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Sharp threshold for blow-up and global existence in a semilinear parabolic equation with variable source

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Abstract
This paper deals with a semilinear parabolic equation with variable source under the case that the initial energy is less than the potential well depth. We deduce a sharp threshold for blow-up and global existence of solutions. Furthermore, we conclude that the global solution decays as the time goes to infinity.

Keywords: semilinear parabolic equations; variable source; initial energy; global existence; blow-up

1 Introduction

In this paper, we consider an initial boundary value problem for the semilinear parabolic equation with variable exponent:

$$\begin{cases} u_t = \Delta u + |u|^{p(x)-2}u, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ ($N \geq 3$), $u_0 \in H_0^1(\Omega)$, and $p(x)$ is a continuous and bounded function satisfying

$$2 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < 2^* = \frac{2N}{N-2}. \tag{1.2}$$

Eq. (1.1) has been used to model a variety of important physical processes, such as electrorheological fluids (where u is the velocity of moving fluids in electro-magnetic fields) [1], thermo-rheological flows or population dynamics [2, 3]. There is a substantial amount of work concerning the case $p(x) \equiv p$, see, for example, [4–6].

To deal with the variable source, it is convenient to introduce a Lebesgue space $L^{p(\cdot)}(\Omega)$, defined as the space of measurable functions u in Ω satisfying $\int_\Omega |u|^{p(x)}\,dx < \infty$. We mention that this kind of Lebesgue space or general Sobolev space with variable exponent and their applications have got a lot of attention, see the monograph [7] and some recent work [8–11] for instance.

With the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}}(\Omega) = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

the space $L^{p(\cdot)}(\Omega)$ is a Banach space and

$$\inf\left\{\|u\|_{p(\cdot)}^{p^+}, \|u\|_{p(\cdot)}^{p^-}\right\} \le \int_\Omega |u|^{p(x)}\,dx \le \max\left\{\|u\|_{p(\cdot)}^{p^+}, \|u\|_{p(\cdot)}^{p^-}\right\}, \tag{1.3}$$

see [12]. Combining Corollary 3.34 in [12] and the Poincaré inequality, we have

$$\|u\|_{p(\cdot)} \le B\|\nabla u\|_2. \tag{1.4}$$

Regarding variable sources, Pinasco [13] proved that the solution of (1.1) blows up in finite time provided that $p^- > 1$ and the initial data is large enough. This result was then extended to $p^+ > 1$ by Ferreira et al. in [14]. For some positive initial energy, Wu et al. [15] gave a blow-up condition.

Proposition 1.1 (Theorem 1.1 in [15]) *Let*

$$E_1 =: \frac{1}{p^-}\left(\frac{p^+ - 2}{2}B^{p^+}\alpha_1^{\frac{p^+}{2}} + \frac{p^- - 2}{2}B^{p^-}\alpha_1^{\frac{p^-}{2}}\right), \tag{1.5}$$

and

$$\overline{E}_1 =: \left(\frac{p^+ - 2}{p^- - 2}\right)^{\frac{2}{p^+}}\left\{\frac{\alpha_1}{2} - \frac{1}{p^-}\left[B^{p^+}\left(\frac{p^+ - 2}{p^- - 2}\right)^{\frac{p^+-2}{p^+}}\alpha_1^{\frac{p^+}{2}} + B^{p^-}\left(\frac{p^+ - 2}{p^- - 2}\right)^{\frac{p^--2}{p^-}}\alpha_1^{\frac{p^-}{2}}\right]\right\} < E_1,$$

where α_1 is defined by

$$\frac{1}{p^-}\left(B^{p^+}p^+\alpha_1^{\frac{p^+-2}{2}} + B^{p^-}p^-\alpha_1^{\frac{p^--2}{2}}\right) = 1. \tag{1.6}$$

Assume $1 < \sqrt{2p^+ - 1} < p^- \le p^+ \le \frac{N+2}{N-2}$ *and* $0 < E(u_0) < \overline{E}_1$. *If* $\|\nabla u_0\|_2^2 > \alpha_1$, *then the solution of Eq.* (1.1) *blows up in finite time.*

Later, another blow-up condition was derived by Wang and He [16].

Proposition 1.2 (Theorem 1 in [16]) *Assume* $1 < p^- \le p^+ \le \frac{N+2}{N-2}$ *and* $0 < E(u_0) < E_2 = \frac{p^--2}{2p^-}B_1^{-\frac{2p^-}{p^--2}}$ *with* $B_1 \ge \max\{B, 1\}$. *If* $\|\nabla u_0\|_2^2 > \alpha_2 = B_1^{-\frac{2p^-}{p^--2}}$, *then the solution of Eq.* (1.1) *blows up in finite time.*

Motivated by the above research, in this paper we have the main purpose to look for a sharp threshold for blow-up and global existence of solutions of (1.1) in general case (1.2) and (1.7) (in Section 4, we show $E_0 > \overline{E}_1$ and $E_0 \ge E_2$). We mainly use the potential well method, which was used to study the case $p(x) \equiv p$ by Payne and Sattinger [17], and was widely used to consider other parabolic models during the last years, see, for example, [18–21]. Similar to [22], local existence and uniqueness of solutions of (1.1) can be obtained by the Banach fixed point theorem as follows.

Proposition 1.3 *Assume that* (1.2) *holds. Then* (1.1) *admits a unique solution* $u \in C([0,T_{\max}); H_0^1(\Omega)) \cap C^1((0,T_{\max}); L^2(\Omega))$, *where* $T_{\max} > 0$ *denotes the maximal existence time. Either* $T_{\max} < +\infty$ *and* $\lim_{t\to T_{\max}} \|u\|^2_{H_0^1(\Omega)} = +\infty$ (*we say that the solution blows up in finite time*), *or* $T_{\max} = +\infty$ (*we say that the solution is global in time*).

Denote the energy functional

$$E(u) = \int_\Omega \left[\frac{1}{2}\left|\nabla u(x,t)\right|^2 - \frac{1}{p(x)}\left|u(x,t)\right|^{p(x)}\right] dx$$

and the Nehari manifold

$$\mathcal{N} = \left\{u \in H_0^1(\Omega) | N(u) = 0, u \neq 0\right\}$$

with

$$N(u) = \left\langle E'(u), u\right\rangle = \int_\Omega \left[\left|\nabla u(x,t)\right|^2 - \left|u(x,t)\right|^{p(x)}\right] dx.$$

In this paper, we assume that the initial energy is less than the potential well depth, namely

$$E(u_0) < E_0 := \inf_{u\in\mathcal{N}} E(u). \tag{1.7}$$

Now, we introduce our main results as follows.

Theorem 1.1 *Assume that* (1.2) *and* (1.7) *hold. Then*

(1) *if* $N(u_0) < 0$, *then the solution of Eq.* (1.1) *blows up in finite time*;

(2) *if* $N(u_0) \geq 0$, *then the solution* u *of Eq.* (1.1) *is global in time and* $u(t) \to 0$ *strongly in* $H_0^1(\Omega)$ *as* $t \to \infty$.

Finally, we consider applications of Theorem 1.1 and derive the following results.

Corollary 1.1 *Assume* (1.2) *and* $0 < E(u_0) < \overline{E}_1$. *The solution to Eq.* (1.1) *is global in time if* $\|\nabla u_0\|_2^2 \leq \alpha_1$.

Corollary 1.2 *Assume* (1.2) *and* $0 < E(u_0) < E_2$. *The solution of Eq.* (1.1) *is global in time if* $\|\nabla u_0\|_2^2 \leq \alpha_2$.

Remark 1.1 Combined with Proposition 1.1 and Proposition 1.2, the above corollaries imply that the blow-up conditions in [15] and [16] are also sharp there.

This paper is organized as follows. In Section 2, we determine the blow-up condition of solutions of Eq. (1.1). In Section 3, we deal with global existence condition and then conclude that the global solution decays as the time goes to infinity. In Section 4, we prove Corollaries 1.1 and 1.2. Finally, we summarize the main results of the current paper.

In the sequel, we use $\|\cdot\|_p$ to denote $L^p(\Omega)$ norm, and denote the inner product in $L^2(\Omega)$ by the symbol $(\cdot,\cdot)$.

2 Finite time blow-up

In this section, we pay attention to studying blow-up of solutions to Eq. (1.1). To deal with Theorem 1.1(1), we first give some preliminary lemmas.

Lemma 2.1 *It holds* $E_0 > 0$.

Proof For $u \in \mathcal{N}$, that is $\int_\Omega |\nabla u|^2\,dx = \int_\Omega |u|^{p(x)}\,dx$, we have

$$\int_\Omega |\nabla u|^2\,dx \le \int_\Omega |u|^{p^-}\,dx + \int_\Omega |u|^{p^+}\,dx.$$

We then apply the Poincáre inequality to find that

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx &\le \int_\Omega |u|^{p^-}\,dx + \int_\Omega |u|^{p^+}\,dx \\ &\le C\left(\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p^-}{2}} + \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p^+}{2}}\right),\end{aligned}$$

thereby obtaining the inequality

$$\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p^-}{2}-1} + \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p^+}{2}-1} \ge \frac{1}{C}.$$

This implies

$$\int_\Omega |\nabla u|^2\,dx \ge \min\left\{\left(\frac{1}{2C}\right)^{\frac{2}{p^--2}}, \left(\frac{1}{2C}\right)^{\frac{2}{p^+-2}}\right\}.$$

Therefore, we deduce

$$\begin{aligned}E(u) &\ge \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p^-}\int_\Omega |u|^{p(x)}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p^-}\right)\int_\Omega |\nabla u|^2\,dx + \frac{1}{p^-}N(u) \\ &\ge \frac{p^- - 2}{2p^-}\min\left\{\left(\frac{1}{2C}\right)^{\frac{2}{p^--2}}, \left(\frac{1}{2C}\right)^{\frac{2}{p^+-2}}\right\}.\end{aligned}$$

Due to the arbitrariness of $u \in \mathcal{N}$, we conclude $E_0 > 0$. □

Lemma 2.2 *Fix* $u \in H_0^1(\Omega)$, $u \neq 0$ *and define* $u^\lambda := \lambda u$ *with* $\lambda > 0$. *Then*

(1) *There exists a unique positive constant* $\lambda_0 > 0$ *such that* $u^{\lambda_0} \in \mathcal{N}$ *and* $E(u^{\lambda_0}) = \max_{\lambda>0} E(u^\lambda)$;

(2) $\lambda_0 < 1$ *if and only if* $N(u) < 0$;

(3) $\lambda_0 = 1$ *if and only if* $N(u) = 0$.

Proof A direct computation yields

$$\frac{d}{d\lambda}E(u^\lambda) = \lambda\int_\Omega |\nabla u|^2\,dx - \int_\Omega \lambda^{p(x)-1}|u|^{p(x)}\,dx = \frac{1}{\lambda}N(\lambda u),$$

and

$$\frac{d^2}{d\lambda^2}E(u^\lambda) = \int_\Omega |\nabla u|^2\,dx - \int_\Omega (p(x)-1)\lambda^{p(x)-2}|u|^{p(x)}\,dx.$$

Set $f(\lambda) := \frac{d}{d\lambda}E(\lambda u)$. There exists a unique $\lambda_1 > 0$ such that

(i) $f(\lambda_1) = \max_{\lambda>0} f(\lambda)$,

(ii) $f(0) = 0$,

(iii) $f(\lambda)$ increases in $(0,\lambda_1)$ and decreases in $(\lambda_1,+\infty)$. Conclusions (1)-(3) are easy consequences of (i)-(iii). □

Denote

$$E^\varepsilon := \inf\{E(u) : N(u) = -\varepsilon, u \in H_0^1(\Omega)\}$$

with $\varepsilon > 0$.

Lemma 2.3 *Let* (1.2) *hold. Then* $E^\varepsilon \geq E_0 - \frac{\varepsilon}{2}$.

Proof Choose a minimized sequence $(u_i) \subseteq H_0^1(\Omega)$ such that

$$N(u_i) = -\varepsilon \quad \text{and} \quad E(u_i) \to E^\varepsilon \quad (\text{as } i \to \infty).$$

It follows from Lemma 2.2 that there exists a positive constant $\mu_i \in (0,1)$ such that

$$N(\mu_i u_i) = 0.$$

Therefore

$$\begin{aligned} E_0 \leq E(\mu_i u_i) &= \frac{1}{2}N(\mu_i u_i) + \int_\Omega \left(\frac{1}{2} - \frac{1}{p(x)}\right)\left|\mu_i u_i(x,t)\right|^{p(x)}\,dx \\ &= \int_\Omega \left(\frac{1}{2} - \frac{1}{p(x)}\right)\left|\mu_i u_i(x,t)\right|^{p(x)}\,dx \\ &\leq \int_\Omega \left(\frac{1}{2} - \frac{1}{p(x)}\right)\left|u_i(x,t)\right|^{p(x)}\,dx \\ &= E(u_i) - \frac{1}{2}N(u_i) \\ &= E(u_i) + \frac{\varepsilon}{2}. \end{aligned}$$

Let $i \to \infty$, then $E_0 \leq E^\varepsilon + \frac{\varepsilon}{2}$. □

Lemma 2.4 (Lemma 2.1 [15]) $\frac{d}{dt}E(u) = -\int_\Omega u_t^2\,dx \leq 0$.

Now we can prove Theorem 1.1(1) via the potential well method [17] and the Kaplan method [23].

Proof of Theorem 1.1(1) By Lemma 2.4, we have $E(u) \leq E(u_0) < E_0$. We claim that the solution u blows up in finite time provided that $N(u_0) < 0$. Otherwise, assume that u is global in time. Choose $\varepsilon = \min\{E_0 - E(u_0), -N(u_0)\}/2$. Then $N(u_0) \leq -2\varepsilon < -\varepsilon, E(u_0) \leq E_0 - 2\varepsilon < E_0 - \frac{\varepsilon}{2} \leq E^\varepsilon$ by Lemma 2.3. Consequently, $N(u) < -\varepsilon$ for all $t > 0$. Indeed, if there exists $t_0 > 0$ such that $N(u(t_0)) = -\varepsilon$, then $E(u(t_0)) \geq E^\varepsilon$ by the definition of E^ε. This is impossible since $E(u_0) < E^\varepsilon$ and Lemma 2.4.

Let $M(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\,dx$. Then by (1.1) we get that

$$\begin{aligned} M'(t) &= \int_\Omega u(x,t)u_t(x,t)\,dx \\ &= -\int_\Omega \bigl|\nabla u(x,t)\bigr|^2\,dx + \int_\Omega \bigl|u(x,t)\bigr|^{p(x)}\,dx \\ &= -N(u) > \varepsilon. \end{aligned} \tag{2.1}$$

Consequently, we have

$$\lim_{t\to\infty} M(t) = +\infty. \tag{2.2}$$

Next, we derive a contradiction by showing that $M(t)$ blows up in finite time. We deal with variable source as

$$\begin{aligned} \int_\Omega |u|^{p(x)}\,dx &= \int_{|u|\leq 1} |u|^{p(x)}\,dx + \int_{|u|\geq 1} |u|^{p(x)}\,dx \\ &\geq \int_{|u|\leq 1} |u|^{p^+}\,dx + \int_{|u|\geq 1} |u|^{p^-}\,dx \\ &\geq \int_{|u|\geq 1} |u|^{p^-}\,dx \\ &= \int_{|u|\leq 1} dx + \int_{|u|\geq 1} |u|^{p^-}\,dx - |\Omega| \\ &\geq \int_{|u|\leq 1} |u|^{p^-}\,dx + \int_{|u|\geq 1} |u|^{p^-}\,dx - |\Omega| \\ &\geq \int_\Omega |u|^{p^-}\,dx - |\Omega|, \end{aligned}$$

where $|\Omega|$ is the measure of Ω. By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega |u|^{p(x)}\,dx &\geq |\Omega|^{\frac{2-p^-}{2}} \left(\int_\Omega |u|^2\,dx\right)^{\frac{p^-}{2}} - |\Omega| \\ &= |\Omega|^{\frac{2-p^-}{2}} M(t)^{\frac{p^-}{2}} - |\Omega|. \end{aligned}$$

Therefore, by Lemma 2.4, we rewrite $M'(t)$ in (2.1) as

$$\begin{aligned} M'(t) &= -2E(u) + \int_\Omega \left(1 - \frac{2}{p(x)}\right)|u|^{p(x)}\,dx \\ &\geq -2E(u_0) + \left(1 - \frac{2}{p^-}\right)\int_\Omega |u|^{p(x)}\,dx \\ &\geq \left(1 - \frac{2}{p^-}\right)|\Omega|^{\frac{2-p^-}{2}} M(t)^{\frac{p^-}{2}} - \left(1 - \frac{2}{p^-}\right)|\Omega| - 2E(u_0). \end{aligned}$$

It follows from (2.2) that there exists $t_0 > 0$ such that

$$\left(1-\frac{2}{p^-}\right)|\Omega|^{\frac{2-p^-}{2}}M(t)^{\frac{p^-}{2}} \geq 2\left[\left(1-\frac{2}{p^-}\right)|\Omega| + 2E(u_0)\right],$$

for $t \geq t_0$. Thus $M'(t) \geq \frac{1}{2}(1-\frac{2}{p^-})|\Omega|^{\frac{2-p^-}{2}}M(t)^{\frac{p^-}{2}}$. This immediately implies that $M(t)$ blows up in finite time, a contradiction. □

3 Global existence of solutions

In this section, we consider global existence of solutions to Eq. (1.1) and study asymptotic behavior of the global solution.

Proof of Theorem 1.1(2) We divide the proof into two cases: (i) $N(u_0) = 0$, (ii) $N(u_0) > 0$. If $N(u_0) = 0$, then $u_0 = 0$. Otherwise, if $u_0 \neq 0$, then $E(u_0) \geq E_0$. This is a contradiction to the assumption condition (1.7). By uniqueness of solutions to (1.1), we have the solution $u(t) = 0$ for all $t \geq 0$.

If $N(u_0) > 0$, if $u(t) = 0$ for some t, then $u(s) = 0$ for $s \geq t$ by uniqueness, and the conclusion is true. Hereafter, we assume that $u(t) \neq 0$ for all $t \in (0, T)$. We claim that $N(u) > 0$ as long as the solution u exists. Otherwise, there exists t_0 such that $N(u(t_0)) = 0$. By Lemma 2.4, $E(u(t_0)) \leq E(u_0) < E_0$, which contradicts the definition of E_0. Therefore,

$$\begin{aligned} E_0 > E(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \frac{1}{p^-}|u|^{p(x)}\,dx \\ &= \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\nabla u|^2\,dx + \frac{1}{p^-}N(u) > \left(\frac{1}{2}-\frac{1}{p^-}\right)\int_\Omega |\nabla u|^2\,dx. \end{aligned}$$

This with the Poincaré inequality yield that $\|u\|_{H_0^1(\Omega)}$ is uniformly bounded. By Proposition 1.3, there exists a global solution of Eq. (1.1).

Next, we investigate long time behavior of the global solution to Eq. (1.1). Basing on the above process, if $N(u_0) = 0$, then $u(t) = 0$ for all $t \geq 0$. The conclusion is true. If $N(u_0) > 0$, then the global solution u is uniformly bounded in $H_0^1(\Omega)$ and $N(u) \geq 0$. Consequently, $E(u) \geq 0$. Noting the equality

$$\int_0^t \int_\Omega u_t^2\,dx\,ds + E(u) = E(u_0)$$

by (1.1), we have

$$\int_0^\infty \int_\Omega u_t^2\,dx\,ds \leq E(u_0).$$

Therefore, there exists a time sequence $\{t_i\}_{i=1}^\infty$ with $t_i \to \infty$, as $i \to \infty$, such that $\|u_t(t_i)\|_2^2 \to 0$. Since $u(t_i)$ is bounded in $H_0^1(\Omega)$ and thus $|u|^{p(\cdot)-2}u(t_i)$ is bounded in $H^{-1}(\Omega)$, going to a subsequence if necessary, still denoted by $u(t_i)$,

$$u(t_i) \rightharpoonup \varphi \quad \text{weakly in } H_0^1(\Omega), \tag{3.1}$$

$$u(t_i) \to \varphi \quad \text{strongly in } L^q(\Omega) \quad \left(2 \leq q < 2^*\right), \tag{3.2}$$

$$\left|u(t_i)\right|^{p(\cdot)-2}u(t_i) \rightharpoonup |\varphi|^{p(\cdot)-2}\varphi \quad \text{weakly in } H^{-1}(\Omega). \tag{3.3}$$

Multiplying (1.1) by $v \in H_0^1(\Omega)$ and integrating, we have

$$\big(u_t(t_i), v\big) + \big(\nabla u(t_i), \nabla v\big) = \big\langle |u|^{p(\cdot)-2} u(t_i), v\big\rangle. \tag{3.4}$$

Letting $i \to \infty$ in the above equality, we obtain that

$$(\nabla \varphi, \nabla v) = \big\langle |\varphi|^{p(\cdot)-2} \varphi, v\big\rangle.$$

Choosing $v = \varphi$ in the above equality, we have $N(\varphi) = 0$. By (3.2) and the mean value theorem, we derive that

$$\lim_{i\to\infty} \int_\Omega \big|u(t_i)\big|^{p(x)}\, dx = \int_\Omega |\varphi|^{p(x)}\, dx. \tag{3.5}$$

By the weak semi-continuity of $H_0^1(\Omega)$ norm and (3.5), we have

$$E(\varphi) \leq \liminf_{i\to\infty} E\big(u(t_i)\big) < E_0.$$

This with $N(\varphi) = 0$ yield that $\varphi = 0$. Setting $v = u(t_i)$ in (3.4), observing that

$$\big|\big(u_t(t_i), u(t_i)\big)\big| \leq \big\|u_t(t_i)\big\|_2 \big\|u(t_i)\big\|_2 \to 0,$$

we get that

$$\lim_{i\to\infty} \big\|\nabla u(t_i)\big\|_2^2 - \int_\Omega \big|u(t_i)\big|^{p(x)}\, dx = 0. \tag{3.6}$$

This with (3.5) imply that

$$\lim_{i\to\infty} \big\|\nabla u(t_i)\big\|_2^2 = \lim_{i\to\infty} \int_\Omega \big|u(t_i)\big|^{p(x)}\, dx = \int_\Omega |\varphi|^{p(x)}\, dx = 0.$$

Consequently, $E(u(t_i)) \to 0$ as $i \to \infty$. This with the fact that $E(u(t))$ is decreasing with respect of time t and $E(u(t)) \geq 0$ imply that $\lim_{t\to\infty} E(u(t)) = 0$.

Rewrite $E(u(t))$ as

$$E\big(u(t)\big) = \frac{1}{2} N\big(u(t)\big) + \int_\Omega \left(\frac{1}{2} - \frac{1}{p(x)}\right) \big|u(t)\big|^{p(x)}\, dx.$$

It follows from $N(u(t)) \geq 0$ that

$$\begin{aligned}\int_\Omega \big|u(t)\big|^{p(x)}\, dx &\leq \frac{2p^-}{p^- - 2} \int_\Omega \left(\frac{1}{2} - \frac{1}{p(x)}\right) \big|u(t)\big|^{p(x)}\, dx \\ &\leq \frac{2p^-}{p^- - 2} E\big(u(t)\big) \to 0, \quad \text{as } t \to \infty.\end{aligned}$$

Therefore,

$$\big\|\nabla u(t)\big\|_2^2 = 2E\big(u(t)\big) + \int_\Omega \frac{2}{p(x)} \big|u(t)\big|^{p(x)}\, dx \to 0, \quad \text{as } t \to \infty.$$

This with the Poincáre inequality imply that $u(t) \to 0$ strongly in $H_0^1(\Omega)$. □

4 Proof of Corollaries 1.1 and 1.2

To compare our results to those in [15, 16], we begin with analyzing E_0 defined in (1.7) more precisely.

Lemma 4.1 *Set*

$$E_3 := \inf\Big\{\max_{\lambda>0} E(\lambda u) : u \in H_0^1(\Omega), \textit{and } u \not\equiv 0\Big\}.$$

It holds $E_0 = E_3$.

Proof For any $u \in \mathcal{N}$, by Lemma 2.2, we have that

$$E(u) = \max_{\lambda>0} E(\lambda u) \geq E_3.$$

Therefore, $E_0 \geq E_3$. On the other hand, for any $u \in H_0^1(\Omega)$, $u(x) \not\equiv 0$, by Lemma 2.2, there exists $\lambda_0 > 0$ such that $\max_{\lambda>0} E(\lambda u) = E(\lambda_0 u)$ and $N(\lambda_0 u) = 0$. This implies that $\max_{\lambda>0} E(\lambda u) \geq E_0$. Because of the arbitrariness of u, we have $E_3 \geq E_0$. The proof is complete. □

Next, we compare E_0 with $\overline{E}_1$ and E_2, where $\overline{E}_1$ is defined in Proposition 1.1 and E_2 defined in Proposition 1.2, respectively.

Lemma 4.2 $E_0 > \overline{E}_1$ *and* $E_0 \geq E_2$.

Proof For any $u \in H_0^1(\Omega)$, $u \not\equiv 0$, define $u^\lambda = \lambda u$ with $\lambda > 0$. It follows from (1.3) and the Poincaré inequality (1.4) that

$$\begin{aligned}
E(\lambda u) &= E\big(u^\lambda\big) \\
&\geq \frac{1}{2}\big\|\nabla u^\lambda\big\|_2^2 - \frac{1}{p^-}\int_\Omega \big|u^\lambda\big|^{p(x)}\,dx \\
&\geq \frac{1}{2}\big\|\nabla u^\lambda\big\|_2^2 - \frac{1}{p^-}\big\|u^\lambda\big\|_{p(\cdot)}^{p^-} - \frac{1}{p^-}\big\|u^\lambda\big\|_{p(\cdot)}^{p^+} \\
&\geq \frac{1}{2}\big\|\nabla u^\lambda\big\|_2^2 - \frac{1}{p^-}B^{p^-}\big\|\nabla u^\lambda\big\|_2^{p^-} - \frac{1}{p^-}B^{p^+}\big\|\nabla u^\lambda\big\|_2^{p^+} \\
&=: h\big(\big\|\nabla u^\lambda\big\|_2\big).
\end{aligned}$$

Choose $\lambda_1 > 0$ such that $\|\nabla u^\lambda\|_2 = \alpha_1$, defined in (1.6), and $h(\|\nabla u^{\lambda_1}\|_2) = E_1$. Thus

$$\max_{\lambda>0} E(\lambda u) \geq E(\lambda_1 u) \geq h\big(\big\|\nabla u^{\lambda_1}\big\|_2\big) = E_1.$$

This with Lemma 4.1 yield that $E_0 \geq E_1$. Since $E_1 > \overline{E}_1$, we have $E_0 > \overline{E}_1$.

Now we prove $E_0 \geq E_2$. For any nontrivial function $u \in \mathcal{N}$, by (1.3), we have

$$\begin{aligned}
\|\nabla u\|_2^2 &= \int_\Omega |u|^{p(x)}\,dx \\
&\leq \max\big\{\|u\|_{p(\cdot)}^{p^+}, \|u\|_{p(\cdot)}^{p^-}\big\} \leq \max\big\{B_1^{p^+}\|\nabla u\|_2^{p^+}, B_1^{p^-}\|\nabla u\|_2^{p^-}\big\}.
\end{aligned}$$

Therefore, $\|\nabla u\|_2 \geq B_1^{-\frac{p^+}{p^+-2}}$, or $\|\nabla u\|_2 \geq B_1^{-\frac{p^-}{p^--2}}$. As $B_1 > 1$ and $p^+ \geq p^-$, we get $\|\nabla u\|_2 \geq B_1^{-\frac{p^-}{p^--2}}$. Consequently,

$$E(u) \geq \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\nabla u\|_2^2 \geq \frac{p^- - 2}{2p^-} B_1^{-\frac{2p^-}{p^--2}} = E_2.$$

Due to the arbitrariness of $u \in \mathcal{N}$, we conclude $E_0 \geq E_2$. □

Finally, we prove Corollaries 1.1 and 1.2.

Proof of Corollary 1.1 Assume $\|\nabla u_0\|_2^2 \leq \alpha_1$, where α_1 is defined in (1.6). Then

$$\frac{1}{p^-}\left(B^{p^+} p^+ \|\nabla u_0\|_2^{p^+-2} + B^{p^-} p^- \|\nabla u_0\|_2^{p^--2}\right) \leq 1.$$

Consequently,

$$\begin{aligned} \|\nabla u_0\|_2^2 &\geq \frac{1}{p^-}\left(B^{p^+} p^+ \|\nabla u_0\|_2^{p^+} + B^{p^-} p^- \|\nabla u_0\|_2^{p^-}\right) \\ &\geq B^{p^+} \|\nabla u_0\|_2^{p^+} + B^{p^-} \|\nabla u_0\|_2^{p^-}. \end{aligned}$$

By (1.3) and the Poincaré inequality (1.4), we have

$$\begin{aligned} \|\nabla u_0\|_2^2 &\geq \|u_0\|_{p(\cdot)}^{p^+} + \|u_0\|_{p(\cdot)}^{p^-} \\ &\geq \int_\Omega |u_0|^{p(x)}\,dx, \end{aligned}$$

thereby we obtain $N(u_0) \geq 0$. Combining Lemma 4.2 and Theorem 1.1(2), we see that the solution of Eq. (1.1) is global. □

Proof of Corollary 1.2 Since $\|\nabla u_0\|_2^2 \leq \alpha_2 = B_1^{-\frac{2p^-}{p^--2}}$, $B_1 \geq 1$ and $p^- \leq p^+$, then $B_1^{-\frac{2p^-}{p^--2}} \geq B_1^{-\frac{2p^+}{p^+-2}}$ and

$$\|\nabla u_0\|_2^2 \leq \min\left\{B_1^{-\frac{2p^-}{p^--2}}, B_1^{-\frac{2p^+}{p^+-2}}\right\}.$$

Consequently,

$$\|\nabla u_0\|_2^2 \geq \max\left\{B_1^{p^-}\|\nabla u_0\|_2^{p^-}, B_1^{p^+}\|\nabla u_0\|_2^{p^+}\right\}.$$

Using (1.3) and the Poincaré inequality (1.4), we obtain that

$$\begin{aligned} \|\nabla u_0\|_2^2 &\geq \max\left\{\|u_0\|_{p(\cdot)}^{p^-}, \|u_0\|_{p(\cdot)}^{p^+}\right\} \\ &\geq \int_\Omega |u_0|^{p(x)}\,dx. \end{aligned}$$

This implies that $N(u_0) \geq 0$. The rest is the same as the proof of Corollary 1.1, and hence is omitted. □

5 Conclusions

The main aim of the current work is to study asymptotic behavior of solutions to the parabolic equation (1.1). We prove that, when the initial energy is smaller than the mountain pass level corresponding to the stationary equation of (1.1) (see (1.7) and Lemma 4.1), the initial Nehari functional plays an important role in determining asymptotic behavior of the solution of (1.1), see Theorem 1.1. That is, if the initial Nehari functional is negative, then the solution of (1.1) blows up in finite time, while there exists a global solution if the initial Nehari functional is nonnegative. Moreover, the global solution decays as the time goes to infinity. This result generalizes the ones in [15] and [16], see Lemma 4.2 for differences between them. As applications of Theorem 1.1, we derive two corollaries which yield that the blow-up conditions in [15] and [16] are also sharp there, see Corollaries 1.1, 1.2 and Remark 1.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the manuscript was completed in cooperation with the same responsibility. All authors read and approved the final manuscript.

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