{6(q+6)}\mu_n\int_{\mathbb{R}^3} \left(u_{\varepsilon,\mu_n}^+\right)^6 \\ &\quad + \frac{q-2}{2(q+6)}\int_{\mathbb{R}^3} V(\varepsilon x)u_{\varepsilon,\mu_n}^2 - \frac{1}{(q+6)}\int_{\mathbb{R}^3} \big(DV(\varepsilon x), \varepsilon x\big)u_{\varepsilon,\mu_n}^2 \\ &\quad + \frac{1}{p}\frac{p-q}{q+6}\mu_n\lambda\int_{\mathbb{R}^3} W(\varepsilon x)\left(u_{\varepsilon,\mu_n}^+\right)^p + \frac{1}{p}\frac{2}{q+6}\mu_n\lambda\int_{\mathbb{R}^3} \big(DW(\varepsilon x), \varepsilon x\big)\left(u_{\varepsilon,\mu_n}^+\right)^p \\ &= c_{\varepsilon,\mu_n} \leq c_{\varepsilon,1-\delta_0} \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{p}\right)a\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^3} V(\varepsilon x)u_{\varepsilon,\mu_n}^2 + \left(\frac{1}{4} - \frac{1}{p}\right)b\left(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2\right)^2 \\ &\quad + \left(\frac{1}{p} - \frac{1}{6}\right)\mu_n\int_{\mathbb{R}^3} \left(u_{\varepsilon,\mu_n}^+\right)^6 = c_{\varepsilon,\mu_n} \leq c_{\varepsilon,1-\delta_0}. \end{aligned} \tag{3.29}$$

By (3.28) and (P_3), $\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2$ must be bounded, then by (3.29), $a\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\mu_n}|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u_{\varepsilon,\mu_n}^2$ is bounded, *i.e.* $\{u_{\varepsilon,\mu_n}\}$ is bounded in $H^1(\mathbb{R}^3)$. Hence, we get

$$\begin{aligned}
&\lim_{n\to\infty} I_{\varepsilon,1}(u_{\varepsilon,\mu_n}) \\
&= \lim_{n\to\infty}\left(I_{\varepsilon,\mu_n}(u_{\varepsilon,\mu_n}) + \frac{1}{p}(\mu_n-1)\lambda\int_{\mathbb{R}^3} W(\varepsilon x)\left(u^+_{\varepsilon,\mu_n}\right)^p + \frac{1}{6}(\mu_n-1)\int_{\mathbb{R}^3}\left(u^+_{\varepsilon,\mu_n}\right)^6\right) \\
&= \lim_{n\to\infty} c_{\varepsilon,\mu_n} = c_{\varepsilon,1}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left\langle I'_{\varepsilon,1}(u_{\varepsilon,\mu_n}),\varphi\right\rangle\right| \\
&= \left|\left\langle I'_{\varepsilon,\mu_n}(u_{\varepsilon,\mu_n}),\varphi\right\rangle + \frac{1}{p}(\mu_n-1)\lambda\int_{\mathbb{R}^3} W(\varepsilon x)\left(u^+_{\varepsilon,\mu_n}\right)^{p-1}\varphi + \frac{1}{6}(\mu_n-1)\int_{\mathbb{R}^3}\left(u^+_{\varepsilon,\mu_n}\right)^5\varphi\right| \\
&\le C(1-\mu_n)\lambda\left(\int_{\mathbb{R}^3}\left(u^+_{\varepsilon,\mu_n}\right)^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^3}|\varphi|^p\right)^{\frac{1}{p}} \\
&\quad + (1-\mu_n)\left(\int_{\mathbb{R}^3}\left(u^+_{\varepsilon,\mu_n}\right)^6\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}|\varphi|^6\right)^{\frac{1}{6}} \\
&= o(1)\|\varphi\|_{H^1(\mathbb{R}^3)}, \quad \forall\varphi\in H^1\left(\mathbb{R}^3\right),
\end{aligned}$$

i.e. $\{u_{\varepsilon,\mu_n}\}$ is, in fact, a bounded $(\mathrm{PS})_{c_{\varepsilon,1}}$ sequence for $I_\varepsilon = I_{\varepsilon,1}$. Using the same argument in *Step 1* with $\mu = 1$, we can easily check that $\exists u_{\varepsilon,1}\in H^1(\mathbb{R}^3)$ such that $u_{\varepsilon,\mu_n}\to u_{\varepsilon,1}$ in $H^1(\mathbb{R}^3)$ and $I'_\varepsilon(u_{\varepsilon,1}) = 0$, $I_\varepsilon(u_{\varepsilon,1}) = c_{\varepsilon,1}$.

Step 3: Next, we prove the existence of a ground-state solution for (3.2). Set

$$m_\varepsilon := \inf\left\{I_\varepsilon(u) | I'_\varepsilon(u) = 0, u\in H^1\left(\mathbb{R}^3\right)\backslash\{0\}\right\}.$$

By (3.28) and (P_3), we see that $0\le m_\varepsilon\le I_\varepsilon(u_{\varepsilon,1}) = c_{\varepsilon,1} < +\infty$. Let $\{u_n\}$ be a sequence of nontrivial critical points of I_ε such that $I_\varepsilon(u_n)\to m_\varepsilon$. By the same argument as in *Step 2*, we see that $\{u_n\}$ is a bounded $(\mathrm{PS})_{m_\varepsilon}$ sequence of I_ε. Similar to the argument in *Step 1*, we see that $\exists w_\varepsilon\in H^1(\mathbb{R}^3)$ such that

$$u_n\to w_\varepsilon \quad \text{in } H^1\left(\mathbb{R}^3\right). \tag{3.30}$$

Next, we will show that $m_\varepsilon > 0$. Since

$$\begin{aligned}
0 &= \left\langle I'_\varepsilon(u_n), u_n\right\rangle \\
&= a\int_{\mathbb{R}^3}|\nabla u_n|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2 + b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 - \lambda\int_{\mathbb{R}^3} W(\varepsilon x)\left(u_n^+\right)^p - \int_{\mathbb{R}^3}\left(u^+_{\varepsilon,\mu_n}\right)^6 \\
&\ge C\|u_n\|^2_{H^1(\mathbb{R}^3)} - C\lambda\|u_n\|^p_{H^1(\mathbb{R}^3)} - C\|u_n\|^6_{H^1(\mathbb{R}^3)},
\end{aligned}$$

which implies that $\|u_n\|_{H^1(\mathbb{R}^3)}\ge C^* > 0$, then by (3.30), $\|w_\varepsilon\|_{H^1(\mathbb{R}^3)}\ge C^* > 0$, *i.e.* $w_\varepsilon\not\equiv 0$. Similar to (3.28), we deduce that $m_\varepsilon > 0$. Hence $I_\varepsilon(w_\varepsilon) = m_\varepsilon > 0$, $I'_\varepsilon(w_\varepsilon) = 0$. By the standard elliptic estimate and the strong maximum principle, we see that $w_\varepsilon > 0$. Set $u_\varepsilon(x) = w_\varepsilon(x/\varepsilon)$, u_ε is in fact a positive ground-state solution of (1.1). □

Next, we will prove the concentration result of Theorem 2.1 by using a similar method related to Proposition 2.2 in [1].

Proof of Theorem 2.1(A)-(a_2) For any $\varepsilon_j \to 0$, similar to (3.28), (3.29), we can easily check that w_{ε_j} is bounded in $H^1(\mathbb{R}^3)$.

By the vanishing theorem, we have $\exists\{y^1_{\varepsilon_j}\} \subset \mathbb{R}^3$, $R, \beta > 0$ such that

$$\int_{B_R(y^1_{\varepsilon_j})} w^2_{\varepsilon_j} \geq \beta > 0.$$

Set $v_{\varepsilon_j}(x) = w_{\varepsilon_j}(x + y^1_{\varepsilon_j})$, then v_{ε_j} satisfies

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\right)\Delta v_{\varepsilon_j} + V\left(\varepsilon_j x + \varepsilon_j y^1_{\varepsilon_j}\right)v_{\varepsilon_j} = \lambda W\left(\varepsilon_j x + \varepsilon_j y^1_{\varepsilon_j}\right)v^{p-1}_{\varepsilon_j} + v^5_{\varepsilon_j}, \tag{3.31}$$

and, up to a subsequence, $\exists v_1 \in H^1(\mathbb{R}^3)\backslash\{0\}$, such that

$$\begin{cases} v_{\varepsilon_j} \rightharpoonup v_1 & \text{in } H^1(\mathbb{R}^3), \\ v_{\varepsilon_j} \to v_1 & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3), 1 \leq s < 6, \\ v_{\varepsilon_j} \to v_1 & \text{a.e.} \end{cases} \tag{3.32}$$

Denote $A^2 := \lim_{j\to\infty} \int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2$, and it is trivial that

$$\int_{\mathbb{R}^3} |\nabla v_1|^2 \leq A^2.$$

Since V and W are bounded with $\tau > 0$ and $\inf_{\mathbb{R}^3} W > 0$, then, up to a subsequence, as $j \to \infty$,

$$V\left(\varepsilon_j y^1_{\varepsilon_j}\right) \to V\left(x^1\right) > 0, \qquad W\left(\varepsilon_j y^1_{\varepsilon_j}\right) \to W\left(x^1\right) > 0,$$

where

$$\varepsilon_j y^1_{\varepsilon_j} \to x^1 \quad \text{as } j \to \infty \ \left(x^1 \text{ might be } \infty\right).$$

In view of the uniformly continuous of V and W in $\mathbb{R}^3$, we can easily check that

$$V\left(\varepsilon_j x + \varepsilon_j y^1_{\varepsilon_j}\right) \to V\left(x^1\right) > 0, \qquad W\left(\varepsilon_j x + \varepsilon_j y^1_{\varepsilon_j}\right) \to W\left(x^1\right) > 0 \quad \text{as } j \to \infty$$

uniformly on any compact set. Consequently, we have

$$\left(a + bA^2\right)\int_{\mathbb{R}^3} \nabla v_1 \cdot \nabla\varphi + V\left(x^1\right)\int_{\mathbb{R}^3} v_1\varphi = \lambda W\left(x^1\right)\int_{\mathbb{R}^3} v_1^{p-1}\varphi + \int_{\mathbb{R}^3} v_1^5\varphi, \quad \forall\varphi \in C^\infty_c\left(\mathbb{R}^3\right),$$

then v_1 solves

$$-\left(a + bA^2\right)\Delta u + V\left(x^1\right)u = \lambda W\left(x^1\right)u^{p-1} + u^5 \tag{3.33}$$

with the energy functional $J_{V(x^1),W(x^1)}$, where the functional is defined as

$$J_{a_0,b_0}(u) = \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{a_0}{2}\int_{\mathbb{R}^3}u^2 - \frac{b_0}{p}\lambda\int_{\mathbb{R}^3}\left(u^+\right)^p - \frac{1}{6}\int_{\mathbb{R}^3}\left(u^+\right)^6,$$
$$u \in H^1\left(\mathbb{R}^3\right), \tag{3.34}$$

a_0, b_0 are positive constants.

Set

$$J_\varepsilon(u) := \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)u^2 - \frac{\lambda}{p}\int_{\mathbb{R}^3}W(\varepsilon x)\left(u^+\right)^p - \frac{1}{6}\int_{\mathbb{R}^3}\left(u^+\right)^6,$$
$$u \in H^1\left(\mathbb{R}^3\right).$$

Similar to (3.8), (3.9), we have

$$J_{\varepsilon_j}(w_{\varepsilon_j}) = I_{\varepsilon_j}(w_{\varepsilon_j}) + \frac{b}{4}A^4 + o(1)$$

and

$$J'_{\varepsilon_j}(w_{\varepsilon_j}) \to 0 \quad \text{in } \left(H^1\left(\mathbb{R}^3\right)\right)^{-1} \text{ as } j \to \infty.$$

Now, we consider $w_{\varepsilon_j,1}(x) = w_{\varepsilon_j}(x) - v_1(x - y^1_{\varepsilon_j})\chi_{\varepsilon_j}(x - y^1_{\varepsilon_j})$, where $\chi_\varepsilon(x) = \chi(\sqrt{\varepsilon}x)$ for $\varepsilon > 0$ small and $\chi(x)$ is a smooth cut-off function with $0 \le \chi(x) \le 1$, $\chi(x) = 1$ on $B_1(0)$, $\chi(x) = 0$ on $\mathbb{R}^3\backslash B_2(0)$ and $|\nabla\chi| \le C$ for some constant $C > 0$. It is easy to verify that $w_{\varepsilon_j,1}(x)$ is bounded in $H^1(\mathbb{R}^3)$. Furthermore, for any $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\|_{H^1(\mathbb{R}^3)} \le 1$, we have

$$\begin{aligned}
\left\langle J'_{\varepsilon_j}(w_{\varepsilon_j,1}), \varphi\right\rangle &= \left\langle J'_{\varepsilon_j}(w_{\varepsilon_j}), \varphi\right\rangle - \left\langle J'_{\varepsilon_j}\left(v_1\left(x - y^1_{\varepsilon_j}\right)\chi_{\varepsilon_j}\left(x - y^1_{\varepsilon_j}\right)\right), \varphi\right\rangle \\
&\quad + \lambda\int_{\mathbb{R}^3}W(\varepsilon_j x)\left(w^{p-1}_{\varepsilon_j}\varphi - \left(w^+_{\varepsilon_j,1}\right)^{p-1}\varphi - \left(v_1\left(x - y^1_{\varepsilon_j}\right)\chi_{\varepsilon_j}\left(x - y^1_{\varepsilon_j}\right)\right)^{p-1}\varphi\right) \\
&\quad + \int_{\mathbb{R}^3}\left(w^5_{\varepsilon_j}\varphi - \left(w^+_{\varepsilon_j,1}\right)^5\varphi - \left(v_1\left(x - y^1_{\varepsilon_j}\right)\chi_{\varepsilon_j}\left(x - y^1_{\varepsilon_j}\right)\right)^5\varphi\right) \\
&= o(1) + (I) + (II) + (III).
\end{aligned} \tag{3.35}$$

First, we see

$$\begin{aligned}
(I) &= -\left\langle J'_{\varepsilon_j}\left(v_1\left(x - y^1_{\varepsilon_j}\right)\chi_{\varepsilon_j}\left(x - y^1_{\varepsilon_j}\right)\right), \varphi\right\rangle \\
&= -\left\langle J'_{\varepsilon_j}\left(v_1\left(x - y^1_{\varepsilon_j}\right)\chi_{\varepsilon_j}\left(x - y^1_{\varepsilon_j}\right)\right), \varphi\right\rangle + \left\langle J'_{V(x^1),W(x^1)}(v_1), \chi_{\varepsilon_j}\varphi\left(x + y^1_{\varepsilon_j}\right)\right\rangle \\
&= -\left(a + bA^2\right)\int_{\mathbb{R}^3}\nabla(v_1\chi_{\varepsilon_j}) \cdot \nabla\varphi\left(x + y^1_{\varepsilon_j}\right) + \left(a + bA^2\right)\int_{\mathbb{R}^3}\nabla v_1 \cdot \nabla\left(\chi_{\varepsilon_j}\varphi\left(x + y^1_{\varepsilon_j}\right)\right) \\
&\quad - \int_{\mathbb{R}^3}V\left(\varepsilon_j x + \varepsilon_j y^1_{\varepsilon_j}\right)v_1\chi_{\varepsilon_j}\varphi\left(x + y^1_{\varepsilon_j}\right) + \int_{\mathbb{R}^3}V\left(x^1\right)v_1\chi_{\varepsilon_j}\varphi\left(x + y^1_{\varepsilon_j}\right) \\
&\quad + \lambda\int_{\mathbb{R}^3}W\left(\varepsilon_j x + \varepsilon_j y^1_{\varepsilon_j}\right)(v_1\chi_{\varepsilon_j})^{p-1}\varphi\left(x + y^1_{\varepsilon_j}\right) - \lambda\int_{\mathbb{R}^3}W\left(x^1\right)v_1^{p-1}\chi_{\varepsilon_j}\varphi\left(x + y^1_{\varepsilon_j}\right) \\
&\quad + \int_{\mathbb{R}^3}(v_1\chi_{\varepsilon_j})^5\varphi\left(x + y^1_{\varepsilon_j}\right) - \int_{\mathbb{R}^3}v_1^5\chi_{\varepsilon_j}\varphi\left(x + y^1_{\varepsilon_j}\right) = o(1),
\end{aligned} \tag{3.36}$$

where we have used (3.33).

Next, we study (II),

$$\begin{aligned}(II) &= \lambda \int_{\mathbb{R}^3} W(\varepsilon_j x)\big(w_{\varepsilon_j}^{p-1}\varphi - \big(w_{\varepsilon_j,1}^{+}\big)^{p-1}\varphi - \big(v_1\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}\big(x-y_{\varepsilon_j}^1\big)\big)^{p-1}\varphi\big) \\ &= \lambda \int_{\mathbb{R}^3} W\big(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}^1\big)\big(v_{\varepsilon_j}^{p-1} - \big(w_{\varepsilon_j,1}^{+}\big)^{p-1}\big(x+y_{\varepsilon_j}^1\big) - (v_1\chi_{\varepsilon_j})^{p-1}\big)\varphi\big(x+y_{\varepsilon_j}^1\big). \end{aligned} \tag{3.37}$$

For any given $\delta > 0$ small, we can choose a bounded domain $\Lambda \subset \mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3\setminus\Lambda} |\nabla v_1|^2 + v_1^2 + v_1^p + v_1^6 \le \delta.$$

Hence,

$$\begin{aligned}&\left|\int_{\mathbb{R}^3\setminus\Lambda} W\big(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}^1\big)\big(v_1(x)\chi_{\varepsilon_j}(x)\big)^{p-1}\varphi\big(x+y_{\varepsilon_j}^1\big)\right| \\ &\quad\le C\int_{\mathbb{R}^3\setminus\Lambda} v_1^{p-1}(x)\big|\varphi\big(x+y_{\varepsilon_j}^1\big)\big| \\ &\quad\le C\left(\int_{\mathbb{R}^3\setminus\Lambda} v_1^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^3\setminus\Lambda} \big|\varphi\big(x+y_{\varepsilon_j}^1\big)\big|^p\right)^{\frac{1}{p}} \\ &\quad\le C\|\varphi\|_{H^1(\mathbb{R}^3)}\delta^{\frac{p-1}{p}} \le C\delta^{\frac{p-1}{p}} \end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^3\setminus\Lambda} W\big(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}^1\big)\big(v_{\varepsilon_j}^{p-1}(x) - \big(w_{\varepsilon_j,1}^{+}\big)^{p-1}\big(x+y_{\varepsilon_j}^1\big)\big)\varphi\big(x+y_{\varepsilon_j}^1\big)\right| \\ &\quad= \left|\int_{\mathbb{R}^3\setminus\Lambda} W\big(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}^1\big)\big(v_{\varepsilon_j}^{p-1}(x) - \big(v_{\varepsilon_j}(x) - v_1(x)\chi_{\varepsilon_j}(x)\big)^{+(p-1)}\big)\varphi\big(x+y_{\varepsilon_j}^1\big)\right| \\ &\quad\le C\int_{\mathbb{R}^3\setminus\Lambda} v_1\big(v_1^{p-2} + v_{\varepsilon_j}^{p-2}\big)\big|\varphi\big(x+y_{\varepsilon_j}^1\big)\big| \\ &\quad\le C\left(\int_{\mathbb{R}^3\setminus\Lambda} v_1^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^3\setminus\Lambda} \big|\varphi\big(x+y_{\varepsilon_j}^1\big)\big|^p\right)^{\frac{1}{p}} \\ &\qquad + C\left(\int_{\mathbb{R}^3\setminus\Lambda} v_1^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^3\setminus\Lambda} v_{\varepsilon_j}^p\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^3\setminus\Lambda} \big|\varphi\big(x+y_{\varepsilon_j}^1\big)\big|^p\right)^{\frac{1}{p}} \\ &\quad\le C\|\varphi\|_{H^1(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3\setminus\Lambda} v_1^p\right)^{\frac{p-1}{p}} + C\|\varphi\|_{H^1(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3\setminus\Lambda} v_1^p\right)^{\frac{1}{p}} \\ &\quad\le C\big(\delta^{\frac{p-1}{p}} + \delta^{\frac{1}{p}}\big). \end{aligned} \tag{3.39}$$

In view of (3.32), $v_{\varepsilon_j} \to v_1$ in $L^p(\Lambda)$. Since $\Lambda \subset B_{1/\sqrt{\varepsilon_j}}(0)$ for ε_j small, we have

$$\begin{aligned}&\left|\int_{\Lambda} W\big(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}^1\big)\big(v_{\varepsilon_j}^{p-1} - \big(w_{\varepsilon_j,1}^{+}\big)^{p-1}\big(x+y_{\varepsilon_j}^1\big) - (v_1\chi_{\varepsilon_j})^{p-1}\big)\varphi\big(x+y_{\varepsilon_j}^1\big)\right| \\ &\quad= \left|\int_{\Lambda} W\big(\varepsilon_j x + \varepsilon_j y_{\varepsilon_j}^1\big)\big(v_{\varepsilon_j}^{p-1} - (v_{\varepsilon_j} - v_1)^{+(p-1)} - v_1^{p-1}\big)\varphi\big(x+y_{\varepsilon_j}^1\big)\right| \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\int_\Lambda |v_{\varepsilon_j}-v_1|^p\right)^{\frac{p-1}{p}}\left(\int_\Lambda \left|\varphi\left(x+y^1_{\varepsilon_j}\right)\right|^p\right)^{\frac{1}{p}}\\
&\quad + C\left(\int_\Lambda \left|v^{p-1}_{\varepsilon_j}-v^{p-1}_1\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_\Lambda \left|\varphi\left(x+y^1_{\varepsilon_j}\right)\right|^p\right)^{\frac{1}{p}}\\
&\le C\|\varphi\|_{H^1(\mathbb{R}^3)}\left(\int_\Lambda |v_{\varepsilon_j}-v_1|^p\right)^{\frac{p-1}{p}} + C\|\varphi\|_{H^1(\mathbb{R}^3)}\left(\int_\Lambda \left|v^{p-1}_{\varepsilon_j}-v^{p-1}_1\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\\
&= o(1).
\end{aligned}
\tag{3.40}
$$

Therefore, (3.37)-(3.40) lead to $(II) = o(1)$. Before studying (III), we first claim that

$$
v_{\varepsilon_j} \to v_1 \quad \text{in } L^6_{\mathrm{loc}}\left(\mathbb{R}^3\right). \tag{3.41}
$$

Indeed, in view of (3.32), we may assume that

$$
|\nabla v_{\varepsilon_j}|^2 \rightharpoonup |\nabla v_1|^2 + \mu \quad \text{and} \quad v^6_{\varepsilon_j} \rightharpoonup v^6_1 + \nu,
$$

where μ and ν are two bounded nonnegative measures on $\mathbb{R}^3$. By the concentration compactness principle II (Lemma 1.1 of [26]), we obtain an at most countable index set Γ, sequence $\{x_i\} \subset \mathbb{R}^3$ and $\{\mu_i\}, \{\nu_i\} \subset (0,\infty)$ such that

$$
\mu \ge \sum_{i\in\Gamma} \mu_i \delta_{x_i}, \qquad \nu = \sum_{i\in\Gamma} \nu_i\delta_{x_i} \quad \text{and} \quad S(\nu_i)^{\frac{1}{3}} \le \mu_i. \tag{3.42}
$$

It suffices to show that, for any bounded domain Ω, $\{x_i\}_{i\in\Gamma} \cap \Omega = \emptyset$. Suppose, by contradiction, that $x_i \in \Omega$ for some $i \in \Gamma$. Define, for $\rho > 0$, the function $\psi_\rho(x) := \psi(\frac{x-x_i}{\rho})$ where ψ is a smooth cut-off function such that $\psi = 1$ on $B_1(0)$, $\psi = 0$ on $\mathbb{R}^3\backslash B_2(0)$, $0 \le \psi \le 1$ and $|\nabla\psi| \le C$. We suppose that ρ is chosen in such a way that the support of ψ_ρ is contained in Ω. By (3.31), we see

$$
\begin{aligned}
&a\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\psi_\rho + a\int_{\mathbb{R}^3} (\nabla v_{\varepsilon_j}\cdot\nabla\psi_\rho)v_{\varepsilon_j} + \int_{\mathbb{R}^3} V\left(\varepsilon_j x+\varepsilon_j y^1_{\varepsilon_j}\right)v^2_{\varepsilon_j}\psi_\rho\\
&\quad + b\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\left(\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\psi_\rho\right) + b\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\int_{\mathbb{R}^3} (\nabla v_{\varepsilon_j}\cdot\nabla\psi_\rho)v_{\varepsilon_j}\\
&\quad = \lambda\int_{\mathbb{R}^3} W\left(\varepsilon_j x+\varepsilon_j y^1_{\varepsilon_j}\right)v^p_{\varepsilon_j}\psi_\rho + \int_{\mathbb{R}^3} v^6_{\varepsilon_j}\psi_\rho.
\end{aligned}
\tag{3.43}
$$

Since

$$
\begin{aligned}
\varlimsup_{j\to\infty}\left|\int_{\mathbb{R}^3} (\nabla v_{\varepsilon_j}\cdot\nabla\psi_\rho)v_{\varepsilon_j}\right| &\le \varlimsup_{j\to\infty}\left(\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\right)^{\frac{1}{2}}\cdot\left(\int_{\mathbb{R}^3} v^2_{\varepsilon_j}|\nabla\psi_\rho|^2\right)^{\frac{1}{2}}\\
&\le C\left(\int_{\mathbb{R}^3} v^2_1|\nabla\psi_\rho|^2\right)^{\frac{1}{2}} \le C\left(\int_{B_{2\rho}(x_i)} v^6_1\right)^{\frac{1}{6}}\left(\int_{B_{2\rho}(x_i)} |\nabla\psi_\rho|^3\right)^{\frac{1}{3}}\\
&\le C\left(\int_{B_{2\rho}(x_i)} v^6_1\right)^{\frac{1}{6}} \to 0 \quad \text{as } \rho\to 0,
\end{aligned}
\tag{3.44}
$$

$$
\varlimsup_{j\to\infty} a\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_j}|^2\psi_\rho \ge a\int_{\mathbb{R}^3} |\nabla v_1|^2\psi_\rho + a\mu_i \to a\mu_i \quad \text{as } \rho\to 0, \tag{3.45}
$$

$$
\begin{aligned}
&\overline{\lim_{j\to\infty}}\, b\int_{\mathbb{R}^3}|\nabla v_{\varepsilon_j}|^2\left(\int_{\mathbb{R}^3}|\nabla v_{\varepsilon_j}|^2\psi_\rho\right)\\
&\quad\geq \overline{\lim_{j\to\infty}}\, b\left(\int_{\mathbb{R}^3}|\nabla v_{\varepsilon_j}|^2\psi_\rho\right)^2\\
&\quad\geq b\left(\int_{\mathbb{R}^3}|\nabla v_1|^2\psi_\rho+\mu_i\right)^2\to b\mu_i^2\quad\text{as }\rho\to 0, \qquad (3.46)\\
&\overline{\lim_{j\to\infty}}\,\lambda\int_{\mathbb{R}^3}W\big(\varepsilon_j x+\varepsilon_j y_{\varepsilon_j}^1\big)v_{\varepsilon_j}^p\psi_\rho\\
&\quad=\lambda\int_{\mathbb{R}^3}W\big(x^1\big)v_1^p\psi_\rho\to 0\quad\text{as }\rho\to 0, \qquad (3.47)
\end{aligned}
$$

and

$$
\overline{\lim_{j\to\infty}}\int_{\mathbb{R}^3}v_{\varepsilon_j}^6\psi_\rho=\int_{\mathbb{R}^3}v_1^p\psi_\rho+\nu_i\to\nu_i\quad\text{as }\rho\to 0, \qquad (3.48)
$$

we obtain from (3.43)

$$
a\mu_i+b\mu_i^2\leq\nu_i.
$$

Combining with (3.42), we have

$$
(\nu_i)^{1/3}\geq\frac{bS^2+\sqrt{b^2S^4+4aS}}{2}.
$$

On the other hand,

$$
\begin{aligned}
m_{\varepsilon_j}&=I_{\varepsilon_j}(w_{\varepsilon_j})-\frac{1}{q+6}\big[\big\langle I'_{\varepsilon_j}(w_{\varepsilon_j}),w_{\varepsilon_j}\big\rangle+2P_{\varepsilon_j}(w_{\varepsilon_j})\big]\\
&=\frac{q+2}{2(q+6)}a\int_{\mathbb{R}^3}|\nabla w_{\varepsilon_j}|^2+\frac{q-2}{4(q+6)}b\left(\int_{\mathbb{R}^3}|\nabla w_{\varepsilon_j}|^2\right)^2+\frac{6-q}{6(q+6)}\int_{\mathbb{R}^3}w_{\varepsilon_j}^6\\
&\quad+\frac{1}{2(q+6)}\left[\int_{\mathbb{R}^3}\big((q-2)V(\varepsilon_j x)-2\big(\nabla V(\varepsilon_j x),\varepsilon_j x\big)\big)w_{\varepsilon_j}^2\right]\\
&\quad+\frac{\lambda}{p(q+6)}\left[\int_{\mathbb{R}^3}\big((p-q)W(\varepsilon_j x)+2\big(\nabla W(\varepsilon_j x),\varepsilon_j x\big)\big)w_{\varepsilon_j}^p\right]\\
&\geq\frac{q+2}{2(q+6)}a\int_{\mathbb{R}^3}|\nabla w_{\varepsilon_j}|^2+\frac{q-2}{4(q+6)}b\left(\int_{\mathbb{R}^3}|\nabla w_{\varepsilon_j}|^2\right)^2+\frac{6-q}{6(q+6)}\int_{\mathbb{R}^3}w_{\varepsilon_j}^6\\
&=\frac{q+2}{2(q+6)}a\int_{\mathbb{R}^3}|\nabla v_{\varepsilon_j}|^2+\frac{q-2}{4(q+6)}b\left(\int_{\mathbb{R}^3}|\nabla v_{\varepsilon_j}|^2\right)^2+\frac{6-q}{6(q+6)}\int_{\mathbb{R}^3}v_{\varepsilon_j}^6\\
&\geq\frac{q+2}{2(q+6)}a\mu_i+\frac{q-2}{4(q+6)}b\mu_i^2+\frac{6-q}{6(q+6)}\nu_i+o(1), \qquad (3.49)
\end{aligned}
$$

where we have used (P_3) and notice that

$$
\begin{aligned}
P_{\varepsilon_j}(w_{\varepsilon_j}):={}&\frac{a}{2}\int_{\mathbb{R}^3}|\nabla w_{\varepsilon_j}|^2+\frac{3}{2}\int_{\mathbb{R}^3}V(\varepsilon_j x)w_{\varepsilon_j}^2+\frac{1}{2}\int_{\mathbb{R}^3}\big(DV(\varepsilon_j x),\varepsilon_j x\big)w_{\varepsilon_j}^2\\
&+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla w_{\varepsilon_j}|^2\right)^2-\frac{3}{p}\lambda\int_{\mathbb{R}^3}W(\varepsilon_j x)w_{\varepsilon_j}^p
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{p}\lambda\int_{\mathbb{R}^3}\big(DW(\varepsilon_j x),\varepsilon_j x\big)w_{\varepsilon_j}^p-\frac{1}{2}\int_{\mathbb{R}^3}w_{\varepsilon_j}^6 \\
&=0
\end{aligned}
$$

is the Pohozaev identity applying to $I'_{\varepsilon_j}(w_{\varepsilon_j})=0$.

Since $m_{\varepsilon_j}\le c_{\varepsilon_j,1}\le c_{V(x_0),W(x_0),1}+o(1)<\frac{1}{4}abS^3+\frac{1}{24}b^3S^6+\frac{1}{24}(b^2S^4+4aS)^{\frac{3}{2}}$ for any $x_0\in\mathbb{R}^3$ and $\varepsilon_j>0$ small, then, up to a subsequence, we may assume that, as $j\to\infty$,

$$
m_{\varepsilon_j}\to\overline{c}<\frac{1}{4}abS^3+\frac{1}{24}b^3S^6+\frac{1}{24}\big(b^2S^4+4aS\big)^{\frac{3}{2}}.
$$

By (3.49),

$$
\begin{aligned}
\bar{c}&\ge\frac{q+2}{2(q+6)}a\mu_i+\frac{q-2}{4(q+6)}b\mu_i^2+\frac{6-q}{6(q+6)}\nu_i \\
&\ge\frac{q+2}{2(q+6)}aS(\nu_i)^{1/3}+\frac{q-2}{4(q+6)}bS^2(\nu_i)^{2/3}+\frac{6-q}{6(q+6)}\nu_i \\
&\ge\frac{q+2}{2(q+6)}aS\frac{bS^2+\sqrt{b^2S^4+4aS}}{2}+\frac{q-2}{4(q+6)}bS^2\left(\frac{bS^2+\sqrt{b^2S^4+4aS}}{2}\right)^2 \\
&\quad+\frac{6-q}{6(q+6)}\left(\frac{bS^2+\sqrt{b^2S^4+4aS}}{2}\right)^3 \\
&=\frac{1}{4}abS^3+\frac{1}{24}b^3S^6+\frac{1}{24}\big(b^2S^4+4aS\big)^{\frac{3}{2}}.
\end{aligned}
$$

This leads to a contradiction, hence (3.41) holds.

Similar to the proof of (II), we can easily check that $(III)=o(1)$. By (3.35), we have

$$
J'_{\varepsilon_j}(w_{\varepsilon_j,1})\to 0\quad\text{in }\big(H^1(\mathbb{R}^3)\big)^{-1}\text{ as }j\to\infty.
$$

We also claim that

$$
J_{\varepsilon_j}(w_{\varepsilon_j,1})\to\overline{c}+\frac{b}{4}A^4-J_{V(x^1),W(x^1)}(v_1)\quad\text{as }j\to\infty. \tag{3.50}
$$

Indeed,

$$
\begin{aligned}
J_{\varepsilon_j}(w_{\varepsilon_j,1})&=J_{\varepsilon_j}(w_{\varepsilon_j})-J_{\varepsilon_j}\big(v_1\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}\big(x-y_{\varepsilon_j}^1\big)\big) \\
&\qquad-\big(a+bA^2\big)\int_{\mathbb{R}^3}\nabla\big(v_1\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}\big(x-y_{\varepsilon_j}^1\big)\big)\cdot\nabla w_{\varepsilon_j,1} \\
&\qquad-\int_{\mathbb{R}^3}V(\varepsilon_j x)v_1\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}\big(x-y_{\varepsilon_j}^1\big)w_{\varepsilon_j,1}(x) \\
&\qquad+\frac{\lambda}{p}\int_{\mathbb{R}^3}W(\varepsilon_j x)\big(w_{\varepsilon_j}^p-\big(w_{\varepsilon_j,1}^+\big)^p-v_1^p\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}^p\big(x-y_{\varepsilon_j}^1\big)\big) \\
&\qquad+\frac{1}{6}\int_{\mathbb{R}^3}\big(w_{\varepsilon_j}^6-\big(w_{\varepsilon_j,1}^+\big)^6-v_1^6\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}^6\big(x-y_{\varepsilon_j}^1\big)\big) \\
&=\overline{c}+\frac{b}{4}A^4+o(1)+(IV)+(V)+(VI)+(VII)+(VIII), \\
(IV)&=-J_{\varepsilon_j}\big(v_1\big(x-y_{\varepsilon_j}^1\big)\chi_{\varepsilon_j}\big(x-y_{\varepsilon_j}^1\big)\big) \\
&=-\frac{a+bA^2}{2}\int_{\mathbb{R}^3}\big|\nabla(v_1\chi_{\varepsilon_j})\big|^2-\frac{1}{2}\int_{\mathbb{R}^3}V\big(\varepsilon_j x+\varepsilon_j y_{\varepsilon_j}^1\big)(v_1\chi_{\varepsilon_j})^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda}{p}\int_{\mathbb{R}^3} W\big(\varepsilon_j x+\varepsilon_j y^1_{\varepsilon_j}\big)(v_1\chi_{\varepsilon_j})^p+\frac{1}{6}\int_{\mathbb{R}^3}(v_1\chi_{\varepsilon_j})^6\\
&=-J_{V(x^1),W(x^1)}(v_1)+o(1),
\end{aligned}
$$

$$
\begin{aligned}
(V)&=-\big(a+bA^2\big)\int_{\mathbb{R}^3}\nabla\big(v_1\big(x-y^1_{\varepsilon_j}\big)\chi_{\varepsilon_j}\big(x-y^1_{\varepsilon_j}\big)\big)\cdot\nabla w_{\varepsilon_j,1}\\
&=-\big(a+bA^2\big)\int_{\mathbb{R}^3}\nabla(v_1\chi_{\varepsilon_j})\cdot\nabla(v_{\varepsilon_j}-v_1\chi_{\varepsilon_j})\\
&=\big(a+bA^2\big)\int_{\mathbb{R}^3}\big|\nabla(v_1\chi_{\varepsilon_j})\big|^2-\big(a+bA^2\big)\int_{\mathbb{R}^3}\nabla(v_1\chi_{\varepsilon_j})\nabla v_{\varepsilon_j}\\
&=\big(a+bA^2\big)\int_{\mathbb{R}^3}|\nabla v_1|^2\chi^2_{\varepsilon_j}-\big(a+bA^2\big)\int_{\mathbb{R}^3}\nabla v_1\nabla v_{\varepsilon_j}\chi_{\varepsilon_j}+o(1)=o(1),
\end{aligned}
$$

where we have used (3.32).

Similar to (V), (II), (III), we can easily check that $(VI)=o(1)$, $(VII)=o(1)$ and $(VIII)=o(1)$, then (3.50) holds.

Next, we repeat the above procedure for $w_{\varepsilon_j,1}$ and so on. It is easy to see that $J_{V(x^i),W(x^i)}(v_i)$ obtained in this process is always larger than the mountain-pass value of $J_{\tau,\kappa}$, therefore, the process will stop at finite k. Similar to the proof of Lemma 3.4, we see that, for $\varepsilon_j\to 0$, there is a sequence of j, a nonnegative integer k and k sequences $\{y^i_{\varepsilon_j}\}$, $1\le i\le k$, such that, as $j\to\infty$,

$$
\Big\|w_{\varepsilon_j}(x)-\sum_{i=1}^k v_i\big(x-y^i_{\varepsilon_j}\big)\chi_{\varepsilon_j}\big(x-y^i_{\varepsilon_j}\big)\Big\|_{H^1(\mathbb{R}^3)}\to 0, \tag{3.51}
$$

$$
\overline{c}+\frac{b}{4}A^4=\sum_{i=1}^k J_{V(x^i),W(x^i)}(v_i)\quad\text{and}\quad A^2=\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v_i|^2, \tag{3.52}
$$

where v_i is a nontrivial critical point of $J_{V(x^i),W(x^i)}$.

Using the same argument as in (3.27), we get

$$
J_{V(x^i),W(x^i)}(v_i)\ge c_{V(x^i),W(x^i),1}+\frac{b}{4}A^2\int_{\mathbb{R}^3}|\nabla v_i|^2,
$$

then in view of (3.52), we have

$$
\begin{aligned}
\overline{c}+\frac{b}{4}A^4&=\sum_{i=1}^k J_{V(x^i),W(x^i)}(v_i)\\
&\ge\sum_{i=1}^k c_{V(x^i),W(x^i),1}+\frac{b}{4}A^2\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v_i|^2\\
&=\sum_{i=1}^k c_{V(x^i),W(x^i),1}+\frac{b}{4}A^4,
\end{aligned}
$$

i.e.

$$
\overline{c}\ge\sum_{i=1}^k c_{V(x^i),W(x^i),1}.
$$

In view of Lemma 3.3 and (3.7), $\overline{c} \leq c_{V(x^1),W(x^1),1}$, then we conclude that $k = 1$, *i.e.*

$$\overline{c} = c_{V(x^1),W(x^1),1}.$$

By (3.51), we have

$$\left\| w_{\varepsilon_j}(x) - v_1\big(x - y^1_{\varepsilon_j}\big)\chi_{\varepsilon_j}\big(x - y^1_{\varepsilon_j}\big)\right\|_{H^1(\mathbb{R}^3)} \to 0,$$

then by the Sobolev inequality, we get

$$\|v_{\varepsilon_j} - v_1\|_{L^6(\mathbb{R}^3)} \leq \|v_{\varepsilon_j} - v_1\chi_{\varepsilon_j}\|_{L^6(\mathbb{R}^3)} + \|v_1\chi_{\varepsilon_j} - v_1\|_{L^6(\mathbb{R}^3)} \to 0 \quad \text{as } j \to \infty.$$

Hence, $v^6_{\varepsilon_j}$ is uniformly integrable near ∞, the Brezis-Kato type argument and the maximum principle yield

$$\lim_{|x|\to\infty} v_{\varepsilon_j}(x) = 0 \quad \text{uniformly for } j. \tag{3.53}$$

Next, we assume that (P_2)-(i) holds.

We claim that $\{\varepsilon_j y^1_{\varepsilon_j}\}$ is bounded. Assuming to the contrary that $|\varepsilon_j y^1_{\varepsilon_j}| \to \infty$, then $V(x^1) = \tau_\infty > \tau$ and $W(x^1) = \kappa_\infty \leq W(x_v)$, hence $c_{V(x^1),W(x^1),1} = c_{\tau_\infty,\kappa_\infty,1} > c_{\tau,W(x_v),1}$. But, from Lemma 3.3, we have

$$c_{V(x^1),W(x^1),1} = \overline{c} = \lim_{j\to\infty} m_{\varepsilon_j} \leq \lim_{j\to\infty} c_{\varepsilon_j,1} \leq c_{V(x_v),W(x_v),1} = c_{\tau,W(x_v),1}, \tag{3.54}$$

a contradiction.

We will show that $x^1 \in \mathcal{A}_v$. In fact, if $x^1 \in \mathcal{V}$, by (3.54), we have

$$c_{\tau,W(x^1),1} \leq c_{V(x^1),W(x^1),1} \leq c_{\tau,W(x_v),1},$$

which implies that $W(x^1) \geq W(x_v)$. By the definition of $W(x_v)$, $W(x^1) \leq \max_{x\in\mathcal{V}} W(x) = W(x_v)$, then $W(x^1) = W(x_v)$.

If $x^1 \notin \mathcal{V}$, then $V(x^1) > \tau$. Assuming to the contrary that $W(x^1) \leq W(x_v)$, then $c_{V(x^1),W(x^1),1} > c_{\tau,W(x_v),1}$, which contradicts (3.54).

Let P_{ε_j} a maximum point of v_{ε_j}, since $\Delta v_{\varepsilon_j}(P_{\varepsilon_j}) \leq 0$, (3.31) implies that

$$V\big(\varepsilon_j P_{\varepsilon_j} + \varepsilon_j y^1_{\varepsilon_j}\big)v_{\varepsilon_j}(P_{\varepsilon_j}) \leq \lambda W\big(\varepsilon_j P_{\varepsilon_j} + \varepsilon_j y^1_{\varepsilon_j}\big)v^{p-1}_{\varepsilon_j}(P_{\varepsilon_j}) + v^5_{\varepsilon_j}(P_{\varepsilon_j})$$

which gives $v_{\varepsilon_j}(P_{\varepsilon_j}) \geq C > 0$. By (3.53), P_{ε_j} must be bounded. Denote $x_{\varepsilon_j} = \varepsilon_j P_{\varepsilon_j} + \varepsilon_j y^1_{\varepsilon_j}$, it is clear that x_{ε_j} is a maximum point of u_{ε_j}, then $x_{\varepsilon_j} \to \mathcal{A}_v$. Since $\{\varepsilon_j\}$ is arbitrary, Theorem 2.1(A)-(a_2) is proved. □

To complete the proof of Theorem 2.1(A), we only need to prove the exponential decay result. Since the proof is standard (see [20], for example), we omit the details for simplicity. Note that all the conclusions of Theorem 2.1(B) can be similarly proved to Theorem 2.1(A). Thus, this completes the proof of Theorem 2.1.

Competing interests
The author declares that he has no competing interests.

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