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On the smoothness of solutions of the third order nonlinear differential equation

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Abstract

In this work we study the following third order differential equation:

$$Ly := y''' + (q(x,y) + \lambda)y = f \in L_2(\mathbb{R}), \quad \mathbb{R} = (-\infty, \infty), \lambda > 0, \tag{1}$$

where $q(x,y) \geq 1$ is a continuous function in all its variables.

We will deal with the following questions:

(a) The existence of a solution to equation (1) in the space $L_2(\mathbb{R})$ where $L_2(\mathbb{R})$ is the space of square summable functions.

(b) Additional conditions on the third derivative of this solution to belong to the space $L_2(\mathbb{R})$.

Keywords: nonlinear equation; solvability; separability; smoothness

1 Introduction

The questions posed in the abstract are equivalent to the so-called 'separability' of the nonlinear differential operator

$$Ly := -y''' + \big(q(x,y) + \lambda\big)y, \quad x \in \mathbb{R}, \lambda > 0,$$

on the domain

$$D(L) = \big\{y : y \in L_2(\mathbb{R}), Ly \in L_2(\mathbb{R}), y''' \in L_{2,\mathrm{loc}}(\mathbb{R})\big\}.$$

Statement of the fundamental problem of separability of the differential operator belongs to Everittand and Giertz [1, 2]. They studied the Sturm-Liouville operator, which is known, in many cases, to be the 'touchstone' for the proposed methods of studies. Their research has been continued by, among others, Atkinson [3], Evans and Zettle [4], Otelbaev [5] and Boymatov [6].

In the last years, many mathematicians have applied the methods of Everita and Giertza [2], which consist in the use of classical techniques for the study of the asymptotic behavior at infinity of the Green's function of the considered operator.

In the second half of the seventies of the last century, the study of the separability problem began to apply new methods proposed by Otelbaev [5]. In order to solve these issues, he proposed some modification on the Titchmarsh method, previously used to solve different problems studied in the works of Kostyuchenko [7], Levitan [8] and Gasymov [9].

Later on, in order to solve the problem on the smoothness of the solutions of differential equations, Otelbaev proposed the special method of the local representation of the resolvent, which is called variational method. For unbounded domains, the existence and smoothness of solutions of nonlinear Sturm-Liouville differential equations (with a singular potential) were studied in [10]. However, the smoothness of solutions of nonlinear differential equations remains poorly studied compared with the efforts devoted to the study of linear differential equations. In this case, there are no existing traditional methods that could be applied to the available large number of problems encountered in applications. Note that in recent years several studies devoted to this area were published in [11–13].

2 Main result

The purpose of this work is to study questions related to the existence and smoothness of solutions of odd order nonlinear differential equations of the type (1).

It is very well known that, for even order operators, the following condition is fulfilled:

$$\|Lu\|_2 \geq c\|u\|_2.$$

Here $\|\cdot\|_2$ is the usual norm in the space $L_2(\mathbb{R})$ and c is a suitable constant independent of u. However, this property fails when an odd order operator is considered. This fact makes their study more difficult to deal with than the one delivered to even order operators.

In the sequel, we introduce the hypotheses on the data of the considered equation.

Assume that function q satisfies the following inequality:

$$q(x,y) \geq s(x) > 0, \quad x \in \mathbb{R}, \forall y \in \mathbb{R}, \tag{2}$$

where $s(x)$ tends to $+\infty$ whenever $|x| \to +\infty$.

Assume that there exist numbers $0 < A < \infty$, $T(A)$ and $B(A)$ such that

$$\sup_{|x-\eta|\leq 1} \sup_{|C_1-C_2|\leq A, |C_1|\leq A} \frac{q(x,C_1)}{q(\eta,C_2)} \leq T(A) < \infty, \tag{3}$$

and they satisfy the Hölder-type conditions

$$\sup_{|x-\eta|\leq 1} \sup_{|C_1-C_2|\leq A, |C_1|\leq A} \frac{|q(x,C_1) - q(\eta,C_2)|}{q^a(x,C_1)[|x-\eta|^2 + |C_1-C_2|^{2\alpha}]} \leq B(A) < \infty, \tag{4}$$

where a is a given real constant satisfying the following inequalities:

$$3 - 3a + \alpha > 0, \quad a > 0, \alpha \in \left[0, \frac{1}{2}\right].$$

Now we enunciate the main result of this paper.

Theorem 1 *Assume that conditions* (2)-(4) *are fulfilled, then, for any right side* $f \in L_2(\mathbb{R})$, *there exists a real number* $\mu = \mu(A,f)$ *satisfying*

$$0 < \max\left\{6\left[T(A)\right]^{\frac{\alpha}{3}+1+a} B(A)(1+A)^{2\alpha}, 160\sqrt[3]{T^2(A)}, 12\left(\frac{T(A)\|f\|_2}{A}\right)^{4/9}\right\} < \mu(A,f),$$

such that for all $\lambda > \mu$ *equation* (1) *has a solution* $y \in W_2^1(\mathbb{R})$ *satisfying* $\|y\|_{2,1} \leq A$ *and having a quadratically summable third order derivative on the entire axis*.

Here and hereinafter, $W_2^1(\mathbb{R})$ denotes the classical space of Sobolev with the usual norm $\|\cdot\|_{2,1}$.

Remark 1 Note that the previous theorem implies the smoothness of solutions of equation (1) for λ large enough.

Example 1 One can verify that an example of an equation whose coefficients satisfy the conditions of Theorem 1 is the following one:

$$-y''' + \left(e^{|x|} + e^{|y|} + 1\right)y + \lambda y = f \in L_2(\mathbb{R}).$$

Let the function $\mathcal{V} \in W_2^1(\mathbb{R})$ be such that

$$\|\mathcal{V}\|_{2,1} \leq A.$$

To prove Theorem 1, we consider the closure $\tilde{L}_{\mathcal{V}}$ on the norm $L_2(\mathbb{R})$ of the differential expression

$$L_0 y = -y''' + q\left(x, \mathcal{V}(x)\right)y,$$

defined on $C_0^\infty(\mathbb{R})$, where $C_0^\infty(\mathbb{R})$ is the set of infinitely many differentiable functions in $\mathbb{R}$ that vanish at $x \pm \infty$.

Before obtaining the main result of this paper, we prove some preliminary lemmas.

Lemma 1 *Suppose that conditions* (2)-(4) *are fulfilled. Then, for all* $\lambda > \mu(A)$ ($\mu(A)$ *is a large enough number*), *the following properties hold*:

(a) *Operator* $\tilde{L}_{\mathcal{V}} + \lambda E$ *defined in the space* $L_2(\mathbb{R})$ *has a bounded inverse operator.*

(b) *Operator* $\tilde{L}_{\mathcal{V}}$ *is separable.*

Moreover, the following estimates hold:

$$\|y\|_2 \leq \frac{40T(A)}{\sqrt[3]{\lambda^2}}\|f\|_2,$$

$$\left\|y'''\right\|_2 \leq 9T(A)\|f\|_2.$$

To prove this lemma, we use the following result.

Assertion 1 ([5], Lemma 2) Let $\mathcal{K}$ be an integral operator in $L_2(\mathbb{R})$ with continuous kernel $K(x,\eta)$:

$$(Kf)(x) = \int_{-\infty}^{\infty} K(x,\eta)f(\eta)\,d\eta.$$

Then

$$\|K\|_2 \leq \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \left(\left|K(x,\eta)\right| + \left|K(\eta,x)\right|\right)dh.$$

In order to prove Lemma 1, we introduce the following kernels for any $\lambda > 0$:

$$M_0(x,\eta,\mathcal{V}(x),\lambda) = \\ = \begin{cases} \frac{\exp[\frac{(x-\eta)}{2}\sqrt[3]{q(x,\mathcal{V}(x))+\lambda}]}{3\sqrt[3]{(q(x,\mathcal{V}(x))+\lambda)^2}} & \text{if } x > \eta, \\ \frac{\exp[\frac{(x-\eta)}{2}\sqrt[3]{q(x,\mathcal{V}(x))+\lambda}]}{3\sqrt[3]{(q(x,\mathcal{V}(x))+\lambda}}[\cos\frac{\sqrt{3}}{2}(q(x,\mathcal{V}(x))+\lambda)^{\frac{1}{2}}(x-\eta) & \\ -\sqrt{3}\sin\frac{\sqrt{3}}{2}(q(x,\mathcal{V}(x)+\lambda)^{\frac{1}{3}}(x-\eta)] & \text{if } x < \eta, \end{cases}$$

$$M_1(x,\eta,\mathcal{V}(x),\lambda) = -[q(x,\mathcal{V}(x)) + q(\eta,\mathcal{V}(\eta))]M_0(x,\eta,\mathcal{V}(x),\lambda)r(\eta - x),$$

$$M_2(x,\eta,\mathcal{V}(x),\lambda) = -3M''_{0\eta}(x,\eta,\mathcal{V}(x),\lambda)r'_\eta(\eta - x) + M'_{0\eta}(x,\eta,\mathcal{V}(x),\lambda)r''_\eta(\eta - x),$$

$$M_3(x,\eta,\mathcal{V}(x),\lambda) = -M_0(x,\eta,\mathcal{V}(x),\lambda)r'''_2(\eta - x),$$

$$M_4(x,\eta,\mathcal{V}(x),\lambda) = M_0(x,\eta,\mathcal{V}(x),\lambda)r(\eta - x),$$

where function $r \in C_0^\infty(\mathbb{R})$ is constructed by using the following function:

$$\omega(z) = \begin{cases} C\exp(-\frac{1}{1-z^2}) & \text{if } |z| < 1, \\ 0 & \text{if } |z| \geq 1, \end{cases}$$

and satisfy the following properties:

$$r(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2, \\ \int_{2t-3}^{2t+3}\omega(z)\,dz & \text{if } t \in (-2,-1)\cup(1,2), \end{cases} \tag{5}$$

and

$$\sup_{|t|\leq 2}\{|r'(t)|,|r''(t)|,|r'''(t)|\} \leq 20.$$

Note that at $x \neq \eta$ the following property is fulfilled:

$$M'''_{0\eta}(x,\eta,\mathcal{V}(x),\lambda) = (q(x,\mathcal{V}(x)) + \lambda)M_0(x,\eta,\mathcal{V}(x),\lambda).$$

Now, let us denote the following operators defined by

$$(M_j(\lambda)f)(\eta) = \int_{-\infty}^{\infty} M_j(x,\eta,\mathcal{V}(x),\lambda)f(x)\,dx \quad (j = 1,2,3,4).$$

Let us prove the following preliminary result.

Lemma 2 *If* $f \in C_0^\infty(\mathbb{R})$, *then the following equality holds*:

$$(\tilde{L}_{\mathcal{V}} + \lambda E)M_4(\lambda)f = f + M_1(\lambda)f + M_2(\lambda)f + M_3(\lambda)f.$$

Proof From the definition of operator $M_4(\lambda)$, we have

$$\begin{aligned}\big(M_4(\lambda)f\big)(\eta) &= \frac{1}{3}\int_\eta^\infty \frac{1}{\sqrt[3]{(q(x,\mathcal{V}(x))+\lambda)^2}}\exp\big[-(x-\eta)\sqrt[3]{q\big(x,\mathcal{V}(x)+\lambda\big)}\big]r(\eta-x)f(x)\,dx\\ &\quad+\frac{1}{3}\int_\eta^\infty \frac{1}{\sqrt[3]{(q(x,\mathcal{V}(x))+\lambda)^2}}\exp\bigg[\frac{(x-\eta)}{2}\sqrt[3]{q\big(x,\mathcal{V}(x)\big)+\lambda}\bigg]\\ &\quad\times\bigg[\cos\frac{\sqrt{3}}{2}\big(q\big(x,\mathcal{V}(x)\big)+\lambda\big)^{\frac{1}{3}}(x-\eta)-\sqrt{3}\sin\frac{\sqrt{3}}{2}\big(q\big(x,\mathcal{V}(x)\big)+\lambda\big)^{\frac{1}{3}}\bigg]\\ &\quad\times r(\eta-x)f(x)\,dx.\end{aligned}$$

From this and (5), it is not difficult to verify that $M_4(\lambda)f\in D(\tilde{L}_{\mathcal{V}})$.
Further

$$(\tilde{L}_{\mathcal{V}}+\lambda E)M_4(\lambda)f=-\big(M_4(\lambda)f\big)_\eta'''+\big(q\big(\eta,\mathcal{V}(\eta)\big)+\lambda\big)M_4(\lambda)f.$$

After simple calculations, we deduce that

$$\begin{aligned}&\big(M_4(\lambda)f\big)_\eta'''+\big(q\big(\eta,\mathcal{V}(\eta)\big)+\lambda\big)M_4(\lambda)f\\ &\quad=f(\eta)-\int_{-\infty}^{\infty}\big[q\big(x,\mathcal{V}(x)\big)-q\big(\eta,\mathcal{V}(\eta)\big)\big]M_0\big(x,\eta,\mathcal{V}(x),\lambda\big)r(\eta-x)f(x)\,dx\\ &\qquad-3\int_{-\infty}^{\infty}M_{0\eta}''\big(x,\eta,\mathcal{V}(x),\lambda\big)f(x)r'(\eta-x)\\ &\qquad+M_{0\eta}'\big(x,\eta,\mathcal{V}(x),\lambda\big)r''(\eta-x)f(x)\,dx\\ &\qquad+\int_{-\infty}^{\infty}M_0\big(x,\eta,\mathcal{V}(x),\lambda\big)r'''(\eta-x)f(x)\,dx\\ &\quad=f+M_1(\lambda)f+M_2(\lambda)f+M_2(\lambda)f.\end{aligned}$$

So, Lemma 2 is proved. □

Proof of Lemma 1 We start the proof by looking for estimates of the norms of the operators $M_j(\lambda)$ $(j=1,2,3,4)$, under the assumptions of Lemma 1.
By virtue of Assertion 1, we have

$$\big\|M_j(\lambda)\big\|\le\sup_{\eta\in\mathbb{R}}\int_{-\infty}^{\infty}\big(\big|M_j\big(x,\eta,\mathcal{V}(x),\lambda\big)\big|+\big|M_j\big(x,\eta,\mathcal{V}(\eta),\lambda\big)\big|\big)\,dx.$$

Since $\mathcal{V}\in W_2^1(\mathbb{R})$, then $\mathcal{V}(\eta)-\mathcal{V}(x)=\int_x^\eta\mathcal{V}'(t)\,dt$.
By Bunyakovskii's inequality, using that $\|\mathcal{V}\|_{2,1}\le A$, we deduce that

$$\big|\mathcal{V}(\eta)-\mathcal{V}(x)\big|\le|x-\eta|^{\frac{1}{2}}A. \tag{6}$$

Using the representation of $M_1(x,\eta,\mathcal{V}(x),\lambda)$, under conditions (3) and (4), and taking (6) into account, we obtain

$$\big\|M_1(\lambda)\big\|_2\le\frac{8}{3}T(A)^{\frac{\alpha}{3}+1+\alpha}B(A)\big(1+A^{2\alpha}\big)\lambda^{-\gamma},\quad\text{with }\gamma=\frac{2}{3}+1-a>0.$$

Moreover, we have that

$$\int_{-\infty}^{\infty} \left(\left| M_2\big(x,\eta,\mathcal{V}(x),\lambda\big) \right| + \left| M_2\big(\eta,x,\mathcal{V}(\eta),\lambda\big) \right| \right) dx$$
$$= 3 \int_{-\infty}^{\infty} \left(\left| M''_{0\eta}\big(x,\eta,\mathcal{V}(x),\lambda\big) r'_{\eta}(\eta - x) + M'_{0\eta}\big(x,\eta,\mathcal{V}(X),\lambda\big) r''(\eta - x) \right| \right.$$
$$\left. + \left| M'_{0\eta}\big(x,\eta,\mathcal{V}(\eta),\lambda\big) r'_{\eta}(\eta - x) + M'_{0\eta}\big(x,\eta,\mathcal{V}(\eta)\big),\lambda\big) r''_{\eta}(\eta - x) \right| dx.$$

So, we can conclude

$$\left\| M_2(\lambda) \right\|_2 \le \frac{160\,T(A)}{\sqrt{\lambda}}.$$

In the last two estimates, we have used condition (3) and equality (5).

The norms of $M_3(\lambda)$ and $M_4(\lambda)$ can be estimated in a similar way.

Denote now

$$S(\lambda) = M_1(\lambda) + M_2(\lambda) + M_3(\lambda).$$

In the sequel, we will prove the following equality:

$$(\tilde{L}_{\mathcal{V}} + \lambda E)^{-1} = M_4(\lambda)\big(E + S(\lambda)\big)^{-1}.$$

Using the norm's estimations of operators $M_j(\lambda)$, it follows that for all $\lambda > \mu(A)$, operator $E + S(\lambda)$ is bounded along with its inverse and, therefore, the set

$$M = \left\{ g, g = \big(E + S(\lambda)\big) f, f \in C_0^{\infty}(\mathbb{R}) \right\}$$

is dense in $L_2(\mathbb{R})$.

From Lemma 2, applied to $g = B(\lambda)f$, with $f \in C_0^{\infty}(\mathbb{R})$ and $B(\lambda) = E + S(\lambda)$, it follows that

$$M_4(\lambda)f = M_4(\lambda)B^{-1}(\lambda)g \in D(\tilde{L}_{\mathcal{V}}).$$

As a consequence,

$$(\tilde{L}_{\mathcal{V}} + \lambda E)M_4(\lambda)B^{-1}(\lambda)g = (\tilde{L}_{\mathcal{V}} + \lambda E)M_4(\lambda)f = B(\lambda)f = g,$$

or, which is the same,

$$(\tilde{L}_{\mathcal{V}} + \lambda E)M_4(\lambda)B^{-1}(\lambda) = E.$$

Since the set M is dense in $L_2(\mathbb{R})$, we obtain that

$$(\tilde{L}_{\mathcal{V}} + \lambda E)^{-1} = M_4(\lambda)B^{-1}(\lambda). \tag{7}$$

Obviously, operator $q(x,\mathcal{V}(x))(\tilde{L}_{\mathcal{V}} + \lambda E)^{-1}$ is bounded if and only if operator $q(x,\mathcal{V}(x))M_4(\lambda)$ is also bounded. Therefore, to ensure the separability of operator $\tilde{L}_{\mathcal{V}}$, it is sufficient to show the boundedness of operator $q(x,\mathcal{V}(x))M_4(\lambda)$.

By virtue of condition (2) and the properties of function r, we deduce that the kernel $M(x,\eta,\mathcal{V}(x),\lambda)$ of operator $q(x,\mathcal{V}(x))M_4(\lambda)$ satisfies the following inequality:

$$\begin{aligned} &\left|M\left(x,\eta,\mathcal{V}(x),\lambda\right)\right| \\ &\quad\le \begin{cases} \frac{1}{3}\sqrt[3]{T(A)q(\eta,\mathcal{V}(\eta))+\lambda}\exp\left[-(x-\eta)\sqrt[3]{\frac{q(\eta,\mathcal{V}(\eta))+\lambda}{T(A)}}\right] & \text{if } x>\eta, |x-\eta|\le 2, \\ \frac{1}{3}\sqrt[3]{T(A)q(\eta,\mathcal{V}(\eta)+\lambda}\exp\left[\frac{(x-\eta)}{2}\sqrt[3]{\frac{q(\eta,\mathcal{V}(\eta))+\lambda}{T(A)}}\right] & \text{if } x>\eta, |x-\eta|\le 2. \end{cases} \end{aligned}$$

Hence, after simple calculations, we conclude that

$$\int_{-\infty}^{\infty}\left(\left|M\left(x,\eta,\mathcal{V}(x),\lambda\right)\right| + \left|M\left(\eta,x,\eta,\mathcal{V}(\eta),\lambda\right)\right|\right)dx \le 4T(A).$$

So, by means of this inequality, we can verify the conditions of Assertion 1 and obtain the boundedness of operator $q(x,\mathcal{V}(x))M_4(\lambda)$, and besides

$$\left\|q\left(x,\mathcal{V}(x)\right)M_4(\lambda)\right\|_2 \le 4T(A). \tag{8}$$

Further, we have

$$\begin{aligned} \left\|y'''\right\|_2 &\le \|\tilde{L}_{\mathcal{V}}\|_2 + \left\|q\left(x,\mathcal{V}(x)\right)y(x)\right\|_2 = \|f\|_2 + \left\|q\left(x,\mathcal{V}(x)\right)(\tilde{L}_{\mathcal{V}}+\lambda E)^{-1}f\right\|_2 \\ &\le \|f\|_2 + 2\left\|q\left(x,\mathcal{V}(x)\right)M_4(\lambda)f\right\|_2 \le \left(1+2\left\|q\left(x,\mathcal{V}(x)M_4(\lambda)\right)\right\|\right)\|f\|_2. \end{aligned}$$

By virtue of (8), it follows that

$$\left\|y'''\right\|_2 \le 9T(A)\|f\|_2.$$

Since $(\tilde{L}_{\mathcal{V}}+\lambda E)^{-1}$ exists and is bounded, we have that

$$y = (\tilde{L}_{\mathcal{V}}+\lambda E)^{-1}f.$$

Therefore, by (7)

$$\begin{aligned} \|y\|_2 &\le \left\|(\tilde{L}_{\mathcal{V}}+\lambda E)^{-1}f\right\|_2 = \left\|M_4(\lambda)\left(E+S(\lambda)\right)^{-1}f\right\|_2 \\ &\le \left\|M_4(\lambda)\right\|_2\left\|\left(E+S(\lambda)\right)^{-1}\right\|_2\|f\|_2. \end{aligned}$$

It is clear that for all $\lambda>\mu(A)$, from the estimations on the norms of $M_j(\lambda)$ ($j=1,2,3,4$), we deduce that $\|S(\lambda)\|_2 \le \frac{1}{2}$, hence, by the well-known theory of inverse operators, we arrive at

$$\left\|E+S(\lambda)\right\|_2 \le 2,$$

and we obtain the inequality

$$\|y\|_2 \le \frac{40T(A)}{\sqrt[3]{\lambda^2}}\|f\|_2,$$

and the proof of Lemma 1 is finished. □

Lemma 3 *Let $\mathcal{V} \in W_2^1(\mathbb{R})$ be such that $\|\mathcal{V}\|_{2,1} \leq A$. Suppose that the conditions of Lemma* 1 *are fulfilled. Then, for all $\lambda > \mu(A)$, the equation*

$$\tilde{L}_{\mathcal{V}} y \equiv -y''' + \big(q\big(x, \mathcal{V}(x)\big) + \lambda\big) y = f \in \tilde{L}_{\mathcal{V}}(\mathbb{R})$$

has a solution $y \in W_2^1(\mathbb{R})$ satisfying

$$\|y\|_{2,1} \leq \frac{1}{2} A. \tag{9}$$

Proof By the embedding theorem, the following inequality holds:

$$\|y'\|_2^2 \leq \|y'''\|_2^2 + \|y\|_2^2. \tag{10}$$

After replacing $x = at$, inequality (10) has the form

$$a^{-2} \|y'\|_2^2 \leq \frac{1}{a^6} \|y'''\|_2^2 + \|y\|_2^2.$$

Hence, from Lemma 1, the following inequality is attained:

$$\begin{aligned} \|y'\|_2^2 &\leq a^{-4} \|y'''\|_2^2 + a^2 \|y\|_2^2 \leq \left[a^{-4} 81 T^2(A) + \frac{a^2 40 T(A)}{\sqrt[3]{\lambda^4}} \right] \|f\|_2^2 \\ &= a^{-4} \left[81 T^2(A) + a^6 \frac{1{,}600 T^2(A)}{\sqrt[3]{\lambda^4}} \right] \|f\|_2^2. \end{aligned}$$

Putting $a^6 = \lambda^{4/3}$, we have

$$\|y'\|_2^2 \leq 1{,}681 \lambda^{-8/9} T^2(A) \|f\|_2^2.$$

From these inequalities we deduce

$$\|y\|_{2,1}^2 = \|y'\|_2^2 + \|y\|_2^2 \leq 72 \lambda^{-\frac{8}{9}} T^2(A) \|f\|_2^2.$$

Therefore, for all $\lambda > 12(\frac{T(A)\|f\|_2}{A})^{\frac{4}{9}}$, we obtain the validity of estimate (9) and the result is proved. □

Lemma 4 *Let conditions* (2)-(4) *be fulfilled and $\lambda \geq \mu(A)$. Then, for a fixed $f \in L_2(\mathbb{R})$, the set*

$$\big\{ y; y = (\tilde{L}_{\mathcal{V}} + \lambda E)^{-1} f, \|\mathcal{V}\|_{2,1} \leq A \big\}$$

is compact in the space $W_2^1(\mathbb{R})$.

Proof Since conditions (3) and (4) are satisfied, using Lemma 1 for any $y \in D(\tilde{L}_{\mathcal{V}})$, it is not difficult to verify the validity of the next inequality

$$\|y'''\|_2 + \big\|q\big(x, \mathcal{V}(x)\big) y\big\|_2 \leq 14 T(A) \|f\|_2 \equiv M.$$

Thus, using equation (8) and Lemma 3, we deduce that

$$y \in W_2^1(\mathbb{R}) \quad \text{and} \quad \|y\|_{2,1} \leq \frac{A}{2}.$$

Now, since y is a continuous function, we conclude that the following inclusion is fulfilled:

$$\{y, y = (L_{\mathcal{V}} + \lambda E)^{-1} f, \|\mathcal{V}\|_{2,1} \leq A\} \subset \{y; \|y'''\|_2 + \|q(x, \mathcal{V}(x))y\|_2 \leq M\} \equiv N.$$

So, to prove Lemma 4, it is enough to verify the compactness of the set N in $W_2^1(\mathbb{R})$. And this, by virtue of condition (2), follows from the results given in [14]. □

So, we are in a position to prove the main result of this paper, Theorem 1.

To this end, we start by proving the continuity of operator

$$Z_f(\mathcal{V}) := (\tilde{L}_{\mathcal{V}} + \lambda E)^{-1} f,$$

which transforms, for a fixed $f \in L_2(\mathbb{R})$, any function $\mathcal{V} \in W_2^1(\mathbb{R})$ into the function $(\tilde{L}_{\mathcal{V}} + \lambda E)^{-1} f \in W_2^1(\mathbb{R})$.

Let

$$\mathcal{V} \in E(0, A) := \{\mathcal{V} \in W_2^1(\mathbb{R}); \|\mathcal{V}\|_{2,1} \leq A\}$$

and

$$\mathcal{V}_n \to \mathcal{V} \in L_2(\mathbb{R}), \quad \mathcal{V}_n \in E(0, A).$$

Suppose that

$$(\tilde{L}_{\mathcal{V}} + \lambda E)y \equiv -y''' + q(x, \mathcal{V}) + \lambda y = f$$

and

$$(\tilde{L}_{\mathcal{V}_n} + \lambda E)y_n \equiv -y_n''' + q(x, \mathcal{V}_n) + \lambda y_n = f,$$

where $f \in L_2(\mathbb{R})$ is a given function.

Then we have

$$-(y_n - y)''' + q(x, \mathcal{V}_n(x))y_n - q(x, \mathcal{V}(x))y + \lambda(y_n - y) = 0,$$

or, which is the same,

$$\begin{aligned}
0 = & -(y_n - y)''' + q(x, \mathcal{V}_n(x))y_n - q(x, \mathcal{V}(x))y_n + q(x, \mathcal{V}(x))y_n \\
& - q(x, \mathcal{V}(x))y + \lambda(y_n - y) \\
= & -(y_n - y)''' + q(x, \mathcal{V}_n(x))(y_n - y) + \lambda(y_n - y) \\
= & [q(x, \mathcal{V}_n(x)) - q(x, \mathcal{V}(x))]y_n,
\end{aligned}$$

that is,

$$y_n - y = (\tilde{L}_{\mathcal{V}} + \lambda E)^{-1}\bigl[q\bigl(x, \mathcal{V}_n(x)\bigr) - q\bigl(x, \mathcal{V}(x)\bigr)\bigr]y_n. \tag{11}$$

By the embedding theorem, $\mathcal{V}(x)$ and $\mathcal{V}_n(x)$ are continuous functions. Then, since $q(x,y)$ is continuous in both arguments, it is clear that

$$q\bigl(x, \mathcal{V}(x)\bigr) \in C_{\mathrm{loc}}(\mathbb{R}). \tag{12}$$

Let $[a,b]$ be any compact interval of $\mathbb{R}$. Then by (11) and (12) we deduce that there is a positive constant C, for which the following inequalities are fulfilled:

$$\begin{aligned}\lim_{n\to\infty} \|y_n - y\|^2_{[a,b]} &\le C \lim_{n\to\infty} \bigl\|\bigl[q\bigl(x, \mathcal{V}_n(x)\bigr) - q\bigl(x, \mathcal{V}(x)\bigr)\bigr]y_n\bigr\|^2_{2,[a,b]} \\ &\le C \lim_{n\to\infty} \sup_{x\in[a,b]} \bigl\|q\bigl(x, \mathcal{V}_n(x)\bigr) - q\bigl(x, \mathcal{V}(x)\bigr)\bigr\| \|y_n\|^2_{2,[a,b]} \\ &\le CA \lim_{n\to\infty} \sup_{x\in[a,b]} \bigl|q\bigl(x, \mathcal{V}_n(x)\bigr) - q\bigl(x, \mathcal{V}(x)\bigr)\bigr| = 0. \end{aligned} \tag{13}$$

On the other hand, by Lemma 3 and the known embedding theorem [15], we conclude that there exists $z \in L_2(\mathbb{R})$ such that $y_n \to z$ strongly in $L_2(\mathbb{R})$.

According to Lemma 1, we have that the following estimation is fulfilled:

$$\|y\|_2 \le \frac{40T(A)}{\sqrt[3]{\lambda^2}}\|f\|_2 \quad \text{and} \quad \bigl\|y'''\bigr\|_2 \le 9T(A)\|f\|_2.$$

Therefore, it is enough to study the behavior of the norm in $L_2(\mathbb{R})$. Taking into account (13), we conclude that $z = y$ and, as a consequence, operator Z_f is continuous.

On the other hand, operator Z_f, as follows from Lemma 1, applies the ball $E(0,A)$ into itself. From Lemma 4 and the continuity of operator Z_f we can deduce that it is completely continuous. Consequently, according to the principle of Schauder [16], operator Z_f has a fixed point in the ball $E(0,A) \subset W_2^1(\mathbb{R})$.

Therefore the equation

$$Ly \equiv -y''' + q(x,y)y + \lambda y = f \in L_2(\mathbb{R})$$

has a solution y which lies in a ball of radius A in $W_2^1(\mathbb{R})$.

As a consequence, all the conditions of Lemma 1 for operator $\tilde{L}_{\mathcal{V}} + \lambda E$ at $\mathcal{V}(x) = y(x)$ are satisfied.

This completes the proof of Theorem 1.

Remark 2 Similar results can be proved for the equation

$$Ly \equiv -y^{(2n+1)} + q(x,y)y + \lambda y = f, \quad \lambda > 0,$$

in the space $L_2(\mathbb{R})$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors have contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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