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Reducibility of beam equations in higher-dimensional spaces

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Abstract
In this paper, we reduce a linear d-dimensional beam equation with an x-periodic and t-quasi-periodic potential for most values of the frequency vector via the KAM theorem. We focus on the measure estimates of small divisor conditions and the estimation on the coordinate transformation.

Keywords: infinite-dimensional Hamiltonian systems; beam equations; reducibility; invariant torus

1 Introduction

Inspired by an old report of the existence of traveling waves on the Golden Gate Bridge in San Francisco in 1938, the study of traveling waves in supported beams was begun in [1] and [2]. The first result is the following beam equation in [2]:

$$u_{tt} + u_{xxxx} + u^{+} = 1, \quad x \in \mathbb{R}^1. \tag{1}$$

The solutions of the form $1 + y(x - ct)$ were found by reducing the partial differential equation (1) to the ordinary differential equation on the real line. Later, many solutions were constructed and calculated by the mountain pass algorithm. Until recently, there has been little progress on the proof of existence of solutions of the beam equations, especially $x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. Recently, Geng and You in [3] obtained the higher-dimensional nonlinear beam equations with periodic boundary conditions by the KAM method, *i.e.*,

$$u_{tt} + \Delta^2 u + \sigma u + f(u) = 0, \tag{2}$$

where the nonlinearity $f(u)$ was a real-analytic function near $u = 0$ with $f(0) = f'(0) = 0$. Notice that the perturbation in [3] satisfied special conditions and did not explicitly containing the space variables and the time variable, which is crucial for the proof. Eliasson, Grébert, Kuksin [4] made a breakthrough in KAM theory for the space-multidimensional beam equation with the perturbation containing the space variables. In [4] they considered the nonlinear beam equation

$$u_{tt} + \Delta^2 u + mu + \partial_u G(x,u) = 0, \quad t \in \mathbb{R}, x \in \mathbb{T}^d,$$

where $G(x,u) = u^4 + O(u^5)$. Thanks to those results of admissible sets and the proof of the KAM theorem we obtain the existence of quasi-periodic solutions of a simper beam equation.

In this paper we focus us our attention on the following non-autonomous, d-dimensional beam equation with quasi-periodic forcing:

$$u_{tt} + (-\Delta + M)^2 u + \varepsilon\psi(\omega t,x)u = 0, \quad M > 0, t \in \mathbb{R}, x \in \mathbb{T}^d, \tag{3}$$

with periodic boundary conditions

$$u(t,x_1,x_2,\ldots,x_d) = u(t,x_1 + 2\pi,x_2,\ldots,x_d) = \cdots = u(t,x_1,x_2,\ldots,x_d + 2\pi), \tag{4}$$

where ε is a small parameter, the frequency vector $\omega = (\omega_1,\ldots,\omega_m) \in [\varrho,2\varrho]^m$, $0 < \varrho < 1$, $\psi(\omega t,x)$ is a real-analytic function with x-periodic and t-quasi-periodic.

The forced problem is an important feature of the classical perturbation for Hamiltonian systems. The reducibility of finite-dimensional systems is interesting itself and remains open in the general case. In [5], Bogoljubov firstly applied KAM-techniques to reduce of non-autonomous finite-dimensional linear systems to constant coefficient equations. In [6], Bambusi and Graffi gave a general proof of reducibility of quasi-periodically forced PDEs. Jianguo Si [7] considered the existence of small-amplitude quasi-periodic solutions of the quasi-periodically forced nonlinear wave equations. In [8], the author of this paper and Si proved the existence of quasi-periodic solutions of quasi-periodically forced nonlinear Schrödinger equations with quasi-periodic inhomogeneous terms, *i.e.*,

$$iu_t - u_{xx} + mu + \phi(t)|u|^2 u = \varepsilon g(t).$$

The reducibility is well developed for one-dimensional Hamiltonian systems. For higher-dimensional Hamiltonian PDEs, there are few results of reducibility via the KAM theorem because of the multiplicity of eigenvalues. It is worth noting that in the higher-dimensional case the multiplicity goes asymptotically to infinity. On one hand, it is due to the unperturbed part and solving the linearized equations being more complicated in a KAM iteration; on the other hand, it makes the measure estimation very difficult since there are so many non-resonance conditions to be satisfied. To overcome this difficulty, Bourgain [9] made the first breakthrough by proving that the two-dimensional nonlinear Schrödinger equations admitted small-amplitude quasi-periodic solutions by developing the Craig-Wayne method. Craig-Wayne-Bourgain's method succeeded in avoiding the multiplicity by using of the explicitly Hamiltonian structure of the systems. Moreover, the KAM approach has its own advantages. For example, the linear stability and zero Liapunov exponents are obtained and the existence results allow one to construct a local normal form in a neighborhood of the obtained solutions, which is useful for better understanding of the dynamics.

Recently, some results were also obtained in higher-dimensional systems by KAM method. In 2009, Eliasson and Kuksin [10] have proved that the following linear d-dimensional Schrödinger equation with an x-periodic and t-quasi-periodic potential was reduced to an autonomous equation for most values of the frequency vector:

$$\dot{u} = -\mathrm{i}\big(\Delta u - \varepsilon V(\varphi_0 + t\omega, x;\omega)u\big), \quad x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d.$$

In 2010, Eliasson and Kuksin [11] also considered the nonlinear d-dimensional Schrödinger equations. Their methods of dealing with the multiplicity of the eigenvalues of the linear operator are the key to our paper. In 2013, Procesi and Xu [12] proved the existence and stability of a quasi-periodic solution of the following nonlinear d-dimensional Schrödinger equations:

$$\mathrm{i}u_t - \Delta u + M_\xi u + f\left(|u|^2\right)u = 0, \quad t \in \mathbb{R}, x \in \mathbb{T}^d,$$

where $f(y)$ is a real-analytic function with $f(0) = 0$, M_ξ is a Fourier multiplier. In 2015, Procesi and Procesi [13] also proved, through a KAM algorithm, the existence of large families of stable and unstable quasi-periodic solutions for the NLS in any number of independent frequencies. Those results are important for KAM theories of space-multidimensional Hamiltonian PDEs.

Different from one-dimensional Hamiltonian systems, there exist two difficulties in higher-dimensional beam equations. One is the measure estimate of small divisor conditions. In fact, the first Melnikov conditions and the second Melnikov conditions are partially violated. To overcome this difficulty we have divided $\psi(\omega t, x)$ by its mean value. Another difficulty is the estimation on the symplectic transformation. We use the special assumption of the forced term $\psi(\omega t, x)$ to overcome this difficulty. We introduce the assumptions.

Throughout this paper, we assume that:

(A1) $\psi(\omega t, x)$ is a real-analytic quasi-periodic function. Moreover,

$$\psi(\theta, x) = \psi_0 + \sum_{k \in \mathbb{Z}^m \setminus \{0\}} \psi_k(x) e^{\langle k, \theta \rangle}, \quad \omega t = \theta, 0 \neq \psi_0 \in \mathbb{R}.$$

(A2) There exists a constant C such that

$$\left|\psi(\omega t, x)\right| \leq C, \qquad |\psi_0| \leq C, \qquad \left|\psi_k(x)\right| \leq C.$$

Remark 1.1 Assumption (A1) on the forcing $\psi(\omega t, x)$ is fundamental in order to deal with the small divisors problem. Indeed it implies in equation (12) that the coefficients $\zeta^{11}_{n_1 n_2} = 0$ for $k = 0$ and $n_1 \neq n_2$ for any $n_1, n_2 \in \mathbb{Z}^d$. An important consequence is that one can solve equations (27) even if the small divisor

$$\langle k, \omega \rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu} = 0, \quad k = 0, n_1 \neq n_2, |n_1| = |n_2|.$$

This is the key point to estimate the generator of the change of coordinates and to have at each step a diagonal normal form. Thanks to this fact at each step the homological equation is a scalar equation when one passes to the Fourier basis. Without such an assumption one could get only a block diagonal normal form with blocks whose dimension grows with $|n|$ and the homological equation would be more difficult. This assumption makes the measure estimate as easy as the one-dimensional case. In order to complete one KAM step, we need to prove that the perturbation always has the special form along the KAM iteration.

The paper is organized as follows. In Section 2, we introduce some notations, the expression of Hamiltonian and the main result. Section 3 is devoted to proving the reducibility. The proof of the measure estimate is given in Section 4.

2 The main result

In the following we introduce some notations. Let $l^{a,\rho}$ be the Banach spaces of complex valued sequences $z = (\dots, z_n, \dots)_{n\in\mathbb{Z}^d}$, and its complex conjugate $\bar{z} = (\dots, \bar{z}_n, \dots)_{n\in\mathbb{Z}^d}$ with finite weighted norm

$$\|z\|_{a,\rho} = \sum_{n\in\mathbb{Z}^d} |z_n||n|^a e^{|n|\rho},$$

where $a \geq 0$, $\rho > 0$, $n = (n_1, \dots, n_d)$, $|n| = \sqrt{n_1^2 + \cdots + n_d^2}$. Denote the average of f by

$$[f] = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(\theta)\,d\theta.$$

Let $A = -\Delta + M$ and $u_t = w$, then we can rewrite equation (3) as follows:

$$u_t = w, \qquad w_t = -A^2 u - \varepsilon\psi(\omega t, x)u.$$

Letting $w = A^{\frac{1}{2}}(\frac{\bar{v}-v}{\sqrt{2}\mathrm{i}})$ and $u = A^{-\frac{1}{2}}(\frac{\bar{v}+v}{\sqrt{2}})$, the equations become

$$v_t = \mathrm{i}\left(Av + \frac{\varepsilon}{\sqrt{2}}A^{-\frac{1}{2}}\left(\psi(\omega t, x)A^{-\frac{1}{2}}\left(\frac{v+\bar{v}}{\sqrt{2}}\right)\right)\right). \tag{5}$$

Equation (5) can be rewritten as the Hamiltonian equation

$$v_t = \mathrm{i}\frac{\partial H}{\partial \bar{v}} \tag{6}$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}\langle Av, \bar{v}\rangle + \frac{\varepsilon}{2}\int_{\mathbb{T}^d}\psi(\omega t, x)\left(A^{-\frac{1}{2}}\left(\frac{v+\bar{v}}{\sqrt{2}}\right)\right)^2 dx. \tag{7}$$

The operator A with the periodic boundary conditions has eigenvalues $\{\lambda_n\}$ and an exponential basis $\{\phi_n(x)\}$ satisfying, respectively,

$$\lambda_n = |n|^2 + M, \qquad \phi_n(x) = \frac{e^{\mathrm{i}\langle n,x\rangle}}{\sqrt{(2\pi)^d}}, \quad n \in \mathbb{Z}^d.$$

Let $v = \sum_{n\in\mathbb{Z}^d} z_n\phi_n(x)$. The system (6) is equivalent to the following equations:

$$\dot{z}_n = \mathrm{i}\left(\lambda_n z_n + \varepsilon\frac{\partial P}{\partial \bar{z}_n}\right), \tag{8}$$

with corresponding Hamiltonian function $H = \sum_{n\in\mathbb{Z}^d}\lambda_n z_n\bar{z}_n + \varepsilon P(t, z, \bar{z})$, where

$$\begin{aligned} P(t, z, \bar{z}) &\equiv \int_{\mathbb{T}^d}\frac{\psi(\omega t, x)}{4}\left(\sum_{n_1\in\mathbb{Z}^d}\frac{z_{n_1}\phi_{n_1}}{\sqrt{\lambda_{n_1}}} + \sum_{n_2\in\mathbb{Z}^d}\frac{\bar{z}_{n_2}\bar{\phi}_{n_2}}{\sqrt{\lambda_{n_2}}}\right)^2 dx \\ &:= \sum_{n_1,n_2\in\mathbb{Z}^d}\left(\zeta^{20}_{n_1n_2}(t)z_{n_1}z_{n_2} + \zeta^{11}_{n_1n_2}(t)z_{n_1}\bar{z}_{n_2} + \zeta^{02}_{n_1n_2}(t)\bar{z}_{n_1}\bar{z}_{n_2}\right) \end{aligned}$$

with

$$\begin{aligned}
\zeta^{20}_{n_1n_2}(t) &= \frac{1}{4\sqrt{\lambda_{n_1}\lambda_{n_2}}}\int_{\mathbb{T}^d}\psi(\omega t,x)\phi_{n_1}(x)\phi_{n_2}(x)\,dx, \\
\zeta^{11}_{n_1n_2}(t) &= \frac{1}{2\sqrt{\lambda_{n_1}\lambda_{n_2}}}\int_{\mathbb{T}^d}\psi(\omega t,x)\phi_{n_1}(x)\bar{\phi}_{n_2}(x)\,dx, \\
\zeta^{02}_{n_1n_2}(t) &= \frac{1}{4\sqrt{\lambda_{n_1}\lambda_{n_2}}}\int_{\mathbb{T}^d}\psi(\omega t,x)\bar{\phi}_{n_1}(x)\bar{\phi}_{n_2}(x)\,dx.
\end{aligned} \tag{9}$$

We introduce a pair of action-angle variables $(J,\theta)\in\mathbb{R}^m\times\mathbb{R}^m$, $\theta=\omega t$ with

$$\dot{\theta}=\omega,\qquad \dot{J}=-\frac{\partial H}{\partial\theta},\qquad \dot{z_n}=\mathrm{i}\frac{\partial H}{\partial\bar{z}_n},\qquad \dot{\bar{z}}_n=-\mathrm{i}\frac{\partial H}{\partial z_n},\quad n\in\mathbb{Z}^d.$$

Thus, from Assumption (A1) and (9), the Hamiltonian of (8) becomes

$$H=\langle\omega,I\rangle+\sum_{n\in\mathbb{Z}^d}\lambda_n z_n\bar{z}_n+\varepsilon P, \tag{10}$$

where

$$P=\sum_{n_1,n_2\in\mathbb{Z}^d}\left(\zeta^{20}_{n_1n_2}(\theta)z_{n_1}z_{n_2}+\zeta^{11}_{n_1n_2}(\theta)z_{n_1}\bar{z}_{n_2}+\zeta^{02}_{n_1n_2}(\theta)\bar{z}_{n_1}\bar{z}_{n_2}\right), \tag{11}$$

with

$$\begin{aligned}
\zeta^{20}_{n_1n_2}(\theta) &= \frac{\psi_0}{4\sqrt{\lambda_{n_1}\lambda_{n_2}}}\int_{\mathbb{T}^d}\phi_{n_1}(x)\phi_{n_2}(x)\,dx \\
&\quad+\frac{1}{4\sqrt{\lambda_{n_1}\lambda_{n_2}}}\sum_{k\in\mathbb{Z}^m\setminus\{0\}}\int_{\mathbb{T}^d}\psi_k(x)\phi_{n_1}(x)\phi_{n_2}(x)\,dx e^{\mathrm{i}\langle k,\theta\rangle} \\
&:= \sum_{k\in\mathbb{Z}^m}\zeta^{k20}_{n_1n_2}e^{\mathrm{i}\langle k,\theta\rangle}, \\
\zeta^{11}_{n_1n_2}(\theta) &= \frac{\psi_0}{2\sqrt{\lambda_{n_1}\lambda_{n_2}}}\int_{\mathbb{T}^d}\phi_{n_1}(x)\bar{\phi}_{n_2}(x)\,dx \\
&\quad+\frac{1}{2\sqrt{\lambda_{n_1}\lambda_{n_2}}}\sum_{k\in\mathbb{Z}^m\setminus\{0\}}\int_{\mathbb{T}^d}\psi_k(x)\phi_{n_1}(x)\bar{\phi}_{n_2}(x)\,dx e^{\mathrm{i}\langle k,\theta\rangle} \\
&:= \sum_{k\in\mathbb{Z}^m}\zeta^{k11}_{n_1n_2}e^{\mathrm{i}\langle k,\theta\rangle}, \\
\zeta^{02}_{n_1n_2}(t) &= \frac{\psi_0}{4\sqrt{\lambda_{n_1}\lambda_{n_2}}}\int_{\mathbb{T}^d}\bar{\phi}_{n_1}(x)\bar{\phi}_{n_2}(x)\,dx \\
&\quad+\frac{1}{4\sqrt{\lambda_{n_1}\lambda_{n_2}}}\sum_{k\in\mathbb{Z}^m\setminus\{0\}}\int_{\mathbb{T}^d}\psi_k(x)\bar{\phi}_{n_1}(x)\bar{\phi}_{n_2}(x)\,dx e^{\mathrm{i}\langle k,\theta\rangle} \\
&:= \sum_{k\in\mathbb{Z}^m}\zeta^{k02}_{n_1n_2}e^{\mathrm{i}\langle k,\theta\rangle}.
\end{aligned} \tag{12}$$

Moreover, we can get

$$\begin{aligned} &\zeta^{k20}_{n_1n_2} = 0, \qquad \zeta^{k02}_{n_1n_2} = 0, \quad \text{if } k = 0, n_1 + n_2 = 0; \\ &\zeta^{k11}_{n_1n_2} = 0, \quad \text{if } k = 0, n_1 - n_2 = 0. \end{aligned} \tag{13}$$

We introduce the following sets. For $0 < \varrho < 1$, let

$$J^0 := [\varrho, 2\varrho]^m.$$

For $0 \neq k \in \mathbb{Z}^m$, let

$$J^1_k = \left\{\omega \in J^0 : \left|\langle k, \omega\rangle\right| \le \frac{\varrho}{C_*|k|^{m+1}}, C_* \gg 1\right\}.$$

It is easy to see that

$$\operatorname{meas} J^1_k \le C|k|^{-1}\varrho^{m-1}\frac{\varrho}{C_*|k|^{m+1}} \le \frac{\varrho^m}{C_*|k|^{m+2}}.$$

Let $J^1 = \bigcup_{0\neq k\in\mathbb{Z}^m} J^1_k$ and $\hat{J} = J^0 \backslash J^1$, there exists a constant $0 < \gamma < 1$ such that

$$\operatorname{meas} \hat{J} = \varrho^m - \sum_{0\neq k\in\mathbb{Z}^m} \operatorname{meas} J^1_k \ge \varrho^m - \frac{\varrho^m}{C_*}\sum_{0\neq k\in\mathbb{Z}^b}\frac{1}{|k|^{m+2}} \ge \left(1 - \frac{\gamma}{3}\right)\varrho^m$$

by the convergence of $\sum_{0\neq k\in\mathbb{Z}^b}\frac{1}{|k|^{m+2}}$.

Theorem 2.1 *For the higher-dimensional beam equation* (3), *there exist a* $0 < \varepsilon^* \ll 1$ *and a set* $\bar{J} \subset \hat{J}$ *with* $\operatorname{meas}\bar{J} \ge \operatorname{meas}\hat{J}(1 - \mathcal{O}(\varrho^m))$, *such that, for any* $0 < \varepsilon < \varepsilon^*, \omega \in \bar{J}$, *equations* (3) *and* (4) *admit solutions of the form*

$$u(t,x) = \sum_{n\in\mathbb{Z}^d}\frac{C\cos(\mu_n t)}{\sqrt{2\lambda_n}}\phi_n(x) \tag{14}$$

where

$$\mu_n = \lambda_n + \varepsilon\big(c_n + \bar{c}_n(\varepsilon)\big), \tag{15}$$

with $c_n = \frac{\psi_0}{2\lambda_n}$ *and* $\bar{c}_n(\varepsilon) \to 0$, *as* $\varepsilon \to 0$.

Remark 2.2 Theorem 2.1 can be directly proved by Lemma 3.1. In fact, from the Hamiltonian (17) in Lemma 3.1, $v = \sum_{n\in\mathbb{Z}^d} z_n\phi_n(x)$ and $u = A^{-\frac{1}{2}}(\frac{\bar{v}+v}{\sqrt{2}})$, it is easy to obtain the solution of the beam equation (3). Thus, in this paper we focus on the proof of the Lemma 3.1.

3 Reducibility

3.1 Notations

We introduce the following notations and spaces.

For given $\sigma > 0$, $r > 0$, $\nu = 1, 2, \ldots$, we define sequences $\{\sigma_\nu\}$ and $\{r_\nu\}$:

(1) $\sigma_0 = \sigma$, $\sigma_\nu = \sigma_0(1-\tau_\nu)$ with $\tau_0 = 0$ and $\tau_\nu = \frac{\sum_{j=1}^{\nu} j^{-2}}{2\sum_{j=1}^{\infty} j^{-2}}$. It is easy to see $\sigma_0 > \sigma_\nu > \sigma_{\nu+1} > \sigma/2$.

(2) $r_0 = r$, $r_\nu = r_0(1-\tau_\nu)$, $\varepsilon_0 = \varepsilon$, $\varepsilon_\nu = \varepsilon^{(1+\delta)^\nu}$. It is easy to see $r_0 > r_\nu > r_{\nu+1} > r/2$.

Denote

$$\Theta(\sigma) = \left\{\theta = (\theta_1,\dots,\theta_m) \in \mathbb{C}^m/2\pi\mathbb{Z}^m : |\operatorname{Im}\theta| < \sigma\right\}$$

and

$$\begin{aligned} D^{a,\rho} &= D^{a,\rho}(\sigma, r) \\ &= \big\{(\theta, J, z, \bar z) \in \mathbb{C}^m/2\pi\mathbb{Z}^m \times \mathbb{C}^m \times l^{a,\rho} \times l^{a,\rho} : \\ &\quad |\operatorname{Im}\theta| < \sigma, |J| < r^2, \|z\|_{a,\rho} < r, \|\bar z\|_{a,\rho} < r\big\}. \end{aligned}$$

Thus, we get a family of domains:

$$\Theta(\sigma_0) \supset \Theta(\sigma_1) \supset \cdots \supset \Theta(\sigma_\nu) \supset \Theta(\sigma_{\nu+1}) \supset \cdots \supset \Theta\left(\frac{\sigma_0}{2}\right)$$

and

$$D^{a,\rho}(\sigma_0, r_0) \supset \cdots \supset D^{a,\rho}(\sigma_\nu, r_\nu) \supset D^{a,\rho}(\sigma_{\nu+1}, r_{\nu+1}) \supset \cdots \supset D^{a,\rho}\left(\frac{\sigma_0}{2}, \frac{r_0}{2}\right).$$

We rewrite $\Theta_l := \Theta(\sigma_l)$, $D_l^{a,\rho} = D^{a,\rho}(\sigma_l, r_l)$, $l = 0, 1, \dots$.

For a one order Whitney smooth function $F(\omega)$ on closed bounded set $\hat{J}$, we define

$$\|F\|_{\hat J}^* = \max\left\{\sup_{\omega\in\hat J}|F|, \sup_{\omega\in\hat J}|\partial_\omega F|\right\}.$$

If $F(\omega)$ is a vector function from $\hat J$ to $l^{a,\rho}$ (or $\mathbb{R}^{b_1\times b_2}$) which is one order Whitney smooth on $\hat J$, we define

$$\|F\|_{a,\rho,\hat J}^* = \left\|\left(\|F_i(\omega)\|_{\hat J}^*\right)_i\right\|_{a,\rho} \quad \left(\text{or } \|F\|_{J^*}^* = \max_{1\le i_1\le b_1}\sum_{1\le i_2\le b_2}\left(\|F_{i_1 i_2}(\omega)\|_{J^*}^*\right)\right).$$

Let $\tilde w = (\theta, J, z, \bar z) \in D^{a,\rho}$, we denote the weighted norm for $\tilde w$

$$|\tilde w|_{a,\rho} = |\theta| + \frac{1}{r^2}|J| + \frac{1}{r}\|z\|_{a,\rho} + \frac{1}{r}\|\bar z\|_{a,\rho}.$$

If $F(\tilde w;\omega)$ is a vector function from $D^{a,\rho}\times\hat J$ to $l^{a,\rho}$ which is one order Whitney smooth on ω, we define

$$\|F\|_{a,\rho,D^{a,\rho}\times\hat J}^* = \sup_{\tilde w\in D^{a,\rho}}\|F\|_{a,\rho,\hat J}^* \quad \text{and} \quad \|F\|_{D^{a,\rho}\times\hat J}^* = \sup_{\tilde w\in D^{a,\rho}}\|F\|_{\hat J}^*.$$

To the function $F(\theta, J, z, \bar z)$, associate a hamiltonian vector field defined as $X_F = \{F_J, -F_\theta, \mathrm{i}F_{\bar z}, -\mathrm{i}F_z\}$, we denote the weighted norm for X_F by letting

$$|X_F|_{a,\rho,D^{a,\rho}\times\hat J}^* = \|F_J\|_{D^{a,\rho}\times\hat J}^* + \frac{1}{r^2}\|F_\theta\|_{D^{a,\rho}\times\hat J}^* + \frac{1}{r}\left(\|F_{\bar z}\|_{a,\rho,D^{a,\rho}\times\hat J}^* + \|F_z\|_{a,\rho,D^{a,\rho}\times\hat J}^*\right). \tag{16}$$

Let $B(\tilde{w};\omega)$ be an operator from $D^{a,\rho}$ to $D^{a,\bar{\rho}}$ for $(\tilde{w};\omega) \in D^{a,\rho} \times \hat{J}$, we define the operator norm

$$\left|B(\tilde{w};\omega)\right|^{op}_{a,\bar{\rho},\rho,D^{a,\rho}\times\hat{J}} = \sup_{(\tilde{w};\omega)\in D^{a,\rho}\times\hat{J}} \sup_{\tilde{w}\neq 0} \frac{|B(\tilde{w};\omega)\tilde{w}|_{a,\bar{\rho}}}{|\tilde{w}|_{a,\rho}},$$

$$\left|B(\tilde{w};\omega)\right|^{*op}_{a,\bar{\rho},\rho,D^{a,\rho}\times\hat{J}} = \max\left\{|B|^{op}_{a,\bar{\rho},\rho,D^{a,\rho}\times\hat{J}}, |\partial_\omega B|^{op}_{a,\bar{\rho},\rho,D^{a,\rho}\times\hat{J}}\right\}.$$

3.2 Reducibility

Now, we are ready to introduce our reducibility and prove it via the KAM iteration.

Lemma 3.1 *For the Hamiltonian H in* (10), *there are a $0 < \varepsilon^* \ll 1$ and a set $\bar{J} \subset \hat{J}$ with $\operatorname{meas}\bar{J} \geq \operatorname{meas}\hat{J}(1 - \mathcal{O}(\varrho^m))$, such that, for any $0 < \varepsilon < \varepsilon^*, \omega \in \bar{J}$, there is a linear symplectic transformation*

$$\Sigma^\infty : D^{a,\rho}(\sigma/2, r/2) \times \bar{J} \to D^{a,\rho}(\sigma, r)$$

such that the following statements hold:

(i) *There are two absolute constants $C > 0$ and $0 < \delta < 1$ such that*

$$\left|\Sigma^\infty - id\right|^*_{a,\rho+1,D^{a,\rho}(\sigma/2,r/2)\times\bar{J}} \leq C\varepsilon^{\frac{\delta}{2}},$$

where id is the identity mapping.

(ii) *The transformation Σ^∞ changes Hamiltonian* (10) *into*

$$H^\infty := H \circ \Sigma^\infty = \langle\omega, J\rangle + \sum_{n\in\mathbb{Z}^d} \mu_n z_n \bar{z}_n, \tag{17}$$

where

$$\mu_n = \lambda_n + \sum_{s=0}^{\infty} \varepsilon_s \tilde{\lambda}_{n,s}, \quad \tilde{\lambda}_{n,0} = \frac{\psi_0}{2\lambda_n} \text{ and } |\tilde{\lambda}_{ns}| \leq C, s = 1,2,3,\ldots. \tag{18}$$

3.3 Proof the Lemma 3.1

Proof We first introduce the measure estimates to treat small divisors in reducing, which will be proved in Lemma 4.1. For $k \in \mathbb{Z}^m$, $n_1, n_2 \in \mathbb{Z}^d$, there exists a family of closed subsets $\bar{J}_l (l = 0, \ldots, \nu)$

$$\bar{J}_\nu \subset \cdots \subset \bar{J}_{l+1} \subset \bar{J}_l \subset \cdots \subset \bar{J}_0 \subset \hat{J} \subset J^0$$

such that, for $\omega \in \bar{J}_l$,

$$\left|\langle k,\omega\rangle \pm (\lambda_{n_1,\nu} \pm \lambda_{n_2,\nu})\right| \geq \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}} \tag{19}$$

and

$$\operatorname{meas}\bar{J}_l \geq \operatorname{meas}\hat{J}\left(1 - C\varrho^m \sum_{i=0}^{l} \frac{1}{1+i^2}\right), \tag{20}$$

where C is a constant depending on d. Moreover, let $\bar{J} = \bigcap_{l=0}^{\infty} \bar{J}_l$, then

$$\operatorname{meas} \bar{J} \geq \operatorname{meas} \hat{J}\bigl(1 - \mathcal{O}\bigl(\varrho^m\bigr)\bigr) \tag{21}$$

provided that ϱ is small enough.

Let $(m_1, m_2) = \{(2,0),(1,1),(0,2)\}$. We construct an iterative series $\{H_l\}$ of Hamiltonian functions of the form

$$H_l = \langle \omega, J\rangle + \sum_{n\in\mathbb{Z}^d} \lambda_{n,l} z_n \bar{z}_n + \varepsilon_l P_l(\theta, J, z, \bar{z}; \omega), \quad l = 0, 1, \ldots, \nu, \tag{$(E)_l$}$$

where

$$P_l(\theta, J, z, \bar{z}) = \sum_{k\in\mathbb{Z}^m, n_1, n_2\in\mathbb{Z}^d} \bigl(\zeta^{k20}_{n_1 n_2, l} z_{n_1} z_{n_2} + \zeta^{k11}_{n_1 n_2, l} z_{n_1} \bar{z}_{n_2} + \zeta^{k02}_{n_1 n_2, l} \bar{z}_{n_1} \bar{z}_{n_2}\bigr) e^{\mathrm{i}\langle k, \theta\rangle}$$

with

$$\zeta^{m_1 m_2}_{n_1 n_2, l} = \sum_{k\in\mathbb{Z}^m, n_1, n_2\in\mathbb{Z}^d} \zeta^{k m_1 m_2}_{n_1 n_2, l} e^{\mathrm{i}\langle k, \theta\rangle}$$

and

$$\begin{aligned} &\zeta^{k20}_{n_1 n_2, l} = 0, \qquad \zeta^{k02}_{n_1 n_2, l} = 0, \quad \text{if } k = 0, n_1 + n_2 = 0; \\ &\zeta^{k11}_{n_1 n_2, l} = 0, \quad \text{if } k = 0, n_1 - n_2 = 0. \end{aligned} \tag{$(3.0)_l$}$$

Furthermore, the functions $\zeta^{m_1 m_2}_{n_1 n_2, l}$ are analytic on the domain $\Theta_l \times \bar{J}_l$,

$$\zeta^{m_1 m_2}_{n_1 n_2, l} = \zeta^{m_1 m_2 *}_{n_1 n_2, l}, \qquad \bigl\| \zeta^{m_1 m_2 *}_{n_1, n_2, l} \bigr\|^*_{\Theta_l \times \bar{J}_l} \leq C, \quad n_1, n_2 \in \mathbb{Z}^d, l = 0, 1, \ldots, \nu, \tag{$(3.1)_l$}$$

and

$$\lambda_{n,0} = \lambda_n, \qquad \lambda_{n,l} = \lambda_n + \sum_{s=0}^{l-1} \varepsilon_s \tilde{\lambda}_{n,s}, \quad l = 1, 2, 3, \ldots, \nu, \tag{$(3.2)_l$}$$

with

$$\tilde{\lambda}_{n,s} = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \zeta^{1,1}_{nn,s}(\theta, \omega)\, d\theta, \quad s = 0, 1, 2, \ldots, l-1. \tag{22}$$

Clearly, we have $H_l|_{l=0} = H$. For $l = 0$, we have $P_0(\theta, J, z, \bar{z}) = P(\theta, J, z, \bar{z})$ defined in (11). From (12) and Assumption (A2), the functions $\zeta^{m_1 m_2}_{n_1 n_2, 0}(\theta; \omega)$ are analytic on the domain $\Theta_0 \times \bar{J}_0$ and satisfy $(3.1)_0$. Thus, we get

$$\tilde{\lambda}_{n,0} = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \zeta^{11}_{nn,0}(\theta; \omega)\, d\theta = \frac{\psi_0}{2\lambda_n}.$$

This implies that $(3.2)_0$ is satisfied.

We look for a change of variables S_ν defined in a domain $D^{a,\rho}_{\nu+1}$ by the time-one map $X^1_{\mathcal{F}_\nu}$ of the Hamiltonian vector field $X_{\mathcal{F}_\nu}$, such that the system $(E)_\nu$ is transformed into the form $(E)_{\nu+1}$ and satisfies $(3.1)_{\nu+1}$, $(3.2)_{\nu+1}$. In fact, the new Hamiltonian $H_{\nu+1}$ can be written as

$$\begin{aligned} H_{\nu+1} &:= H_\nu \circ X^1_{\mathcal{F}_\nu} \\ &= \langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d}\lambda_{n,\nu}z_n\bar{z}_n + \Big\{\langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d}\lambda_{n,\nu}z_n\bar{z}_n,\mathcal{F}_\nu\Big\} \\ &\quad + \varepsilon_\nu P_\nu(\theta,J,z,\bar{z};\omega)\big\{\varepsilon_\nu P_\nu(\theta,J,z,\bar{z};\omega),\mathcal{F}_\nu\big\} \\ &\quad + \int_0^1(1-t)\big\{\{H_\nu,\mathcal{F}_\nu\},\mathcal{F}_\nu\big\}\circ X^t_{\mathcal{F}_\nu}\,dt. \end{aligned} \tag{23}$$

Let $\mathcal{F}_\nu = \varepsilon_\nu F_\nu$, and $F^{m_1m_2}_{n_1n_2,\nu} = \sum_{k\in\mathbb{Z}^m}F^{km_1m_2}_{n_1n_2,\nu}e^{\mathrm{i}\langle k,\theta\rangle}$ with

$$\begin{aligned} &F^{k20}_{n_1n_2,\nu} = 0, \qquad F^{k02}_{n_1n_2,\nu} = 0, \quad \text{if } k=0, n_1+n_2=0; \\ &F^{k11}_{n_1n_2,\nu} = 0, \quad \text{if } k=0, n_1-n_2=0. \end{aligned}$$

We shall find a function F of the form

$$\begin{aligned} F_\nu(\theta,J,z,\bar{z};\omega) = &\sum_{\substack{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d,\\ |k|+||n_1|-|n_2||\neq 0}} F^{k11}_{n_1n_2,\nu}z_{n_1}\bar{z}_{n_2}e^{\mathrm{i}\langle k,\theta\rangle} \\ &+ \sum_{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d}\big(F^{k20}_{n_1n_2,\nu}z_{n_1}z_{n_2} + F^{k02}_{n_1n_2,\nu}\bar{z}_{n_1}\bar{z}_{n_2}\big)e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned} \tag{24}$$

satisfying the homological equation

$$\Big\{\langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d}\lambda_{n,\nu}z_n\bar{z}_n,F_\nu\Big\} + P_\nu(\theta,J,z,\bar{z};\omega) = \sum_{n\in\mathbb{Z}^d}\big[\zeta^{11}_{nn,\nu}\big]z_n\bar{z}_n. \tag{25}$$

It follows that

$$\begin{aligned} &\Big\{\langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d}\lambda_{n,\nu}z_n\bar{z}_n,F_\nu\Big\} \\ &\quad = \mathrm{i}\sum_{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d}\big(\langle k,\omega\rangle - \lambda_{n_1,\nu} - \lambda_{n_2,\nu}\big)F^{k20}_{n_1n_2,\nu}z_{n_1}z_{n_2}e^{\mathrm{i}\langle k,\theta\rangle} \\ &\qquad + \mathrm{i}\sum_{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d}\big(\langle k,\omega\rangle + \lambda_{n_1,\nu} + \lambda_{n_2,\nu}\big)F^{k02}_{n_1n_2,\nu}z_{n_1}z_{n_2}e^{\mathrm{i}\langle k,\theta\rangle} \\ &\qquad + \mathrm{i}\sum_{\substack{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d,\\ |k|+||n_1|-|n_2||\neq 0}}\big(\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu}\big)F^{k11}_{n_1n_2,\nu}z_{n_1}\bar{z}_{n_2}e^{\mathrm{i}\langle k,\theta\rangle}. \end{aligned} \tag{26}$$

By (25), it follows that, for $k \in \mathbb{Z}^m, n_1, n_2 \in \mathbb{Z}^d$, the $F^{km_1m_2}_{n_1n_2,\nu}$ are determined by the following linear algebraic system:

$$\begin{aligned} &\bigl(\langle k,\omega\rangle - \lambda_{n_1,\nu} - \lambda_{n_2,\nu}\bigr)F^{k20}_{n_1n_2,\nu} = \mathrm{i}\varsigma^{k20}_{n_1n_2,\nu}, \\ &\bigl(\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu}\bigr)F^{k11}_{n_1n_2,\nu} = \mathrm{i}\varsigma^{k11}_{n_1n_2,\nu}, \quad |k| + \bigl||n_1| - |n_2|\bigr| \neq 0, \\ &\bigl(\langle k,\omega\rangle + \lambda_{n_1,\nu} + \lambda_{n_2,\nu}\bigr)F^{k02}_{n_1n_2,\nu} = \mathrm{i}\varsigma^{k02}_{n_1n_2,\nu}. \end{aligned} \tag{27}$$

For $n_1, n_2 \in \mathbb{Z}^d$, we get

$$\begin{aligned} F^{11}_{n_1n_2,\nu} &= \sum_{\substack{k\in\mathbb{Z}^m,\\ |k|+||n_1|-|n_2||\neq 0}} \frac{\mathrm{i}\zeta^{k11}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle}, \\ F^{20}_{n_1n_2,\nu} &= \sum_{k\in\mathbb{Z}^m} \frac{\mathrm{i}\zeta^{k20}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} - \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle}, \\ F^{02}_{n_1n_2,\nu} &= \sum_{k\in\mathbb{Z}^m} \frac{\mathrm{i}\zeta^{k02}_{n_1n_2,\nu}}{\langle k,\omega\rangle + \lambda_{n_1,\nu} + \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle}. \end{aligned} \tag{28}$$

By Cauchy's estimate and $(3.1)_\nu$, we get

$$\bigl|\zeta^{km_1m_2}_{n_1n_2,\nu}\bigr| \le \bigl\|\zeta^{m_1m_2}_{n_1n_2,\nu}\bigr\|^*_{\Theta_\nu\times\bar{J}_\nu} e^{-|k|\sigma_\nu} \tag{29}$$

and

$$\bigl|\partial_\omega\zeta^{km_1m_2}_{n_1n_2,\nu}\bigr| \le \bigl\|\zeta^{m_1m_2}_{n_1n_2,\nu}\bigr\|^*_{\Theta_\nu\times\bar{J}_\nu} e^{-|k|\sigma_\nu}. \tag{30}$$

Note that (19) and (28), for $n_1, n_2 \in \mathbb{Z}^d$, we get

$$\sup_{(\theta;\omega)\in\Theta_{\nu+1}\times\bar{J}_\nu} \bigl|F^{11}_{n_1n_2,\nu}\bigr| \le C\bigl\|\zeta^{11}_{n_1n_2,\nu}\bigr\|^*_{\Theta_\nu\times\bar{J}_\nu}\bigl(1+\nu^2\bigr)\Biggl(\sum_{\substack{|k|+||n_1|-|n_2||\neq 0,\\ k\in\mathbb{Z}^m}} \bigl(|k|+1\bigr)^{m+3} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}\Biggr),$$

and for $(m_1, m_2) = \{(2,0),(0,2)\}$,

$$\sup_{(\theta;\omega)\in\Theta_{\nu+1}\times\bar{J}_\nu} \bigl|F^{m_1m_2}_{n_1n_2,\nu}\bigr| \le C\bigl\|\zeta^{m_1m_2}_{n_1n_2,\nu}\bigr\|^*_{\Theta_\nu\times\bar{J}_\nu}\bigl(1+\nu^2\bigr)\sum_{k\in\mathbb{Z}^m}\bigl(|k|+1\bigr)^{m+3} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}.$$

Furthermore, using Lemma 3.3 in [14], for $(\theta;\omega) \in \Theta_{\nu+1}\times\bar{J}_\nu$, we get

$$\bigl|F^{m_1m_2}_{n_1n_2,\nu}\bigr| \le C\bigl\|\zeta^{m_1m_2}_{n_1n_2,\nu}\bigr\|^*_{\Theta_\nu\times\bar{J}_\nu}(\nu+1)^{6m+24}, \quad (m_1,m_2) = \bigl\{(2,0),(1,1),(0,2)\bigr\}. \tag{31}$$

From $(3.2)_l$, we have

$$\bigl|\partial_\omega(\lambda_{n_1,\nu}\pm\lambda_{n_2,\nu})\bigr| \le C\varepsilon, \qquad |\partial_\omega\lambda_{n_1,\nu}| \le C\varepsilon.$$

Thus, in view of (28)-(31), and using Lemma 3.3 in [14], we have, for $(\theta;\omega) \in \Theta_{\nu+1} \times \bar{J}_\nu$, $n_1, n_2 \in \mathbb{Z}^d$,

$$\begin{aligned}
\left|\partial_\omega F^{11}_{n_1n_2,\nu}\right| &\le \sum_{\substack{k\in\mathbb{Z}^m,\\ |k|+||n_1|-|n_2||\neq 0}} \left|\frac{\partial_\omega \zeta^{k11}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu}}\right| \left|e^{\mathrm{i}\langle k,\theta\rangle}\right| \\
&\quad + \sum_{\substack{k\in\mathbb{Z}^m,\\ |k|+||n_1|-|n_2||\neq 0}} \left|\frac{\zeta^{k11}_{n_1n_2,\nu}\partial_\omega(\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu})}{(\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu})^2}\right| \left|e^{\mathrm{i}\langle k,\theta\rangle}\right| \\
&\le C\left\|\zeta^{11}_{n_1n_2,\nu}\right\|^*_{\Theta_\nu\times\bar{J}_\nu} \left(1+l^2\right)^2 \sum_{k\in\mathbb{Z}^m} 2\left(|k| + C\varepsilon\right)\left(|k|+1\right)^{2m+6} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})} \\
&\le C\left\|\zeta^{11}_{n_1n_2,\nu}\right\|^*_{\Theta_\nu\times\bar{J}_\nu} (\nu+1)^{6m+24},
\end{aligned} \tag{32}$$

and for $(m_1, m_2) = \{(2,0),(0,2)\}$,

$$\begin{aligned}
\left|\partial_\omega F^{m_1m_2}_{n_1n_2,\nu}\right| &\le \sum_{k\in\mathbb{Z}^m} \left|\frac{\partial_\omega \zeta^{km_1m_2}_{n_1n_2,\nu}}{\langle k,\omega\rangle \pm (\lambda_{n_1,\nu} + \lambda_{n_2,\nu})}\right| \left|e^{\mathrm{i}\langle k,\theta\rangle}\right| \\
&\quad + \sum_{k\in\mathbb{Z}^m} \left|\frac{\zeta^{km_1m_2}_{n_1n_2,\nu}\partial_\omega(\langle k,\omega\rangle \pm (\lambda_{n_1,\nu} + \lambda_{n_2,\nu}))}{(\langle k,\omega\rangle \pm (\lambda_{n_1,\nu} + \lambda_{n_2,\nu}))^2}\right| \left|e^{\mathrm{i}\langle k,\theta\rangle}\right| \\
&\le C\left\|\zeta^{m_1m_2}_{n_1n_2,\nu}\right\|^*_{\Theta_\nu\times\bar{J}_\nu} \left(1+l^2\right)^2 \sum_{k\in\mathbb{Z}^m} 2\left(|k| + C\varepsilon\right)\left(|k|+1\right)^{2m+6} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})} \\
&\le C\left\|\zeta^{m_1m_2}_{n_1n_2,\nu}\right\|^*_{\Theta_\nu\times\bar{J}_\nu} (\nu+1)^{6m+24}.
\end{aligned} \tag{33}$$

It follows immediately that, for $(m_1, m_2) = \{(2,0),(1,1),(0,2)\}$,

$$\left|\partial_\omega F^{m_1m_2}_{n_1n_2,\nu}\right| \le C\left\|\zeta^{m_1m_2}_{n_1n_2,\nu}\right\|^*_{\Theta_\nu\times\bar{J}_\nu} (\nu+1)^{6m+24}. \tag{34}$$

From (31), (33), (32) and (34), we have, for $(m_1, m_2) = \{(2,0),(1,1),(0,2)\}$,

$$\left\|F^{m_1m_2}_{n_1n_2,\nu}\right\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\left\|\zeta^{m_1m_2}_{n_1n_2,\nu}\right\|^*_{\Theta_\nu\times\bar{J}_\nu} (\nu+1)^{6m+24}. \tag{35}$$

In view of (28), we have

$$\begin{aligned}
\partial_\theta F^{11}_{n_1n_2,\nu} &= \sum_{|k|+||n_1|-|n_2||\neq 0} \frac{-\zeta^{k11}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle} \cdot k, \\
\partial_\theta F^{20}_{n_1n_2,\nu} &= \sum_{k\in\mathbb{Z}^m} \frac{-\zeta^{k20}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} - \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle} \cdot k, \\
\partial_\theta F^{02}_{n_1n_2,\nu} &= \sum_{k\in\mathbb{Z}^m} \frac{-\zeta^{k02}_{n_1n_2,\nu}}{\langle k,\omega\rangle + \lambda_{n_1,\nu} + \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle} \cdot k, \\
\partial_{\theta\theta} F^{11}_{n_1n_2,\nu} &= \sum_{|k|+||n_1|-|n_2||\neq 0} \frac{-\zeta^{k11}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} + \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle} \cdot \mathrm{i}kk^T, \\
\partial_{\theta\theta} F^{20}_{n_1n_2,\nu} &= \sum_{k\in\mathbb{Z}^m} \frac{-\zeta^{k20}_{n_1n_2,\nu}}{\langle k,\omega\rangle - \lambda_{n_1,\nu} - \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle} \cdot \mathrm{i}kk^T,
\end{aligned}$$

$$\partial_{\theta\theta} F^{02}_{n_1 n_2,\nu} = \sum_{k\in\mathbb{Z}^m} \frac{-\zeta^{k02}_{n_1 n_2,\nu}}{\langle k,\omega\rangle + \lambda_{n_1,\nu} + \lambda_{n_2,\nu}} e^{\mathrm{i}\langle k,\theta\rangle} \cdot \mathrm{i}kk^T,$$

where k is a m column vector and kk^T is a $m\times m$ matrix. Similar to the above discussion, we get the following estimates:

$$\begin{aligned} &\big\|\partial_\theta F^{m_1 m_2}_{n_1 n_2,\nu}\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\big\|\zeta^{m_1 m_2}_{n_1 n_2,\nu}\big\|^*_{\Theta_\nu\times\bar{J}_\nu}(\nu+1)^{6m+24}, \\ &\big\|\partial_{\theta\theta} F^{m_1 m_2}_{n_1 n_2,\nu}\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\big\|\zeta^{m_1 m_2}_{n_1 n_2,\nu}\big\|^*_{\Theta_\nu\times\bar{J}_\nu}(\nu+1)^{6m+24}. \end{aligned} \tag{36}$$

We will give estimates of the flow $X^t_{\mathcal{F}_\nu}$.

For $\nu = 0,1,\ldots$, there exists a constant $0<\delta<1$ such that

$$\big|\varepsilon_\nu^{1-\delta}(\nu+1)^{6m+24}\big| \le C, \tag{37}$$

as $\varepsilon<1$, where C is an absolute constant independent on ν,ε. From (35), for $(\theta,\omega)\in\Theta_{\nu+1}\times\bar{J}_\nu$, we obtain

$$\big\|\varepsilon_\nu F_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\varepsilon_\nu^\delta\big\|P_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_\nu\times\bar{J}_\nu}. \tag{38}$$

By (24), (36) and (37) we obtain

$$\big\|\varepsilon_\nu\partial_\theta F_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\varepsilon_\nu^\delta\big\|P_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_\nu\times\bar{J}_\nu} \tag{39}$$

and

$$\big\|\varepsilon_\nu\partial_{\theta\theta} F_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\varepsilon_\nu^\delta\big\|P_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_\nu\times\bar{J}_\nu}. \tag{40}$$

It follows from (24) and (38) that

$$\begin{aligned} &\big\|(\mathcal{F}_\nu)_{z_{n_1}}\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \\ &\quad= \varepsilon_\nu\Bigg\|\sum_{\substack{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d,\\ |k|+||n_1|-|n_2||\neq 0}} F^{k11}_{n_1 n_2,\nu}\bar{z}_{n_2}e^{\mathrm{i}\langle k,\theta\rangle} + \varepsilon_\nu\sum_{k\in\mathbb{Z}^m,n_1,n_2\in\mathbb{Z}^d}\big(F^{k20}_{n_1 n_2,\nu} + F^{k20}_{n_2 n_1,\nu}\big)z_{n_2}e^{\mathrm{i}\langle k,\theta\rangle}\Bigg\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \\ &\quad\le \big\|\varepsilon_\nu F_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \\ &\quad\le C\varepsilon_\nu^\delta\big\|P_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_\nu\times\bar{J}_\nu} \end{aligned} \tag{41}$$

and similarly

$$\big\|(\mathcal{F}_\nu)_{\bar{z}_{n_1}}\big\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\varepsilon_\nu^\delta\big\|P_\nu(\theta,J,z,\bar{z};\omega)\big\|^*_{\Theta_\nu\times\bar{J}_\nu}. \tag{42}$$

Therefore, by using of (39)-(42), we obtain

$$\begin{aligned} |X_{\mathcal{F}_\nu}|^*_{a,\rho+1,D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}} &= \big\|(\mathcal{F}_\nu)_J\big\|^*_{D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}} + \frac{1}{r^2_{\nu+1}}\big\|(\mathcal{F}_\nu)_\theta\big\|^*_{D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}} \\ &\quad + \frac{1}{r_{\nu+1}}\big(\|F_{\bar{z}}\|^*_{a,\rho,D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}} + \|F_z\|^*_{a,\rho,D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}}\big) \end{aligned}$$

$$\begin{aligned} &\le C\varepsilon_\nu^\delta \big\| X_{P_\nu}(\theta, J, z, \bar{z}; \omega)\big\|^*_{\Theta_\nu \times \bar{J}_\nu} \\ &\le C\varepsilon_\nu^{\frac{\delta}{2}}, \end{aligned} \tag{43}$$

by $\varepsilon_\nu^{\frac{\delta}{2}} \|X_{R_\nu}(\theta, J, z, \bar{z}; \omega)\|^*_{\Theta_\nu \times \bar{J}_\nu} \le C$, as $\varepsilon < 1$, where C is an absolute constant independent on ν, ε.

To get the estimates for $X^t_{\mathcal{F}_\nu}$, we consider the integral equation

$$X^t_{\mathcal{F}_\nu} = id + \int_0^t X_{\mathcal{F}_\nu} \circ X^s_{\mathcal{F}_\nu}\, ds, \quad 0 \le t \le 1.$$

Hence, we obtain from (43)

$$\big|X^1_{\mathcal{F}_\nu} - id\big|^*_{a,\rho+1,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \le |X_{\mathcal{F}_\nu}|^*_{a,\rho+1,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \le C\varepsilon_\nu^{\frac{\delta}{2}}. \tag{44}$$

Let

$$\begin{aligned} &\big|D^s F\big|^{op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \\ &\quad = \max\left\{ \left| \frac{\partial^{|j|+|i|+|\alpha|+|\beta|}}{\partial J^j \theta^i \partial z_n^\alpha \partial \bar{z}_n^\beta} F \right|^{op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}}, |j| + |i| + |\alpha| + |\beta| = s \ge 2 \right\}. \end{aligned}$$

Notice that F is a polynomial of degree 2 in $z, \bar{z}$. By (16), (43) and the Cauchy inequality, it follows that, for any $s \ge 2$,

$$\big|D^s F\big|^{op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \le C\varepsilon_\nu^{\frac{\delta}{2}}. \tag{45}$$

From $\phi^t_F = id + \int_0^t X_F \circ \phi^s_F\, ds$, we have $\phi^t_F : D^{a,\rho}_{\nu+1} \to D^{a,\rho}_\nu$, $-1 \le t \le 1$, which follows directly from (45). Since

$$D\phi^t_F = \mathrm{Id} + \int_0^t (DX_F) D\phi^s_F\, ds = \mathrm{Id} + \int_0^t \mathcal{J}\big(D^2 F\big) D\phi^s_F\, ds,$$

where $\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, it follows that

$$\big|D\phi^t_F - \mathrm{Id}\big|^{op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \le 2\big|D^2 F\big|^{op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \le C\varepsilon_\nu^{\frac{\delta}{2}}. \tag{46}$$

Similarly,

$$\big|DX^1_{\mathcal{F}_\nu} - \mathrm{Id}\big|^{*op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1} \times \bar{J}_{\nu+1}} \le C\varepsilon_\nu^{\frac{\delta}{2}}. \tag{47}$$

We now estimate the smaller term $P_{\nu+1}$ and we will finish one cycle of the iteration. Let

$$\tilde{\lambda}_{n,\nu} = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \zeta^{11}_{nn,\nu}(\theta; \omega)\, d\theta$$

and

$$\lambda_{n,\nu+1} = \lambda_{n,\nu} + \varepsilon_\nu \tilde{\lambda}_{n,\nu},$$

then it is easy to see that $\lambda_{n,\nu+1}$ satisfies the conditions $(3.2)_{\nu+1}$. Moreover, from (23) and (25), we know that

$$H_{\nu+1} = \langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d} \lambda_{n,\nu+1} z_n \bar{z}_n + \varepsilon_{\nu+1} P_{\nu+1}(\theta,J,z,\bar{z};\omega),$$

where

$$\begin{aligned}&\varepsilon_{\nu+1} P_{\nu+1}(\theta,J,z,\bar{z};\omega)\\ &\quad= \bigl\{\varepsilon_\nu P_\nu(\theta,J,z,\bar{z};\omega),\mathcal{F}_\nu\bigr\} + \int_0^1 (1-t)\bigl\{\{H_\nu,\mathcal{F}_\nu\},\mathcal{F}_\nu\bigr\}\circ X^t_{\mathcal{F}_\nu}\,dt.\end{aligned} \tag{48}$$

By a direct calculation we get

$$\varepsilon_{\nu+1} P_{\nu+1}(\theta,J,z,\bar{z};\omega) = \varepsilon_\nu^2 \sum_{n_1,n_2\in\mathbb{Z}^d} \sum_{m_1,m_2} \tilde{\zeta}^{m_1 m_2}_{n_1 n_2,\nu+1}(\theta;\omega) z_{n_1}^{m_1} \bar{z}_{n_2}^{m_2},$$

where $\tilde{\zeta}^{m_1 m_2}_{n_1 n_2,\nu+1}(\theta;\omega)$'s are a linear combination of the product of $F^{m_1 m_2}_{n_1 n_2,\nu}(\theta;\omega)$ and $\zeta^{\tilde{n}_1\tilde{n}_2}_{n_1 n_2,\nu}(\theta;\omega)'s$, with (m_1,m_2) or $(\tilde{n}_1,\tilde{n}_2) = \{(2,0),(1,1),(0,2)\}$. Thus, by using of $(3.1)_\nu$, (31) and (35),

$$\bigl\|\tilde{\zeta}^{m_1 m_2}_{n_1 n_2,\nu+1}\bigr\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C\bigl\|\zeta^{m_1 m_2}_{n_1 n_2,\nu}\bigr\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}(\nu+1)^{6m+24} \tag{49}$$

is true. In view of $\varepsilon_\nu^{2-(1-\delta)} = \varepsilon_{\nu+1}$, and $C\varepsilon_\nu^{1-\delta}\|\zeta^{m_1 m_2}_{n_1 n_2,\nu+1}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}(\nu+1)^{6m+24} \le 1$, as $\varepsilon < 1$, we can suppose that

$$\zeta^{m_1 m_2}_{n_1 n_2,\nu+1} := \varepsilon_\nu^{1-\delta}\tilde{\zeta}^{m_1 m_2}_{n_1 n_2,\nu+1}. \tag{50}$$

It follows from (49) that

$$\bigl\|\zeta^{m_1 m_2}_{n_1 n_2,\nu+1}\bigr\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu} \le C.$$

This implies $(E)_{\nu+1}$ as defined in $D^{a,\rho}_{\nu+1}$ and the $\zeta^{m_1 m_2}_{n_1 n_2,\nu+1}$ satisfy $(3.1)_{\nu+1}$.

The perturbation P_l satisfying $(3.0)_l$ is used to guarantee that the normal form at each KAM step has the same form as in the first step. In order to complete one KAM step, we need to prove that the new perturbation $P_{\nu+1}$ still has the special form $(3.0)_{\nu+1}$.

From (25), we get

$$\Bigl\{\langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d}\lambda_{n,\nu} z_n\bar{z}_n, F_\nu\Bigr\} = -P_\nu(\theta,J,z,\bar{z};\omega) + \sum_{n\in\mathbb{Z}^d}\bigl[\zeta^{11}_{nn,\nu}\bigr] z_n\bar{z}_n.$$

It is easy to see that $\{\langle\omega,J\rangle + \sum_{n\in\mathbb{Z}^d}\lambda_{n,\nu} z_n\bar{z}_n, F_\nu\}$ satisfies $(3.0)_{\nu+1}$. Thus, from (48), we only to consider $\{P_\nu,F_\nu\}$ satisfies $(3.0)_{\nu+1}$. Let $B_\nu = \{P_\nu,F_\nu\}$. Taking $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \{(e_{n_1}+e_{n_2},0),(e_{n_1},e_{n_2}),(0,e_{n_1}+e_{n_2})\}$, we can assume that

$$P_\nu = \sum_{k,\alpha_1,\beta_1} \zeta_{k\alpha_1\beta_1,\nu} e^{\mathrm{i}\langle k,\theta\rangle} z^{\alpha_1}\bar{z}^{\beta_1}, \qquad F_\nu = \sum_{k,\alpha_2,\beta_2} F_{k\alpha_2\beta_2,\nu} e^{\mathrm{i}\langle k,\theta\rangle} z^{\alpha_2}\bar{z}^{\beta_2}, \tag{51}$$

with

$$\zeta_{k\alpha_1\beta_1,\nu}=0,\quad \text{if } k=0, \sum_{n\in\mathbb{Z}^d}(\alpha_{1n}-\beta_{1n})n=0;$$
$$F_{k\alpha_1\beta_1,\nu}=0,\quad \text{if } k=0, \sum_{n\in\mathbb{Z}^d}(\alpha_{1n}-\beta_{1n})n=0.$$

Since

$$\begin{aligned}
\{P_\nu,F_\nu\} &= \mathrm{i}\sum_m\sum_{a_1}\zeta_{k\alpha_1\beta_1,\nu}F_{k\alpha_1\beta_1,\nu}e^{2\mathrm{i}\langle k,\theta\rangle}z^{\alpha_1-e_m}\bar{z}^{\beta_1}z^{\alpha_2}\bar{z}^{\beta_2-e_m}\\
&\quad -\mathrm{i}\sum_m\sum_{a_2}\zeta_{k\alpha_1\beta_1,\nu}F_{k\alpha_1\beta_1,\nu}e^{2\mathrm{i}\langle k,\theta\rangle}z^{\alpha_1}\bar{z}^{\beta_1-e_m}z^{\alpha_2-e_m}\bar{z}^{\beta_2}\\
&\quad +\mathrm{i}\sum_m\sum_{k\neq 0}\zeta_{k\alpha_1\beta_1,\nu}F_{k\alpha_1\beta_1,\nu}e^{2\mathrm{i}\langle k,\theta\rangle}z^{\alpha_1-e_m}\bar{z}^{\beta_1}z^{\alpha_2}\bar{z}^{\beta_2-e_m}\\
&\quad -\mathrm{i}\sum_m\sum_{k\neq 0}\zeta_{k\alpha_1\beta_1,\nu}F_{k\alpha_1\beta_1,\nu}e^{2\mathrm{i}\langle k,\theta\rangle}z^{\alpha_1}\bar{z}^{\beta_1-e_m}z^{\alpha_2-e_m}\bar{z}^{\beta_2}\\
&= \mathrm{i}\sum_m\sum_{a_3}B_{k(\alpha_1+\alpha_2-e_m)(\beta_1+\beta_2-e_m),\nu}e^{2\mathrm{i}\langle k,\theta\rangle}z^{\alpha_1+\alpha_2-e_m}\bar{z}^{\beta_1+\beta_2-e_m}\\
&\quad +\mathrm{i}\sum_m\sum_{k\neq 0}B_{k(\alpha_1+\alpha_2-e_m)(\beta_1+\beta_2-e_m),\nu}e^{2\mathrm{i}\langle k,\theta\rangle}z^{\alpha_1+\alpha_2-e_m}\bar{z}^{\beta_1+\beta_2-e_m},
\end{aligned}$$

where a_1 denotes

$$k=0,\quad (\alpha_{1m}-1-\beta_{1m})m+\sum_{n\in\mathbb{Z}^d\setminus\{m\}}(\alpha_{1n}-\beta_{1n})n=-m,$$
$$\bigl(\alpha_{2m}-(\beta_{2m}-1)\bigr)m+\sum_{n\in\mathbb{Z}^d\setminus\{m\}}(\alpha_{2n}-\beta_{2n})n=m,$$

a_2 denotes

$$k=0,\quad \bigl(\alpha_{1m}-(\beta_{1m}-1)\bigr)m+\sum_{n\in\mathbb{Z}^d\setminus\{m\}}(\alpha_{1n}-\beta_{1n})n=m,$$
$$(\alpha_{2m}-1-\beta_{2m})m+\sum_{n\in\mathbb{Z}^d\setminus\{m\}}(\alpha_{2n}-\beta_{2n})n=-m,$$

a_3 denotes

$$k=0,\quad \bigl((\alpha_{1m}+\alpha_{2m}-1)-(\beta_{1m}+\beta_{2m}-1)\bigr)m+\sum_{n\in\mathbb{Z}^d\setminus\{m\}}(\alpha_{1n}+\alpha_{2n}-\beta_{1n}-\beta_{2n})n=0.$$

Thus, one finds that $\{P_\nu,F_\nu\}$ satisfies $(3.0)_{\nu+1}$. Moreover, $P_{\nu+1}$ satisfies $(3.0)_{\nu+1}$.

Finally, we consider the convergence of transformations Σ^N.

In view of (43) and (47), by letting

$$S_\nu=X^1_{\mathcal{F}_\nu}:D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}\longmapsto D^{a,\rho}_\nu\times\bar{J}_\nu \tag{52}$$

we have

$$|S_\nu - id|^*_{a,\rho+1,D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}} \le C\varepsilon_\nu^{\frac{\delta}{2}}, \qquad |DS_\nu - \mathrm{Id}|^{*op}_{a,\rho+1,\rho,D^{a,\rho}_{\nu+1}\times\bar{J}_{\nu+1}} \le C\varepsilon_\nu^{\frac{\delta}{2}}. \tag{53}$$

Now we are ready to prove the limiting transformation $S_0 \circ S_1 \circ \cdots$ convergent to a linear symplectic transformation Σ^∞, which integrates equation (10). For any $\omega \in \bar{J}, N \ge 1$, we denote by Σ^N the map

$$\Sigma^N(\cdot;\omega) = S_0(\cdot;\omega) \circ \cdots \circ S_{N-1}(\cdot;\omega) : D^{a,\rho}_N \longmapsto D^{a,\rho}(\sigma, r)$$

as usual, Σ^0 is the identity mapping. From the second inequality of (53), we have

$$\left|D\Sigma^N\right|^{*op}_{a,\rho+1,\rho,D^{a,\rho}_N\times\bar{J}} \le \prod_{\mu=0}^{N-1} |DS_\mu|^{*op}_{a,\rho+1,\rho,D^{a,\rho}_{\mu+1}\times\bar{J}} \le \prod_{\mu\ge 0}\left(1 + C\varepsilon_N^{\frac{\delta}{2}}\right) \le 2$$

provided that ε is small enough. Thus, by using the first inequality of (53), we have

$$\begin{aligned}\left|\Sigma^{N+1} - \Sigma^N\right|^*_{a,\rho+1,D^{a,\rho}_{N+1}\times\bar{J}} &\le \left|D\Sigma^N\right|^{*op}_{a,\rho+1,\rho,D^{a,\rho}_N\times\bar{J}} \cdot |S_N - id|^*_{a,\rho+1,D^{a,\rho}_{N+1}\times\bar{J}} \\ &\le C\varepsilon_N^{\frac{\delta}{2}}.\end{aligned}$$

So the sequence $\{\Sigma^N\}$ converges uniformly in $D^{a,s}_N$ to an analytic map

$$\Sigma^\infty : D^{a,\rho}(\sigma/2, r/2) \longmapsto D^{a,\rho}(\sigma, r).$$

We remark that the Hamiltonian (10) satisfies $(E)_\nu$, $(3.1)_\nu$ and $(3.2)_\nu$ with $\nu = 0$, the above iterative procedure can run repeatedly. So

$$\mu_n = \lambda_n + \frac{\varepsilon\psi_0}{2\lambda_n} + \sum_{k=1}^{\infty} \varepsilon_k \tilde{\lambda}_{n,k},$$

where $|\tilde{\lambda}_{n,k}| \le C, k = 1, 2, 3, \ldots$. So (i) and (ii) are obtained. This completes the proof. □

4 The small divisors lemma

Now we prove the following the small divisors lemma which has been applied in proving the above reducibility theorem.

Lemma 4.1 *For $k \in \mathbb{Z}^m$, $n_1, n_2 \in \mathbb{Z}^d$, there exists a family of closed subsets $\bar{J}_l$ ($l = 0, \ldots, \nu$)*

$$\bar{J}_\nu \subset \cdots \subset \bar{J}_{l+1} \subset \bar{J}_l \subset \cdots \subset \bar{J}_0 \subset \hat{J} \subset J^0$$

such that, for $\omega \in \bar{J}_l$,

$$\left|\langle k, \omega\rangle \pm (\lambda_{n_1,\nu} \pm \lambda_{n_2,\nu})\right| \ge \frac{\varrho \,\mathrm{meas}\, \hat{J}}{(1 + l^2)(|k| + 1)^{m+3}} \tag{54}$$

and

$$\operatorname{meas}\bar{J}_l \geq \operatorname{meas}\hat{J}\left(1 - C\varrho^m \sum_{i=0}^{l} \frac{1}{1+i^2}\right), \tag{55}$$

where C is a constant depending on d. Moreover, let $\bar{J} = \bigcap_{l=0}^{\infty} \bar{J}_l$, then

$$\operatorname{meas}\bar{J} \geq \operatorname{meas}\hat{J}\big(1 - \mathcal{O}\big(\varrho^m\big)\big) \tag{56}$$

provided that ϱ is small enough.

Proof First of all, from $(3.2)_l$, we have

$$\pm\lambda_{n_1,l} \pm \lambda_{n_2,l} = \begin{cases} \pm(|n_1|^2 + M) \pm (|n_2|^2 + M) + \mathcal{O}(\varepsilon), & l = 1, 2, \ldots, \nu, \\ \pm(|n_1|^2 + M) \pm (|n_2|^2 + M), & l = 0, \end{cases} \tag{57}$$

and

$$\langle k, \omega \rangle = k_1\omega_1 + \cdots + k_m\omega_m. \tag{58}$$

By (22), it follows that

$$|\partial_\omega \tilde{\lambda}_{n_1,l}| \leq C\varepsilon\lambda_{n_1}^{-1}.$$

Case 1. $k = 0$. From (28), we need to estimate $\langle k, \omega \rangle \pm (\lambda_{n_1,l} + \lambda_{n_2,l})$.

$$\begin{aligned} \big|\langle k, \omega \rangle \pm (\lambda_{n_1,l} + \lambda_{n_2,l})\big| &= |\lambda_{n_1,l} + \lambda_{n_2,l}| \\ &= \big|\big(|n_1|^2 + M\big) + \big(|n_2|^2 + M\big) + \mathcal{O}(\varepsilon)\big| \\ &\geq C|\sqrt{M}| \geq \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}} \end{aligned}$$

holds provided that ε and ϱ are small enough.

Case 2. $k \neq 0$.

Case 2.1. We consider the following set:

$$\bar{J}_{kn_1n_2}^{l-} := \left\{\omega \in \hat{J} : \big|\langle k, \omega \rangle - \lambda_{n_1,l} + \lambda_{n_2,l}\big| < \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}}\right\}.$$

We suppose that $|n_2|^2 - |n_1|^2 = a \geq 0$, then

$$|\lambda_{n_2} - \lambda_{n_1} - a| \leq \mathcal{O}\big(|n_1|^{-\bar{\delta}}\big).$$

Let $f_{kn_1n_2}^{l-} := \langle k, \omega \rangle - \lambda_{n_1,l} + \lambda_{n_2,l}, f_{kan_1}^{l-} := \langle k, \omega \rangle + a$ and

$$\bar{J}_{kan_1}^{l-} := \left\{\omega \in \hat{J} : \big|\langle k, \omega \rangle + a\big| < \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}} + \mathcal{O}\big(|n_1|^{-\bar{\delta}}\big)\right\}.$$

It is easy to see that $\overline{J}^{l-}_{kn_1n_2} \subseteq \overline{J}^{l-}_{kan_1}$, and $\overline{J}^{l-}_{kan_1} \subseteq \overline{J}^{l-}_{kam_0}$ for $|n_1| \geq |m_0|$.

Now we estimate meas $\overline{J}^{l-}_{kn_1n_2}$ and $\overline{J}^{l-}_{kan_1}$ by the Fubini theorem. It is sufficient to estimate the one-dimensional measure of the intersection of $\overline{J}^{l-}_{kan_1}$ with every line parallel with some fixed direction. In particular, in the direction given by the vector $k|k|^{-1}$. The intersection of $\overline{J}^{l-}_{kan_1}$ with the line $L_\eta = \{\eta + tk|k|^{-1} : t \in \mathbb{R}, \eta \in \mathbb{R}^m\}$ is equal to the set

$$\left\{t \in \mathbb{R} : \left|\omega(t)\right| \leq \frac{\varrho \operatorname{meas} \hat{J}}{(1+l^2)(|k|+1)^{m+3}}\right\} \tag{59}$$

where $\omega(t) = (\langle k, \omega\rangle + a)|_{\omega=\eta+tk|k|^{-1}}$. It is easy to see that $\frac{\partial\langle k,\omega\rangle}{\partial t} = |k|$, so for $t_1 > t_2$, we get

$$\omega(t_1) - \omega(t_2) \geq |k|(t_1 - t_2)$$

as ε small enough. Thus, by Appendix C in [15], we see that the measure of the set (59) is no larger than $\frac{\varrho \operatorname{meas} \hat{J}}{(1+l^2)|k|(|k|+1)^{m+3}}$. This estimate jointly with the Fubini theorem implies that

$$\operatorname{meas} \overline{J}^{l-}_{kan_1} \leq \frac{\varrho^{m-1}}{C(1+l^2)}\left(\frac{\varrho \operatorname{meas} \hat{J}}{|k|(|k|+1)^{m+3}} + \frac{\mathcal{O}(|m_0|^{-\bar{\delta}})}{|k|}\right).$$

Similarly, we also have

$$\operatorname{meas} \overline{J}^{l-}_{kn_1n_2} \leq \frac{\varrho^m \operatorname{meas} \hat{J}}{(1+l^2)|k|(|k|+1)^{m+3}}.$$

Let

$$\overline{J}^-_l = \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{n_1,n_2\in\mathbb{Z}^d} \overline{J}^{l-}_{kn_1n_2}.$$

It yields

$$\begin{aligned}
\operatorname{meas} \overline{J}^-_l &= \operatorname{meas}\left(\bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{n_1,n_2\in\mathbb{Z}^d} \overline{J}^{l-}_{kn_1n_2}\right) = \operatorname{meas}\left(\bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{|n_1|^2-|n_2|^2=a} \overline{J}^{l-}_{kn_1n_2}\right) \\
&\leq \sum_{0\neq k\in\mathbb{Z}^m} \sum_{|n_1|\leq|m_0|} \operatorname{meas} \overline{J}^{l-}_{kn_1n_2} + \sum_{0\neq k\in\mathbb{Z}^m} \operatorname{meas} \overline{J}^{l-}_{kam_0} \\
&\leq \frac{C\varrho^{m-1} \operatorname{meas} \hat{J}}{1+l^2}\left(\varrho|m_0|^{C(d)} \sum_{0\neq k\in\mathbb{Z}^m} \frac{1}{(|k|+1)^{m+4}} + \mathcal{O}\left(|m_0|^{-\bar{\delta}}\right) + \sum_{0\neq k\in\mathbb{Z}^m} \frac{1}{|k|+1}\right) \\
&\leq \frac{C\varrho^m \operatorname{meas} \hat{J}}{1+l^2}\left(\varrho|m_0|^{C(d)} + \mathcal{O}\left(|m_0|^{-\bar{\delta}}\right)\right)
\end{aligned}$$

by using of the convergence of $\sum_{0\neq k\in\mathbb{Z}^m} \frac{1}{(|k|+1)^{m+4}}$ and $\sum_{0\neq k\in\mathbb{Z}^m} \frac{1}{|k|+1}$. By choosing $\varrho|m_0|^{C(d)} = \mathcal{O}(|m_0|^{-\bar{\delta}})$, *i.e.*,

$$|m_0| = \varrho^{-\frac{1}{C(d)+\bar{\delta}}},$$

we have $\varrho|m_0|^{C(d)} = |m_0|^{-\bar{\delta}} = \varrho^{\frac{\bar{\delta}}{C(d)+\bar{\delta}}}$. It follows that

$$\operatorname{meas}\bar{J}_l^- \le \frac{C\varrho^m \operatorname{meas}\hat{J}}{1+l^2}\varrho^{\frac{\bar{\delta}}{C(d)+\bar{\delta}}}.$$

Case 2.2. We consider the following set:

$$\bar{J}_{kn_1n_2}^{l+} := \left\{\omega \in \hat{J} : \left|\langle k,\omega\rangle \pm (\lambda_{n_1,l} + \lambda_{n_2,l})\right| < \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}}\right\}.$$

Let $f_{kn_1n_2}^{l+} := \langle k,\omega\rangle \pm (\lambda_{n_1,l} + \lambda_{n_2,l})$. When $|n_1|^2$ or $|n_2|^2 \ge |k||\omega| + 1$, then

$$\begin{aligned}\left|\langle k,\omega\rangle \pm (\lambda_{n_1,l} + \lambda_{n_2,l})\right| &= \left|\langle k,\omega\rangle \pm \left(\left(|n_1|^2 + M\right) + \left(|n_2|^2 + M\right) + \mathcal{O}(\varepsilon)\right)\right| \\ &\ge \left|\left(|n_1|^2 + M\right) + \left(|n_2|^2 + M\right)\right| - |k||\omega| - \left|\mathcal{O}(\varepsilon)\right| \\ &\ge 1 - \left|\mathcal{O}(\varepsilon)\right| \ge \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}}\end{aligned}$$

holds provided that ε and ϱ are small enough, which implies the set $\bar{J}_{kn_1n_2}^{l+}$ is empty. So, we only need to consider the case $|n_1|^2, |n_2|^2 < |k||\omega| + 1$.

Now we estimate $\bar{J}_{kn_1n_2}^{l+}$ by the Fubini theorem. It is sufficient to estimate the one-dimensional measure of the intersection of $\bar{J}_{kn_1n_2}^{l+}$ with every line parallel with some fixed direction. In particular, in the direction given by the vector $k|k|^{-1}$. The intersection of $\bar{J}_{kn_1n_2}^{l+}$ with the line $L_\eta = \{\eta + tk|k|^{-1} : t \in \mathbb{R}, \eta \in \mathbb{R}^m\}$ is equal to the set

$$\left\{t \in \mathbb{R} : \left|\omega(t)\right| \le \frac{\varrho \operatorname{meas}\hat{J}}{(1+l^2)(|k|+1)^{m+3}}\right\} \tag{60}$$

where $\omega(t) = (\langle k,\omega\rangle \pm (\lambda_{n_1,l} + \lambda_{n_2,l}))|_{\omega=\eta+tk|k|^{-1}}$. It is easy to see that $\frac{\partial\langle k,\omega\rangle}{\partial t} = |k|$, so for $t_1 > t_2$, we get

$$\omega(t_1) - \omega(t_2) \ge |k|(t_1 - t_2) - \varepsilon(t_1 - t_2) \ge \frac{|k|}{2}(t_1 - t_2)$$

as ε small enough. Thus, by Appendix C in [15], we see that the measure of the set (60) is no larger than $\frac{2\varrho \operatorname{meas}\hat{J}}{(1+l^2)|k|(|k|+1)^{m+3}}$. This estimate jointly with the Fubini theorem implies that

$$\operatorname{meas}\bar{J}_{kn_1n_2}^{l+} \le \frac{2\varrho^m \operatorname{meas}\hat{J}}{(1+l^2)|k|(|k|+1)^{m+3}}.$$

Let

$$\bar{J}_l^+ := \bigcup_{0\ne k\in\mathbb{Z}^m} \bigcup_{n_1,n_2\in\mathbb{Z}^d} \bar{J}_{kn_1n_2}^{l+}.$$

It yields

$$\operatorname{meas}\bar{J}_l^+ = \operatorname{meas}\left(\bigcup_{0\ne k\in\mathbb{Z}^m} \bigcup_{n_1,n_2\in\mathbb{Z}^d} \bar{J}_{kn_1n_2}^{l+}\right)$$

$$
\begin{aligned}
&= \operatorname{meas}\Biggl(\bigcup_{0\neq k\in\mathbb{Z}^m}\ \bigcup_{|n_1|^2,|n_2|^2<|k||\omega|+1} \bar{J}^{l+}_{kn_1n_2}\Biggr)\\
&\le \sum_{0\neq k\in\mathbb{Z}^m}\ \sum_{|n_1|^2,|n_2|^2<|k||\omega|+1} \operatorname{meas}\bar{J}^{l+}_{kn_1n_2}\\
&\le \frac{C\varrho^m \operatorname{meas}\hat{J}}{1+l^2}\sum_{0\neq k\in\mathbb{Z}^m}\frac{1}{(|k|+1)^{m+2}} \le \frac{C\varrho^m \operatorname{meas}\hat{J}}{1+l^2}
\end{aligned}
$$

by using of the convergence of $\sum_{0\neq k\in\mathbb{Z}^m}\frac{1}{(|k|+1)^{m+2}}$.

Letting

$$
\bar{J}_0 = \hat{J}\setminus(\bar{J}^0_0+\bar{J}^-_0+\bar{J}^+_0),\qquad \bar{J}_{l+1} = \bar{J}_l\setminus(\bar{J}^0_{l+1}+\bar{J}^-_{l+1}+\bar{J}^+_{l+1}),\quad l=0,1,\ldots,\nu-1, \tag{61}
$$

then (54) and (56) hold true. Let $\bar{J}=\bigcap_{l=0}^{\infty}\bar{J}_l$, then

$$
\begin{aligned}
\operatorname{meas}\bar{J} &= \lim_{l\to\infty}\operatorname{meas}\bar{J}_l\\
&\ge \operatorname{meas}\hat{J}\Biggl(1 - C\varrho^m\sum_{i=0}^{\infty}\frac{1}{1+i^2} - C\varrho^{m+\frac{\bar{\delta}}{C(d)+\bar{\delta}}}\sum_{i=0}^{\infty}\frac{1}{1+i^2} - C\varrho^m\sum_{i=0}^{\infty}\frac{1}{1+i^2}\Biggr)\\
&\ge \operatorname{meas}\hat{J}\Biggl(1-3C\varrho^m\sum_{i=0}^{\infty}\frac{1}{1+i^2}\Biggr)\\
&\ge \operatorname{meas}\hat{J}\bigl(1-\mathcal{O}(\varrho^m)\bigr).
\end{aligned} \tag{62}
$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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