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# Application of $p$ -regularity theory to the Duffing equation

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## Abstract

The paper studies a solution existence problem of the nonlinear Duffing equation of the form

$$F(x, \mu, \beta) = x'' + x + \mu x^3 - \beta \sin t = 0, \quad \beta > 0, \mu \neq 0,$$

where  $F : C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \rightarrow C[0, 2\pi]$  and  $x(0) = x(2\pi) = 0$  using the  $p$ -regularity theory.

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## 1 Introduction

This paper is a continuation of the work by Grzegorzczyk et al. [1]. We study the structure of solutions to the Duffing oscillation equation of the form

$$F(x, \mu, \beta) = x'' + x + \mu x^3 - \beta \sin t = 0, \quad \beta > 0, \mu \neq 0, \quad (1)$$

where  $F : C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \rightarrow C[0, 2\pi]$  and  $x(0) = x(2\pi) = 0$  applying some methods of  $p$ -regularity theory [2–5].

We consider a special case of the general Duffing equation given by

$$x'' + \delta x' + \alpha x + \mu x^3 = \beta \cos \omega t, \quad (2)$$

where the unknown function  $x = x(t)$  is the displacement at time  $t$ ,  $x'$  is the first derivative of  $x$  with respect to time, *i.e.*, velocity, and  $x''$  is the second time-derivative of  $x$ , *i.e.*, acceleration. The numbers  $\delta$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\omega$  are given constants.

In general, the Duffing equation does not admit an exact symbolic solution. However, there are some approximate methods, *i.e.*, expansion in a Fourier series, the Frobenius method, numeric methods such as Euler's and Runge-Kutta methods, the homotopy analysis method [6, 7]. In this paper we present a new approach, the so-called  $p$ -regularity method, and show that this approximate method works well and is effective.

The Duffing equation is a degenerate problem since there exists  $x^*$  such that it is a solution, but  $\text{Im} F'(x^*) \neq Y$ . The construction of  $p$ -regularity gives possibilities for solving

such nontrivial problems. This paper shows how to determine the asymptotic solutions in the case of degeneracy of order higher than two. In the work by Grzegorzczuk et al. [1], the authors studied the 2-regularity case. Now we examine 4-regularity.

## 2 The main results of $p$ -regularity theory

The apparatus of  $p$ -regularity is a new tool for studying nonlinear problems. In this section, we present some definitions, notations and theorems of  $p$ -regularity theory to be used in what follows (see [1–5, 8, 9]).

This paper deals with the nonlinear problem

$$F(x) = 0, \tag{3}$$

where  $F$  is a sufficiently smooth mapping between Banach spaces  $X$  and  $Y$ . Apply  $p$ -regularity theory to this equation.

Assume that  $x^* \in U \subseteq X$ ,  $U$  is a neighborhood of the element  $x^*$ . Suppose that there is singularity at the point  $x^*$ . Assume that the space  $Y$  can be decomposed into a direct sum of subspaces

$$Y = Y_1 \oplus \dots \oplus Y_p, \tag{4}$$

where  $Y_1 = \text{cl}(\text{Im } F'(x^*))$  and  $Z_1 = Y$ . As  $Z_2$  we use the closed complement of  $Y_1$  in  $Y$ . Let  $P_{Z_2} : Y \rightarrow Z_2$  be the projector onto  $Z_2$  along  $Y_1$ . By  $Y_2$  we denote the closure of a linear span of the image of the quadratic mapping  $P_{Z_2}F''(x^*)[\cdot]^2$ . Then, inductively,

$$Y_i = \text{cl}(\text{span Im } P_{Z_i}F^{(i)}(x^*)[\cdot]^i) \subseteq Z_i, \quad i = 2, \dots, p - 1, \tag{5}$$

where  $Z_i$  is a closed complement of  $Y_1 \oplus \dots \oplus Y_{i-1}$ ,  $i = 2, \dots, p$ , with respect to  $Y$ , and  $P_{Z_i} : Y \rightarrow Z_i$  is a projector onto  $Z_i$  along  $Y_1 \oplus \dots \oplus Y_{i-1}$ ,  $i = 2, \dots, p$ , with respect to  $Y$ . Finally,  $Y_p = Z_p$ . The order  $p$  is the minimal number (if it exists) for which the decomposition (4) holds. Define the mapping

$$f_i : U \rightarrow Y_i, \quad f_i(x) = \Pi_i F(x), \quad i = 1, \dots, p, \tag{6}$$

where  $\Pi_i : Y \rightarrow Y_i$  is the projector onto  $Y_i$  along  $Y_1 \oplus \dots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \dots \oplus Y_p$ . Then the mapping  $F$  can be represented as

$$F(x) = f_1(x) + \dots + f_p(x). \tag{7}$$

**Definition 1** The linear operator mapping  $X$  to  $Y$

$$\Psi_p(h) = \Psi_p(x^*, h) = f_1'(x^*) + f_2''(x^*)[h] + \dots + f_p^{(p)}(x^*)[h]^{p-1} \tag{8}$$

such that

$$\Psi_p(h)x = \Psi_p(x^*, h)x = f_1'(x^*)x + f_2''(x^*)[h]x + \dots + f_p^{(p)}(x^*)[h]^{p-1}x, \quad x \in X, \tag{9}$$

is called  $p$ -factor operator.

**Definition 2** A mapping  $F$  is called  $p$ -regular at  $x^*$  along  $h$  ( $p > 1$ ) if  $\text{Im } \Psi_p(h) = Y$  (i.e., the operator  $\Psi_p(h)$  is surjective).

**Definition 3** The  $p$ -kernel of the operator  $\Psi_p(h)$  is defined as a set

$$\begin{aligned} H_p(x^*) &= \text{Ker}^p \Psi_p(h) = \{h \in X : \Psi_p(h)[h] = 0\} \\ &= \{h \in X : f_1'(x^*)[h] + f_2''(x^*)[h]^2 + \dots + f_p^{(p)}(x^*)[h]^p = 0\}. \end{aligned}$$

Note that

$$\text{Ker}^p \Psi_p(h) = \left\{ \bigcap_{i=1}^p \text{Ker}^{i} f_i^{(i)}(x^*) \right\}.$$

**Definition 4** A mapping  $F$  is called  $p$ -regular at  $x^*$  ( $p > 1$ ) if either it is  $p$ -regular along every  $h \in H_p(x^*) \setminus \{0\}$  or  $H_p(x^*) = \{0\}$ .

Define the solution set for the mapping  $F$  as the set

$$M = M(x^*) = \{x \in X : F(x) = F(x^*) = 0\}, \tag{10}$$

and let  $T_{x^*}M$  denote the tangent cone to the set  $M$  at the point  $x^*$ , i.e.,

$$T_{x^*}M = \{h \in X : x^* + \varepsilon h + r(\varepsilon) \in M, \|r(\varepsilon)\| = o(\varepsilon), \varepsilon \in [0, \delta], \delta > 0\}. \tag{11}$$

The following theorem describes the tangent cone to the solutions sets of equation (3) in the  $p$ -regular case.

**Theorem 5** (Generalized Lyusternik theorem, [2]) *Let  $X$  and  $Y$  be the Banach spaces, and let the mapping  $F \in C^p(X, Y)$  be  $p$ -regular at  $x^* \in M$ . Then*

$$T_{x^*}M = H_p(x^*). \tag{12}$$

Let us explain that here (for Banach spaces  $X$  and  $Y$ )  $F \in C^p(X, Y)$  means that  $F : X \rightarrow Y$  is  $p$  times continuously Frechét differentiable.

### 3 Solutions to the Duffing equation

Consider the oscillation Duffing equation

$$F(x, \mu, \beta) = x'' + x + \mu x^3 - \beta \sin t = 0, \quad \beta > 0, \mu \neq 0, \tag{13}$$

where  $F : C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \rightarrow C[0, 2\pi]$  and  $x(0) = x(2\pi) = 0$ .

Note that for  $\mu = 0, \beta > 0$ , solutions of the above equation do not exist, but for  $\beta = 0$ , there exists zero solution. We are interested in nontrivial solutions of this equation of course. The following theorem holds.

**Theorem 6** For sufficiently small  $\beta > 0$ , the Duffing equation (1) has nontrivial solution tangent to elements from the 4-kernel of the form

$$x(\beta, t) = \frac{\beta^{\frac{1}{4}}}{\bar{\beta}^{\frac{1}{4}}} \left( \sqrt[3]{\frac{4\bar{\beta}}{3\bar{\mu}}} \sin t \right) + o(\beta^{\frac{1}{4}}), \tag{14}$$

$$\mu(\beta) = \frac{\beta^{\frac{1}{4}}}{\bar{\beta}^{\frac{1}{4}}} \bar{\mu} + o(\beta^{\frac{1}{4}}), \tag{15}$$

where  $\bar{\mu}, \bar{\beta}$  are fixed numbers for which the following conditions hold:  $3\sqrt[3]{\frac{4\bar{\beta}}{3\bar{\mu}}} + |\bar{\mu}| + |\bar{\beta}^{\frac{1}{4}}| = 1$ ,  $\bar{\mu} = \bar{\mu}(\bar{\beta}^{\frac{1}{4}})$ ,  $\bar{\beta} = \bar{\alpha}^4$  and  $0 < \bar{\beta}^{\frac{1}{4}} < 1$ .

*Proof* In order to apply the theory of  $p$ -regularity, let us put  $\beta^{\frac{1}{4}} = \alpha$  and go to an equivalent equation of the form

$$\bar{F}(x, \mu, \alpha) = x'' + x + \mu x^3 - \alpha^4 \sin t = 0. \tag{16}$$

Exactly now we show that the mapping  $\bar{F}$  is 4-regular, and we will describe solutions. Of course,  $x^* = (0, 0, 0)$  is the trivial solution of the above equation. We have

$$\begin{aligned} \bar{F}'(x, \mu, \alpha) &= (\bar{F}'_x(x, \mu, \alpha), \bar{F}'_\mu(x, \mu, \alpha), \bar{F}'_\alpha(x, \mu, \alpha)) \\ &= \left( \frac{d^2}{dt^2} + 1 + 3\mu x^2, x^3, -4\alpha^3 \sin t \right) \end{aligned} \tag{17}$$

and

$$\bar{F}'(0, 0, 0) = (\bar{F}'_x(0, 0, 0), \bar{F}'_\mu(0, 0, 0), \bar{F}'_\alpha(0, 0, 0)) = \left( \frac{d^2}{dt^2} + 1, 0, 0 \right).$$

Note that  $\text{Ker } \bar{F}'_x(0, 0, 0) = \{x \in C^2[0, 2\pi] : \frac{d^2x}{dt^2} + x = 0\}$ . The general solutions of the equation  $x'' + x = 0$  are  $x(t) = c_1 \cos t + c_2 \sin t$ . Taking into account the boundary conditions, we obtain  $c_1 = 0$ ,  $x(t) = c_2 \sin t$  and  $\text{Ker } \bar{F}'_x(0, 0, 0) = \text{span}\{\sin t\}$ .

The image of the operator  $\bar{F}'_x(0, 0, 0)$  is defined as follows:

$$\begin{aligned} \text{Im } \bar{F}'_x(0, 0, 0) &= \{y \in C[0, 2\pi] : \exists x \in C^2[0, 2\pi] \bar{F}'_x(0, 0, 0)x = y, x(0) = x(2\pi) = 0\} \\ &= \{y \in C[0, 2\pi] : \exists x \in C^2[0, 2\pi] x'' + x = y, x(0) = x(2\pi) = 0\}. \end{aligned}$$

The general solution of the equation  $x'' + x = y$  has the form

$$x(t) = c_1 \cos t + c_2 \sin t - \cos t \int_0^t y(\tau) \sin \tau \, d\tau + \sin t \int_0^t y(\tau) \cos \tau \, d\tau.$$

In view of the boundary condition, we obtain  $c_1 = 0$  i  $\int_0^{2\pi} y(\tau) \sin \tau \, d\tau = 0$ , then

$$\begin{aligned} \text{Im } \bar{F}'_x(0, 0, 0) &= \left\{ y \in C[0, 2\pi] : \int_0^{2\pi} y(\tau) \sin \tau \, d\tau = 0 \right\} \\ &= \{y \in C[0, 2\pi] : \langle y, \sin t \rangle = 0, \sin t \in \text{Ker } \bar{F}'_x(0, 0, 0)\}. \end{aligned}$$

One can easily show that the boundary value problem  $x'' + x = \sin t$ ,  $x(0) = x(2\pi) = 0$  does not have a solution. This implies that the operator  $\bar{F}'_x(0, 0, 0)$  is not surjective and  $\text{Im } \bar{F}'_x(0, 0, 0) \neq C[0, 2\pi] = Y$ . Then  $Y = Y_1 \oplus Z_2$ , where  $Y_1 = \text{Im } \bar{F}'_x(0, 0, 0)$ ,  $Z_2 = Y_1^\perp$ .

The projector  $P_{Z_2} : Y \rightarrow Z_2$  can be described as

$$P_{Z_2}y = \frac{1}{\pi} \sin t \langle y, \sin t \rangle = \frac{1}{\pi} \sin t \int_0^{2\pi} y(\tau) \sin \tau \, d\tau, \quad y \in Y.$$

This implies that

$$\begin{aligned} Y_2 &= \text{span}(\text{Im } P_{Z_2} \bar{F}''(0, 0, 0)[\cdot]^2) \\ &= \text{span} \left\{ y(t) \in Y : y(t) = \frac{1}{\pi} \sin t \int_0^{2\pi} \bar{F}''(0, 0, 0)[z(\tau)]^2 \sin \tau \, d\tau, z(\tau) \in C^2[0, 2\pi] \right\} \\ &\subseteq Z_2. \end{aligned}$$

Let us evaluate the second derivative of the mapping  $\bar{F}$

$$\bar{F}''(x, \mu, \alpha) = ((6\mu x, 3x^2, 0), (3x^2, 0, 0), (0, 0, -12\alpha^2 \sin t)).$$

From this we obtain

$$\bar{F}''(0, 0, 0) = 0$$

and  $Y_2 = \{0\}$ ,  $Z_3 = (Y_1 \oplus \{0\})^\perp$ . We continue

$$\begin{aligned} P_{Z_3}y &= \frac{1}{\pi} \sin t \langle y, \sin t \rangle = \frac{1}{\pi} \sin t \int_0^{2\pi} y(\tau) \sin \tau \, d\tau, \quad y \in Y, \\ Y_3 &= \text{span}(\text{Im } P_{Z_3} \bar{F}'''(0, 0, 0)[\cdot]^3) \\ &= \text{span} \left\{ y(t) \in Y : y(t) = \frac{1}{\pi} \sin t \int_0^{2\pi} \bar{F}'''(0, 0, 0)[z(\tau)]^3 \sin \tau \, d\tau, z(\tau) \in C^2[0, 2\pi] \right\} \\ &\subseteq Z_3. \end{aligned}$$

Let us evaluate the third derivative of the mapping  $\bar{F}$ :

$$\begin{aligned} \bar{F}'''(x, \mu, \alpha) &= (((6\mu, 6x, 0)(6x, 0, 0), (0, 0, 0)), ((6x, 0, 0), (0, 0, 0), (0, 0, 0)), \\ &\quad ((0, 0, 0), (0, 0, 0), (0, 0, -24\alpha \sin t))). \end{aligned}$$

From this

$$\bar{F}'''(0, 0, 0) = 0$$

and  $Y_3 = \{0\}$  i  $Y = Y_1 \oplus \{0\} \oplus \{0\} \oplus Z_4$ ,  $Z_4 = (Y_1 \oplus \{0\} \oplus \{0\})^\perp$ . Next we have

$$P_{Z_4}y = \frac{1}{\pi} \sin t \langle y, \sin t \rangle = \frac{1}{\pi} \sin t \int_0^{2\pi} y(\tau) \sin \tau \, d\tau, \quad y \in Y$$

and

$$\begin{aligned}
 Y_4 &= \text{span}(\text{Im } P_{Z_4} \bar{F}^{(4)}(0, 0, 0)[\cdot]^4) \\
 &= \text{span}\left\{y(t) \in Y : y(t) = \frac{1}{\pi} \sin t \int_0^{2\pi} \bar{F}^{(4)}(0, 0, 0)[z(\tau)]^4 \sin \tau \, d\tau, z(\tau) \in C^2[0, 2\pi]\right\}.
 \end{aligned}$$

Let us evaluate the fourth derivative:

$$\begin{aligned}
 \bar{F}^{(4)}(x, \mu, \alpha) &= (((((0, 6, 0), (6, 0, 0), (0, 0, 0)), ((6, 0, 0), (0, 0, 0), (0, 0, 0)), \\
 &\quad ((0, 0, 0), (0, 0, 0), (0, 0, 0))), (((6, 0, 0), (0, 0, 0), (0, 0, 0)), \\
 &\quad ((0, 0, 0), (0, 0, 0), (0, 0, 0)), ((0, 0, 0), (0, 0, 0), (0, 0, 0))), \\
 &\quad (((0, 0, 0), (0, 0, 0), (0, 0, 0)), ((0, 0, 0), (0, 0, 0), (0, 0, 0)), \\
 &\quad ((0, 0, 0), (0, 0, 0), (0, 0, -24 \sin t)))) \\
 &= \bar{F}^{(4)}(0, 0, 0).
 \end{aligned}$$

Note that  $\bar{F}^{(5)}(x, \mu, \alpha) = 0$ ,  $Z_4 = Y_4$  and  $P_{Z_4}y = \Pi_{Y_4}y$ . Therefore we will examine 4-regularity.

Now let us take  $z(\tau) = (h_x, h_\mu, h_\alpha)$ . For such a defined vector  $z(\tau)$ , the following relation holds:

$$\bar{F}^{(4)}(0, 0, 0)[z(\tau)]^3 = (18h_x^2 h_\mu, 6h_x^3, -24 \sin \tau h_\alpha^3), \tag{18}$$

$$\bar{F}^{(4)}(0, 0, 0)[z(\tau)]^4 = 24(h_x^3 h_\mu - \sin \tau h_\alpha^4), \tag{19}$$

and we can describe the subspace

$$\begin{aligned}
 Y_4 &= \text{span}\left\{y(t) \in Y : y(t) = \frac{24}{\pi} \sin t \int_0^{2\pi} (h_x^3 h_\mu - \sin \tau h_\alpha^4) \sin \tau \, d\tau\right\} \\
 &= \text{span}\{\sin t\} = \text{Ker } \bar{F}'_x(0, 0, 0),
 \end{aligned}$$

4-factor operator

$$\begin{aligned}
 \forall u &= (h_u, h_\lambda, h_\beta) \in C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \\
 \Psi_4(h)[u] &= \Psi_4((0, 0, 0), (h_x, h_\mu, h_\alpha))(h_u, h_\lambda, h_\beta) \\
 &= \frac{d^2 h_u}{dt^2} + h_u + \frac{1}{\pi} \sin t \int_0^{2\pi} (18h_x^2 h_\mu h_u + 6h_x^3 h_\lambda - 24 \sin \tau h_\alpha^3 h_\beta) \sin \tau \, d\tau
 \end{aligned}$$

and the 4-kernel of 4-factor operator  $\Psi_4(h)$

$$\begin{aligned}
 \text{Ker}^4 \Psi_4(h) &= \left\{h = (h_x, h_\mu, h_\alpha) \in C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R} : \right. \\
 &\quad \left. \frac{d^2 h_x}{dt^2} + h_x + \frac{24}{\pi} \sin t \int_0^{2\pi} (h_x^3 h_\mu - \sin \tau h_\alpha^4) \sin \tau \, d\tau = 0\right\}.
 \end{aligned}$$

Taking into account the equations

$$\int_0^{2\pi} \sin^4 \tau \, d\tau = \frac{3}{4}\pi, \quad \int_0^{2\pi} \sin^2 \tau \, d\tau = \pi$$

and the fact that  $h_x = z \sin \tau$ , since  $h_x \in \text{Ker } \overline{F}'_x(0, 0, 0)$ , we solve the following equation of unknowns  $z, h_\mu, h_\alpha$ :

$$\frac{24}{\pi} \sin t \left[ z^3 h_\mu \int_0^{2\pi} \sin^4 \tau \, d\tau - h_\alpha^4 \int_0^{2\pi} \sin^2 \tau \, d\tau \right] = 0. \tag{20}$$

From here  $\frac{3}{4}z^3 h_\mu - h_\alpha^4 = 0$  and  $z = \sqrt[3]{\frac{4h_\alpha^4}{3h_\mu}}$  for  $h_\mu \neq 0$ . We assume here that  $\|h\| = \|(h_x, h_\mu, h_\alpha)\| = 1$  (the vector  $h$  lies on a ball of radius 1), where we define the norm  $\|\cdot\|$  in the space  $C^2[0, 2\pi]$  as follows:

$$\forall x \in C^2[0, 2\pi] \quad \|x\| = \sup_{t \in [0, 2\pi]} |x(t)| + \sup_{t \in [0, 2\pi]} |x'(t)| + \sup_{t \in [0, 2\pi]} |x''(t)|, \tag{21}$$

while the norm in the space  $\mathbb{R} \times \mathbb{R}$  as  $\|(y, z)\| = |y| + |z|$ . We can accept the norm in the space  $C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R}$  as standard, *i.e.*,

$$\|(x, y, z)\| = \|x\|_{C^2[0, 2\pi]} + \|(y, z)\|_{\mathbb{R} \times \mathbb{R}}. \tag{22}$$

Then we describe the space  $\text{Ker}^4 \Psi_4(h)$  clearly as follows:

$$\text{Ker}^4 \Psi_4(h) = \left\{ \left( \sqrt[3]{\frac{4\alpha^4}{3\mu}} \sin t, \mu, \alpha \right) \right\} \cup \{(c \sin t, 0, 0)\} \cup \{(0, \mu, 0)\}, \|(h_x, \mu, \alpha)\| = 1,$$

where  $h_x$  equals  $\sqrt[3]{\frac{4\alpha^4}{3\mu}} \sin t, c \sin t$  for  $c \in \mathbb{R}$  or 0.

At the end we will examine the surjectivity of 4-factor operator on the 4-kernel. Note that vectors  $(c \sin t, 0, 0)$  and  $(0, \mu, 0)$  are solutions of the Duffing equation, then we are interested in vectors of the form  $H = (\sqrt[3]{\frac{4\alpha^4}{3\mu}} \sin t, \mu, \alpha)$ , where  $3\sqrt[3]{\frac{4\alpha^4}{3\mu}} + |\mu| + |\alpha| = 1$ . So we will verify whether

$$\forall y \in C[0, 2\pi] \exists u = [h_u, h_\lambda, h_\beta] \in C^2[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \quad \Psi_4(H)[u] = y.$$

We have

$$\begin{aligned} \Psi_4(H)[u] &= \frac{d^2 h_u}{dt^2} + h_u + \frac{1}{\pi} \sin t \int_0^{2\pi} \left( 18 \left( \sqrt[3]{\frac{4\alpha^4}{3\mu}} \right)^2 \sin^2 \tau \mu h_u \right. \\ &\quad \left. + 8 \frac{\alpha^4}{\mu} \sin^3 \tau h_\lambda - 24 \sin \tau \alpha^3 h_\beta \right) \sin \tau \, d\tau \\ &= y. \end{aligned}$$

Using now Lemma 10, p.6 from [8], *i.e.*, putting  $y = y_4 = a \sin t \in Y_4$ , we find the element  $h_u = b \sin t \in \text{Ker } \overline{F}'_x(0, 0, 0)$ . Namely, we obtain

$$\frac{1}{\pi} \sin t \int_0^{2\pi} \left( 18 \left( \sqrt[3]{\frac{4\alpha^4}{3\mu}} \right)^2 \mu b \sin^4 \tau + 8 \frac{\alpha^4}{\mu} \sin^4 \tau h_\lambda - 24 \alpha^3 h_\beta \sin^2 \tau \right) d\tau = a \sin t.$$

This implies  $b = \frac{1}{9\sqrt[3]{6\alpha^8\mu}}(a + 24\alpha^3h_\beta - \frac{6\alpha^4}{\mu}h_\lambda)$  and

$$h_u = \frac{1}{9\sqrt[3]{6\alpha^8\mu}} \left( a + 24\alpha^3h_\beta - \frac{6\alpha^4}{\mu}h_\lambda \right) \sin t.$$

The solutions of the above equation exist, hence 4-factor operator is a surjection on the element  $H$ .

Summing up, we conclude from the generalized Lyusternic theorem: for  $x^* = (0, 0, 0)$ , the tangent cone  $T_{x^*}M$  to the solutions set  $M$  coincides with the kernel of the fourth derivative of the mapping  $\Psi_4$ , i.e., with the set  $H_4(x^*) = \text{Ker}^4 \Psi_4(H)$ . Thus there exist nontrivial solutions of the Duffing equation (1), and we can write them in the form  $x(\varepsilon, t) = x^* + \varepsilon H + r(\varepsilon)$ , with  $\|r(\varepsilon)\| = o(\varepsilon)$ , for  $\varepsilon \in (0, \delta)$ , where  $\delta > 0$  is sufficiently small. Let  $\bar{\beta}, \bar{\mu}$  be fixed numbers such that the equation  $3\sqrt[3]{\frac{4\bar{\beta}}{3\bar{\mu}}} + |\bar{\mu}| + |\bar{\beta}^{\frac{1}{4}}| = 1$  holds, where  $\bar{\mu} = \bar{\mu}(\bar{\beta}^{\frac{1}{4}})$ ,  $\bar{\beta} = \bar{\alpha}^4$  and  $0 < \bar{\beta}^{\frac{1}{4}} < 1$ . Putting  $x^* = (0, 0, 0)$ , and  $\varepsilon = \frac{\beta^{\frac{1}{4}}}{\bar{\beta}^{\frac{1}{4}}}$ , we get formulas (14), (15), and this finishes the proof of the theorem.  $\square$

#### 4 Conclusion

In this paper we used  $p$ -regularity theory to prove the existence of solutions of the nonlinear Duffing equation and gave an approximate description of the solutions set. Compared with the latter paper by Grzegorzczuk et al. [1], where only 2-regularity was studied, the results of the present paper allow one to study the regularity of higher order (4-regularity) and apply it for seeking approximative solutions of nonlinear boundary value problems. We showed that the method of  $p$ -regularity may have an effective application to the study of solutions of advanced differential equations. This construction may be generalized for investigation of the case  $p > 4$ .

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The authors declare that they have no competing interests.

#### Authors' contributions

All authors read and approved the final manuscript.

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