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Blow-up and global existence for solution of quasilinear viscoelastic wave equation with strong damping and source term

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Abstract

In this paper we consider a quasilinear viscoelastic wave equation with initial-boundary conditions, strong damping and source term. Under suitable assumptions on the initial data and the relaxation function, we establish a blow-up result of a solution for negative initial energy and some positive initial energy if the influence of the source term is greater than the dissipation. We show that the solution exists globally for any initial data if the influence of dissipation is greater than the source term.

Keywords: viscoelasticity wave equation; strong damping; blow-up; global existence

1 Introduction

In this work, we study the following quasilinear viscoelastic wave equation with initial-boundary value conditions, strong damping and source term:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \Delta u_t = |u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, $\rho > 0$ and $p > 2$ are constants. The relaxation function g is a given function to be specified later.

As is well known, the wave equation with memory has been extensively studied. Berrimi and Messaoudi [1] considered the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau = |u|^\gamma u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, γ is a positive constant, and g is a nonnegative and decreasing function. They obtained a local existence result and proved, for certain initial data and suitable conditions on g and γ (under weaker conditions than those in [2, 3]), that the solution is global and decays uniformly (exponentially or polynomially depending on the decay rate of the relaxation function g) if the initial data is small enough. For further work on the existence and the decay of solutions, we refer the reader to [4–8]. Messaoudi [9] discussed the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)\,d\tau + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, $m \geq 1$, $p > 2$, $a, b > 0$ are constants and $g : R^+ \to R^+$ is a positive nonincreasing function. Under suitable conditions on g, he proved that solutions with negative initial energy blow up in finite time if $p > m$, and continue to exist if $p \leq m$. For the same problem, Messaoudi [10] extended this result to certain solutions with initial positive energy. A similar result was also obtained by Lu and Li [11], Guo and Lin [12].

Recently, Song and Zhong [13] studied a nonlinear viscoelastic problem with strong damping:

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)\,d\tau - \triangle u_t = |u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.4}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, $2 < p < \frac{2n-2}{n-2}$. They proved that solutions with positive initial energy blow up in finite time using the potential well method introduced by Payne and Sattinger [14]. Furthermore, Song and Xue [15] extended this result to arbitrarily high initial energy.

In the same direction Cavalcanti *et al.* [16] considered the following initial-boundary value problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t-\tau)\triangle u(\tau)\,d\tau - \gamma\triangle u_t = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.5}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$ and $\rho > 0$. They proved a global existence result for $\gamma \geq 0$ and an exponential decay result for $\gamma > 0$. Cavalcanti *et al.* [17] studied problem (1.5) with $\rho \geq 0$ and $\gamma \geq 0$. The authors showed that the energy decays to zero uniformly with the rate that is determined from the solutions of the ODE quantifying the behavior of $g(t)$, and they improved many previous results. In the case of $\gamma = 0$, Liu [18] discussed the following problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t-\tau)\triangle u(\tau)\,d\tau = b|u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.6}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, and $\rho, b > 0$, $p > 2$ are constants. He obtained a general decay of the solution for certain class of relaxation functions and initial data in the stable set, and showed that the solution blows up in a larger class of initial positive energy. Furthermore, Song [19] studied the following problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)\,d\tau + |u_t|^{m-2}u_t = b|u|^{p-2}u, & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.7}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with smooth boundary $\partial\Omega$, $m > 2$, $g : R^+ \to R^+$ is a positive nonincreasing function. He proved the nonexistence of global solution of (1.7) with initial positive energy.

Motivated by the above pioneering work, we consider the problem (1.1). Under suitable assumptions on the initial data and the relaxation function g, we obtain a blow-up result for the solution with negative initial energy and some positive initial energy if $p > \rho + 2$, and get a global existence result for any initial data if $p \leq \rho + 2$ using the perturbed energy functional technique. This paper is organized as follows. In Section 2, we present some assumptions and preliminaries. Section 3 is devoted to the blow-up result. In Section 4, we obtain the global existence result.

2 Preliminaries

In this section, we shall give some notations and preliminaries used throughout this paper. Denote by $\|\cdot\|_p$ the usual norm in $L^p(\Omega)$ $(p \geq 2)$. Let B be the best embedding constant such that $\|\phi\|_p \leq B\|\nabla\phi\|_2$, $\phi \in H_0^1(\Omega)$. Besides, C and C_i $(i \in N^+)$ denote general positive constants, which may be different in different estimates.

Now, we make the following assumptions.

(G1) $g(t)$: $R^+ \to R^+$ is a C^1 function satisfies

$$g'(s) \leq 0, \tag{2.1}$$

$$1 - \int_0^\infty g(s)\,ds = l > 0. \tag{2.2}$$

(G2) For the nonlinear term, we assume

$$\begin{cases} 2 < p < \infty, \quad \text{if } n = 1,2 \quad \text{and} \quad 2 < p \leq \frac{2(n-1)}{n-2}, \quad \text{if } n \geq 3, \\ 0 < \rho < \infty, \quad \text{if } n = 1,2 \quad \text{and} \quad 2 < \rho \leq \frac{2}{n-2}, \quad \text{if } n \geq 3. \end{cases} \tag{2.3}$$

We first state, without a proof, a local existence theorem which can be established by the Faedo-Galerkin method. The interested reader can refer to Cavalcanti *et al.* [5] for details.

Theorem 2.1 *Assume* (G1) *and* (G2) *hold and* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *is given. Then problem* (1.1) *has a unique local solution*

$$u \in C\big([0,T]; H_0^1(\Omega)\big), \qquad u_t \in C\big([0,T]; H_0^1(\Omega)\big). \tag{2.4}$$

Lemma 2.2 *Assume* (G1) *and* (G2) *hold. Let* $u(t)$ *be a solution of* (1.1). *Then* $E(t)$ *is non-increasing. Moreover, for* $t > 0$, *the following inequality holds*:

$$E'(t) = -\|\nabla u_t(t)\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0, \tag{2.5}$$

where

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u(t)\|_p^p$$

and

$$(g \circ v) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2\,d\tau.$$

Proof Multiplying the first equation of (1.1) by u_t, integrating over Ω, we obtain (2.5). □

Lemma 2.3 ([18], Lemma 2.2) *Assume* (G1) *and* (G2) *hold. Let* $u(t)$ *be a solution of* (1.1). *Assume further that*

$$E(0) < E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)B_1^{-\frac{2p}{p-2}}$$

and

$$\|\nabla u_0\|_2 \ge B_1^{-\frac{p}{p-2}},$$

where $B_1 = \frac{B}{\sqrt{l}}$. *Then there exists a constant* $\beta > B_1^{-\frac{p}{p-2}}$ *such that for* $t > 0$

$$\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 \ge \beta^2 \tag{2.6}$$

and

$$\|u\|_p \ge B_1\beta. \tag{2.7}$$

Lemma 2.4 *For* $2 \le p \le \rho + 2$, *we have*

$$\|u_t\|_p^p \le \|u_t\|_p^2 + \|u_t\|_{\rho+2}^{\rho+2}. \tag{2.8}$$

Proof If $\|u_t\|_p < 1$, then we get $\|u_t\|_p^p \le \|u_t\|_p^2$. If $\|u_t\|_p \ge 1$, then have

$$\|u_t\|_p^p \le C\|u_t\|_p^{\rho+2} \le C\|u_t\|_{\rho+2}^{\rho+2}.$$

Together with the two cases, we obtain (2.8). □

3 Blow-up result

In this section we state and prove the blow-up result.

Theorem 3.1 *Assume that* (G1) *and* (G2) *hold and*

$$\int_0^\infty g(s)\,ds < \frac{p/2-1}{p/2-1+1/2p}. \tag{3.1}$$

Assume further that $p > \rho+2$, *and* $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ *is given. Then the solution* $u(t)$ *of problem* (1.1) *blows up in finite time, i.e. there exists* $T_0 < +\infty$ *such that*

$$\lim_{t\to T_0^-}\left(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|u\|_p^p\right) = \infty, \tag{3.2}$$

if

$$E(0) < \left(1 - \frac{1}{p(p-2)}\frac{1-l}{l}\right)\left(\frac{1}{2}-\frac{1}{p}\right)B_1^{-\frac{2p}{p-2}}, \tag{3.3}$$

and

$$\|\nabla u_0\|_2 > B_1^{-\frac{p}{p-2}}. \tag{3.4}$$

Proof Assume that there exists some positive constant C such that for $t > 0$ the solution $u(t)$ of (1.1) satisfies

$$\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|u\|_p^p \le C. \tag{3.5}$$

We set

$$H(t) = E_2 - E(t),$$

where the constant $E_2 \in (E(0), E_1)$ shall be chosen later. By Lemma 2.2,

$$H'(t) = -E'(t) \ge 0. \tag{3.6}$$

Then, for $0 \le s \le t$, we have

$$0 < H(0) \le H(s) \le H(t) = E_2 - E(t). \tag{3.7}$$

From (2.6), we have

$$\begin{aligned} H(t) &= E_2 - E(t) \\ &= E_2 - \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{p}\left\|u(t)\right\|_p^p \\ &\le E_2 - \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|\nabla u(t)\right\|_2^2 + \frac{1}{p}\left\|u(t)\right\|_p^p \end{aligned}$$

$$
\begin{aligned}
&\le E_1 - \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p \\
&\le E_1 - \frac{1}{2}B_1^{-\frac{2p}{p-2}} + \frac{1}{p}\|u(t)\|_p^p \\
&\le \left(\frac{1}{2} - \frac{1}{p}\right)B_1^{-\frac{2p}{p-2}} - \frac{1}{2}B_1^{-\frac{2p}{p-2}} + \frac{1}{p}\|u(t)\|_p^p \\
&\le \frac{1}{p}\|u(t)\|_p^p.
\end{aligned}
\tag{3.8}
$$

Define

$$
L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2\,dx,
\tag{3.9}
$$

where the constant $\varepsilon > 0$ shall be chosen later and the constant σ satisfies

$$
0 < \sigma < \frac{1}{\rho+2} - \frac{1}{p}.
\tag{3.10}
$$

Taking a derivative of (3.9) and using Lemma 2.2, we have

$$
\begin{aligned}
L'(t) &= (1-\sigma)H^{-\sigma}(t)\left(\|\nabla u_t(t)\|_2^2 - \frac{1}{2}\left(g' \circ \nabla u\right)(t) + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2\right) \\
&\quad + \frac{\varepsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon\int_\Omega |u_t|^\rho u_{tt} u\,dx + \varepsilon\int_\Omega \nabla u_t \nabla u\,dx \\
&\ge (1-\sigma)H^{-\sigma}(t)\|\nabla u_t(t)\|_2^2 + \frac{\varepsilon}{\rho+1}\|u\|_{\rho+2}^{\rho+2} - \varepsilon\|\nabla u\|_2^2 + \varepsilon\|u\|_p^p \\
&\quad - \varepsilon\int_\Omega\int_0^t g(t-\tau)\triangle u(\tau)\,d\tau\, u(t)\,dx.
\end{aligned}
\tag{3.11}
$$

For the last term on the right side of (3.11), using the Green formula, we get

$$
\begin{aligned}
&-\int_\Omega\int_0^t g(t-\tau)\triangle u(\tau)\,d\tau\, u(t)\,dx \\
&\quad = \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau \\
&\quad = \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\nabla\big(u(\tau) - u(t)\big)\,dx\,d\tau + \int_0^t g(t-\tau)\|\nabla u(t)\|_2^2\,d\tau \\
&\quad = \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\big(\nabla u(\tau) - \nabla u(t)\big)\,dx\,d\tau + \int_0^t g(\tau)\,d\tau\,\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{3.12}
$$

Substituting (3.12) into (3.11), we obtain

$$
\begin{aligned}
L'(t) &\ge (1-\sigma)H^{-\sigma}(t)\|\nabla u_t(t)\|_2^2 + \frac{\varepsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon\|u\|_p^p \\
&\quad + \varepsilon\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\big(\nabla u(\tau) - \nabla u(t)\big)\,dx\,d\tau \\
&\quad - \varepsilon\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{3.13}
$$

Using the Cauchy inequality, for $0 < \varepsilon_1 < 1$ we have

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\big(\nabla u(\tau)-\nabla u(t)\big)\,dx\,d\tau \\
&\quad\geq -\frac{p(1-\varepsilon_1)}{2}\int_0^t g(t-\tau)\big\|\nabla u(\tau)-\nabla u(t)\big\|_2^2\,d\tau \\
&\qquad -\frac{1}{(1-\varepsilon_1)2p}\int_0^t g(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2.
\end{aligned} \tag{3.14}$$

By (3.14), we know

$$\begin{aligned}
L'(t) &\geq (1-\sigma)H^{-\sigma}(t)\big\|\nabla u_t(t)\big\|_2^2+\frac{\varepsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}+\varepsilon\|u\|_p^p \\
&\quad -\varepsilon\left(1-\int_0^t g(\tau)\,d\tau\right)\big\|\nabla u(t)\big\|_2^2 \\
&\quad -\varepsilon\left(\frac{p(1-\varepsilon_1)}{2}\int_0^t g(t-\tau)\big\|\nabla u(\tau)-\nabla u(t)\big\|_2^2\,d\tau\right. \\
&\quad \left.+\frac{1}{(1-\varepsilon_1)2p}\int_0^t g(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2\right) \\
&\geq (1-\sigma)H^{-\sigma}(t)\big\|\nabla u_t(t)\big\|_2^2+\varepsilon\left(\frac{1}{\rho+1}+\frac{p(1-\varepsilon_1)}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2}+\varepsilon(1-\varepsilon_1)p\big(E_2-E(t)\big) \\
&\quad +\varepsilon\left(\left(\frac{p(1-\varepsilon_1)}{2}-1\right)\left(1-\int_0^t g(\tau)\,d\tau\right)\big\|\nabla u(t)\big\|_2^2\right. \\
&\quad \left.-\frac{1}{(1-\varepsilon_1)2p}\int_0^t g(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2\right) \\
&\quad -\varepsilon(1-\varepsilon_1)pE_2+\varepsilon\varepsilon_1\|u\|_p^p \\
&= (1-\sigma)H^{-\sigma}(t)\big\|\nabla u_t(t)\big\|_2^2+\varepsilon\left(\frac{1}{\rho+1}+\frac{p(1-\varepsilon_1)}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2}+\varepsilon(1-\varepsilon_1)pH(t) \\
&\quad +\varepsilon\left(\left(\frac{p(1-\varepsilon_1)}{2}-1\right)\left(1-\int_0^t g(\tau)\,d\tau\right)\big\|\nabla u(t)\big\|_2^2\right. \\
&\quad \left.-\frac{1}{(1-\varepsilon_1)2p}\int_0^t g(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2\right) \\
&\quad -\varepsilon(1-\varepsilon_1)pE_2+\varepsilon\varepsilon_1\|u\|_p^p.
\end{aligned} \tag{3.15}$$

For the fourth term on the right side of (3.15), by (2.2) and Lemma 2.3, we obtain

$$\begin{aligned}
&\left(\frac{p(1-\varepsilon_1)}{2}-1\right)\left(1-\int_0^t g(\tau)\,d\tau\right)\big\|\nabla u(t)\big\|_2^2-\frac{1}{(1-\varepsilon_1)2p}\int_0^t g(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2 \\
&\quad=\frac{(\frac{p(1-\varepsilon_1)}{2}-1)(1-\int_0^t g(\tau)\,d\tau)-\frac{1}{(1-\varepsilon_1)2p}\int_0^t g(\tau)\,d\tau}{1-\int_0^t g(\tau)\,d\tau}\left(1-\int_0^t g(\tau)\,d\tau\right)\big\|\nabla u(t)\big\|_2^2 \\
&\quad\geq \frac{(\frac{p(1-\varepsilon_1)}{2}-1)l-\frac{1}{(1-\varepsilon_1)2p}(1-l)}{1-\int_0^t g(\tau)\,d\tau}\beta^2.
\end{aligned} \tag{3.16}$$

Then, by (3.15) and (3.16), we have

$$\begin{aligned} L'(t) \geq{}& (1-\sigma)H^{-\sigma}(t)\left\|\nabla u_t(t)\right\|_2^2 + \varepsilon\left(\frac{1}{\rho+1}+\frac{p(1-\varepsilon_1)}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon(1-\varepsilon_1)pH(t) \\ &+ \frac{(\frac{p(1-\varepsilon_1)}{2}-1)l - \frac{1}{(1-\varepsilon_1)2p}(1-l)}{1-\int_0^t g(\tau)\,d\tau}\varepsilon\beta^2 - \varepsilon(1-\varepsilon_1)pE_2 + \varepsilon\varepsilon_1\|u\|_p^p. \end{aligned} \tag{3.17}$$

Since

$$\int_0^\infty g(s)\,ds < \frac{p/2-1}{p/2-1+1/2p},$$

we have

$$\left(\frac{p}{2}-1\right)\left(1-\int_0^\infty g(\tau)\,d\tau\right) - \frac{1}{2p}\int_0^\infty g(\tau)\,d\tau > 0.$$

It is easy to see that there exists $\varepsilon_1^* > 0$, such that, for $0 < \varepsilon_1 < \varepsilon_1^*$,

$$\frac{(\frac{p(1-\varepsilon_1)}{2}-1)l - \frac{1}{(1-\varepsilon_1)2p}(1-l)}{1-\int_0^t g(\tau)\,d\tau}\beta^2 > \frac{(\frac{p(1-\varepsilon_1)}{2}-1)l - \frac{1}{(1-\varepsilon_1)2p}(1-l)}{1-\int_0^t g(\tau)\,d\tau}B_1^{-\frac{2p}{p-2}}. \tag{3.18}$$

Since

$$E(0) < \left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\frac{1}{p(p-2)}\frac{1-l}{l}\right)B_1^{-\frac{2p}{p-2}} = \frac{(\frac{p}{2}-1)l - \frac{1}{2p}(1-l)}{pl}B_1^{-\frac{2p}{p-2}},$$

we may choose $0 < \varepsilon_1 < 1$ sufficiently small, and $E_2 \in (E(0), E_1)$ sufficiently near E(0), such that

$$\frac{(\frac{p(1-\varepsilon_1)}{2}-1)l - \frac{1}{(1-\varepsilon_1)2p}(1-l)}{1-\int_0^t g(\tau)\,d\tau}B_1^{-\frac{2p}{p-2}} - p(1-\varepsilon_1)E_2 \geq 0. \tag{3.19}$$

Then, for $t > 0$, by (3.17) and (3.19), we obtain

$$\begin{aligned} L'(t) \geq{}& (1-\sigma)H^{-\sigma}(t)\left\|\nabla u_t(t)\right\|_2^2 + \varepsilon\left(\frac{1}{\rho+1}+\frac{p(1-\varepsilon_1)}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} \\ &+ \varepsilon(1-\varepsilon_1)pH(t) + \varepsilon\varepsilon_1\|u\|_p^p. \end{aligned} \tag{3.20}$$

From the above estimate we know there exists a constant $\gamma > 0$ such that

$$L'(t) \geq \varepsilon\gamma\left(\|u_t\|_{\rho+2}^{\rho+2} + H(t) + \|u\|_p^p\right) \geq 0, \quad t > 0, \tag{3.21}$$

where

$$\gamma = \min\left\{\left(\frac{1}{\rho+1}+\frac{p(1-\varepsilon_1)}{\rho+2}\right), (1-\varepsilon_1)p, \varepsilon_1\right\}.$$

Since

$$L(0) = H^{1-\sigma}(0) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_1 u_0\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla u_0|^2\,dx > 0,$$

combining (3.21), we have

$$L(t) \geq L(0) > 0, \quad t > 0.$$

We now estimate the term $\int_\Omega |u_t|^\rho u_t u\,dx$ as follows:

$$\left|\int_\Omega |u_t|^\rho u_t u\,dx\right| \leq \|u_t\|_{\rho+2}^{\rho+1}\|u\|_{\rho+2} \leq C\|u_t\|_{\rho+2}^{\rho+1}\|u\|_p.$$

Using Young's inequality, we have

$$\left(\left|\int_\Omega |u_t|^\rho u_t u\,dx\right|\right)^{\frac{1}{1-\sigma}} \leq C\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}}\|u\|_p^{\frac{1}{1-\sigma}} \leq C\big(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\mu} + \|u\|_p^{\frac{1}{1-\sigma}\theta}\big),$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. By choosing

$$\mu = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1,$$

we have

$$\frac{\theta}{1-\sigma} = \frac{\rho+2}{(1-\sigma)(\rho+2)-(\rho+1)}.$$

By (3.10), we know

$$\frac{\theta}{1-\sigma} < p. \tag{3.22}$$

Then, by (3.8), we have

$$\|u\|_p^{\frac{\theta}{1-\sigma}} = \|u\|_p^{p-(p-\frac{\theta}{1-\sigma})} = \|u\|_p^p\|u\|_p^{-k} \leq \|u\|_p^p CH^{-\frac{k}{p}}(t) \leq C\|u\|_p^p H^{-\frac{k}{p}}(0), \tag{3.23}$$

where $k = p - \frac{\theta}{1-\sigma}$ is a positive constant. Now, from (3.23), we have

$$\left(\left|\int_\Omega |u_t|^\rho u_t u\,dx\right|\right)^{\frac{1}{1-\sigma}} \leq C\big(\|u_t\|_{\rho+2}^{\rho+2} + \|u\|_p^p H^{-\frac{p}{k}}(0)\big).$$

Therefore it follows

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \frac{\varepsilon}{2}\int_\Omega \big|\nabla u(t)\big|^2\,dx\right)^{\frac{1}{1-\sigma}} \\ &\leq C\left(H(t) + \left|\int_\Omega |u_t|^\rho u_t u\,dx\right|^{\frac{1}{1-\sigma}} + \left|\int_\Omega \big|\nabla u(t)\big|^2\,dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\big(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|u\|_p^p + \big\|\nabla u(t)\big\|_2^{\frac{2}{1-\sigma}}\big). \end{aligned} \tag{3.24}$$

From (3.5) and (3.7), we have

$$\big\|\nabla u(t)\big\|_2^{\frac{2}{1-\sigma}} \leq C^{\frac{1}{1-\sigma}} \leq \frac{C^{\frac{1}{1-\sigma}}}{H(0)}H(t). \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$L^{\frac{1}{1-\sigma}}(t) \le C\big(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|u\|_p^p\big). \tag{3.26}$$

Combining (3.21) and (3.26), we arrive at

$$L'(t) > \frac{\varepsilon\gamma}{C} L^{\frac{1}{1-\sigma}}(t), \quad t > 0. \tag{3.27}$$

By a simple integration of (3.27) over $(0, t)$, we obtain

$$L^{\sigma/(1-\sigma)}(t) \ge \frac{1}{L^{-\sigma/(1-\sigma)}(0) - \varepsilon\gamma t\sigma/[C(1-\sigma)]}, \quad t > 0.$$

This shows that $L(t)$ blows up in finite time T_0, and

$$T_0 \le \frac{C(1-\sigma)}{\varepsilon\gamma\sigma L^{\sigma/(1-\sigma)}(0)}.$$

Furthermore, we have

$$\lim_{t\to T_0^-} \big(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|u\|_p^p\big) = \infty. \tag{3.28}$$

This leads to a contradiction with (3.5). Thus, the solution of problem (1.1) blows up in finite time. □

4 Global existence

In this section we show that the solution of (1.1) is global if $\rho + 2 \ge p$.

Theorem 4.1 *Assume that* (G1), (G2) *hold and* $2 < p \le \rho + 2$. *Assume further*

$$\rho + 2 \le \frac{2(n-1)}{n-2}.$$

Then for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *the solution of problem* (1.1) *exists globally.*

Proof We set

$$\begin{aligned} F(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p}\big\|u(t)\big\|_p^p \\ &= E(t) + \frac{2}{p}\big\|u(t)\big\|_p^p. \end{aligned} \tag{4.1}$$

Differentiating $F(t)$ and using (2.5), we get

$$F'(t) = E'(t) + 2\int_\Omega |u|^{p-2}uu_t\,dx \le -\|\nabla u_t\|_2^2 + 2\int_\Omega |u|^{p-2}uu_t\,dx. \tag{4.2}$$

Using the Hölder inequality and Young's inequality, we obtain the estimate

$$\left|\int_\Omega |u|^{p-2}uu_t\,dx\right| \le \tilde{\varepsilon}\|u_t\|_p^p + C(\tilde{\varepsilon})\|u\|_p^p, \tag{4.3}$$

in which $\tilde{\varepsilon}$ is a small positive constant to be chosen later, and $C(\tilde{\varepsilon})$ is a positive constant depending on $\tilde{\varepsilon}$.

Using Lemma 2.4, (4.3) and the embedding theorem, we obtain

$$\begin{aligned}\left|\int_\Omega |u|^{p-2}uu_t\,dx\right| &\le \tilde{\varepsilon}\|u_t\|_p^2 + C_1\tilde{\varepsilon}\|u_t\|_{\rho+2}^{\rho+2} + C(\tilde{\varepsilon})\|u\|_p^p \\ &\le \tilde{\varepsilon}B\|\nabla u_t\|_2^2 + C_1\tilde{\varepsilon}\|u_t\|_{\rho+2}^{\rho+2} + C(\tilde{\varepsilon})\|u\|_p^p. \end{aligned} \tag{4.4}$$

Substituting (4.4) to (4.2), we have

$$F'(t) \le -(1-\tilde{\varepsilon}B)\|\nabla u_t\|_2^2 + C_1\tilde{\varepsilon}\|u_t\|_{\rho+2}^{\rho+2} + C(\tilde{\varepsilon})\|u\|_p^p. \tag{4.5}$$

Choosing $\tilde{\varepsilon} = \frac{1}{2B}$ in (4.5), we arrive at

$$F'(t) \le C_2F(t).$$

Furthermore we obtain

$$F(t) \le C_3e^{C_2t}.$$

This completes the proof of the global existence result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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