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Multiple positive solutions to some second-order integral boundary value problems with singularity on space variable

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Abstract

This article deals with integral boundary value problems of the second-order differential equations

$$\begin{cases} u''(t)+a(t)u'(t)+b(t)u(t)+f(t,u(t))=0, & t\in J_+,\\ u(0)=\int_0^1 g(s)u(s)\,ds, & u(1)=\int_0^1 h(s)u(s)\,ds, \end{cases}$$

where $a\in C(J)$, $b\in C(J,R_-)$, $f\in C(J_+\times R_+,R^+)$ and $g,h\in L^1(J)$ are nonnegative. The result of the existence of two positive solutions is established by virtue of fixed point index theory on cones. Especially, the nonlinearity f permits the singularity on the space variable.

MSC: 34B15; 34B18

Keywords: integral boundary value; cone; two positive solutions; singularity

1 Introduction

The aim of this article is to study the existence of two positive solutions to the following nonlinear second-order differential equation involving integral boundary value conditions:

$$\begin{cases} u''(t)+a(t)u'(t)+b(t)u(t)+f(t,u(t))=0, & t\in J_+,\\ u(0)=\int_0^1 g(s)u(s)\,ds, & u(1)=\int_0^1 h(s)u(s)\,ds, \end{cases} \tag{1}$$

where $a\in C(J)$, $b\in C(J,R_-)$, $f\in C(J_+\times R_+,R^+)$ and $g,h\in L^1(J)$ are nonnegative, $J=[0,1]$, $J_+=(0,1)$, $R^+=[0,+\infty)$, $R_+=(0,+\infty)$, $R_-=(-\infty,0)$. Singularities of the nonlinearity f are related to both $t=0,1$ and $u=0$.

Recently, there has been a considerable increase in the investigation of nonlocal boundary value problems; see [1–11] for integer order and [12–22] for fractional differential equations. Based on a specially constructed cone, the existence as well as nonexistence results on positive solutions for the following second-order integral BVPs are obtained in an abstract space in Feng *et al.* [1]:

$$\begin{cases} u''(t) + f(t, u(t)) = \theta, \quad t \in J_+, \\ u(0) = \int_0^1 g(t)u(t)\,\mathrm{d}t, \qquad u(1) = \theta, \\ \quad \text{or} \quad u(0) = 0, \qquad u(1) = \int_0^1 g(t)u(t)\,\mathrm{d}t, \end{cases}$$

here $f \in C([0,1] \times P, P), \theta$ represents the zero element of E, and g is nonnegative and integrable. Also, in an abstract space, by the fixed point theorem of strict set contraction operators, Zhang *et al.* [2] obtained the existence results of solutions for some second-order integral boundary value problems with the impulsive effect.

For general differential operator, when the nonlinearity f is continuous, Feng and Ge [3] studied multiple positive solutions for the following singular m-point boundary value problems:

$$\begin{cases} Lu = \lambda w(t) f(t, u), \quad t \in J_+, \\ u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \end{cases}$$

here $\lambda > 0$, $\xi_i \in (0,1)$, $\alpha_i \in R_+$ $(i = 1, 2, \ldots, m-2)$ are known constants and L represents the linear operator

$$Lu := -u'' - au' + bu,$$

where $a \in C(J)$ and $b \in C(J, R_+), f \in C(J \times R^+, R^+)$, $w \in C(J_+, R_+)$. As is well known, two, three and multi-point BVPs may be looked upon as a special case of integral boundary value problems. For integral conditions, with some so-called first eigenvalue of the related linear operator, Liu *et al.* [4] formulated the existence results for BVP (1). The whole discussion relied on the fixed point index theorems. With the impulsive effect, under different combinations of super-linear and sub-linear condition on nonlinear term and the impulses, some results of existence of multiple positive solutions as well as nonexistence results for BVP (1) are obtained in Hao *et al.* [5].

We attempt in this article to study the existence of two positive solutions of BVP (1). The interesting points focus on two aspects. First, singularities of the nonlinearity f are related not only to the time but also to the space variables. Second, compared with [4], the method and conditions used to get result of multiple positive solutions are quite different from that used in [4]. The integral of the nonlinearity on some special bounded set is considered in this paper. The tools used to obtain the main result are fixed point index theorems on cones. Obviously, the result obtained in this paper can be analogously given for the Riemann-Stieltjes integral case after some minor modifications.

2 Preliminaries and several lemmas

Let ψ_1 and ψ_2 be the unique solution of the BVP

$$\begin{cases} \psi_1''(t) + a(t)\psi_1'(t) + b(t)\psi_1(t) = 0, \\ \psi_1(0) = 0, \qquad \psi_1(1) = 1, \end{cases}$$

and

$$\begin{cases} \psi_2''(t) + a(t)\psi_2'(t) + b(t)\psi_2(t) = 0, \\ \psi_2(0) = 1, \qquad \psi_2(1) = 0, \end{cases}$$

respectively. By [8], we know that ψ_1, ψ_2 are strictly increasing and strictly decreasing on J, respectively.

We adopt the following assumptions throughout this article.

(H_1) $a \in C(J)$, $b \in C(J, R_-)$;

(H_2) $g, h : J_+ \to R^+$ are integrable, and $k_1 > 0$, $k_4 > 0$, $k > 0$, where

$$k_1 = 1 - \int_0^1 \psi_2(s)g(s)\,\mathrm{d}s, \qquad k_2 = \int_0^1 \psi_1(s)g(s)\,\mathrm{d}s,$$
$$k_3 = \int_0^1 \psi_2(s)h(s)\,\mathrm{d}s, \qquad k_4 = 1 - \int_0^1 \psi_1(s)h(s)\,\mathrm{d}s,$$
$$k = k_1k_4 - k_2k_3;$$

(H_3) $f \in C(J_+ \times R_+, R^+)$;

(H_4) there exist three functions $\widehat{a}, \widehat{b} \in C(J_+, R^+)$, $\widehat{g} \in C(R_+, R^+)$ satisfying

$$f(t,u) \le \widehat{a}(t)\widehat{g}(u) + \widehat{b}(t), \quad \forall t \in J_+, u \in R_+,$$

where, in addition,

$$\widehat{a}_r^* = \int_0^1 \mathcal{H}(t)\widehat{a}(t)\widehat{g}_r(t)\,\mathrm{d}t < +\infty, \qquad \widehat{b}^* = \int_0^1 \mathcal{H}(t)\widehat{b}(t)\,\mathrm{d}t < +\infty,$$

and

$$\widehat{g}_r(t) = \max\bigl\{\widehat{g}(u) : \gamma(t)r \le u \le r\bigr\}, \quad \forall r > 0,$$

here, $\gamma(t)$ is defined in (6), $\mathcal{H}(t)$ is defined in (7);

(H_5) there exists a function $\widehat{c} \in C(J_+, R^+)$ satisfying

$$\frac{f(t,u)}{\widehat{c}(t)u} \to +\infty \quad \text{as } u \to +\infty$$

uniformly for $t \in J_+$, and in addition,

$$\widehat{c}^* = \int_0^1 \widehat{c}(t)\,\mathrm{d}t < +\infty;$$

(H_6) there exists a function $\widehat{d} \in C(J_+, R^+)$ satisfying

$$\frac{f(t,u)}{\widehat{d}(t)} \to +\infty \quad \text{as } u \to 0^+$$

uniformly for $t \in J_+$, and in addition,

$$\widehat{d}^* = \int_0^1 \widehat{d}(t)\,\mathrm{d}t < +\infty.$$

Lemma 1 ([4]) *Assume that* (H_1) *and* (H_2) *hold. Then, for any* $y \in C(J_+) \cap L^1(J)$, *the BVP*

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + y(t) = 0, & t \in J_+, \\ u(0) = \int_0^1 g(s)u(s)\,\mathrm{d}s, & u(1) = \int_0^1 h(s)u(s)\,\mathrm{d}s, \end{cases} \tag{2}$$

has a unique solution u that can be expressed in the form

$$u(t) = \int_0^1 H(t,s)y(s)\,\mathrm{d}s, \quad t \in J,$$

where

$$\begin{aligned} H(t,s) &= G(t,s)p(s) + \frac{\psi_1(t)k_3 + \psi_2(t)k_4}{k} \int_0^1 G(\tau,s)p(s)g(\tau)\,\mathrm{d}\tau \\ &\quad + \frac{\psi_1(t)k_1 + \psi_2(t)k_2}{k} \int_0^1 G(\tau,s)p(s)h(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3}$$

$$p(t) = \exp\left(\int_0^t a(s)\,\mathrm{d}s\right),$$

$$G(t,s) = \frac{1}{\rho} \begin{cases} \psi_1(t)\psi_2(s), & 0 \le t \le s \le 1, \\ \psi_1(s)\psi_2(t), & 0 \le s \le t \le 1, \end{cases} \qquad \rho = \psi_1'(0). \tag{4}$$

Moreover, $u(t) \ge 0$ *on J provided* $y \ge 0$.

By Remark 2.1 in [4], we have

Lemma 2 [4] *Suppose that* (H_1) *and* (H_2) *hold, then, for any* $t, s \in J$, *we have*

$$0 \le G(t,s) \le G(s,s), \qquad 0 \le H(t,s) \le \mathcal{H}(s), \tag{5}$$

$$H(t,s) \ge \gamma(t)\mathcal{H}(s), \tag{6}$$

where $\gamma(t) = \min\{\psi_1(t), \psi_2(t)\}$, $t \in J$ *and*

$$\mathcal{H}(s) = G(s,s)p(s) + \frac{k_3 + k_4}{k} \int_0^1 G(\tau,s)p(s)g(\tau)\,\mathrm{d}\tau + \frac{k_1 + k_2}{k} \int_0^1 G(\tau,s)p(s)h(\tau)\,\mathrm{d}\tau. \tag{7}$$

Let $E = C(J)$ be the standard Banach space with the maximum norm and P be the typical cone of nonnegative continuous functions in the form

$$P = \{u \in E : u(t) \ge \gamma(t)\|u\|, t \in J\}.$$

Let $P_{mn} = \{u \in P, m \le \|u\| \le n\}$, $P_r = \{u \in P : \|u\| \le r\}$ for $n > m > 0$, $r > 0$.

First, we give an operator $T : P \setminus \{0\} \to C(J)$ as follows:

$$(Tu)(t) = \int_0^1 H(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \quad t \in J. \tag{8}$$

Lemma 3 *If conditions* (H_1) *and* (H_2) *are satisfied, then, for any* $n > m > 0$, $T : P_{mn} \to P$ *is a completely continuous operator.*

Proof For any given $u \in P_{mn}$, we have $m \le \|u\| \le n$. From the construction of P, we have

$$\gamma(t)m \le u(t) \le n, \quad \forall t \in J. \tag{9}$$

Clearly, for any $n > m > 0$, condition (H_4) means that

$$\widehat{a}^*_{mn} = \int_0^1 \mathcal{H}(t)\widehat{a}(t)\widehat{g}_{mn}(t)\,\mathrm{d}t < +\infty, \tag{10}$$

where

$$\widehat{g}_{mn}(t) = \max\bigl\{\widehat{g}(u) : \gamma(t)m \le u \le n\bigr\}. \tag{11}$$

It follows from (8), (H_3), (H_4) and Lemma 2 that

$$f\bigl(t,u(t)\bigr) \le \widehat{a}(t)\widehat{g}_{mn}(t) + \widehat{b}(t), \quad \forall t \in J_+, u \in P_{mn}, \tag{12}$$

and

$$\begin{aligned}(Tu)(t) &= \int_0^1 H(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s \\ &\le \int_0^1 \mathcal{H}(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s \\ &\le \int_0^1 \mathcal{H}(s)\bigl[\widehat{a}(s)\widehat{g}_{mn}(s) + \widehat{b}(s)\bigr]\,\mathrm{d}s \\ &= \widehat{a}^*_{mn} + \widehat{b}^*, \quad \forall t \in J,\end{aligned} \tag{13}$$

which shows that T makes sense. According to Lemma 2, we have for any $t \in J$

$$\begin{aligned}(Tu)(t) &= \int_0^1 H(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s \\ &\le \int_0^1 \mathcal{H}(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s.\end{aligned}$$

Hence,

$$\|Tu\| \le \int_0^1 \mathcal{H}(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s. \tag{14}$$

At the same time, by Lemma 2 and (14), we get

$$\begin{aligned}(Tu)(t) &= \int_0^1 H(t,s)f\big(s,u(s)\big)\,\mathrm{d}s \\ &\geq \gamma(t)\int_0^1 \mathcal{H}(s)f\big(s,u(s)\big)\,\mathrm{d}s \\ &\geq \gamma(t)\|Tu\|, \quad \forall t\in J. \end{aligned} \tag{15}$$

This indicates that T maps P_{mn} into P.

Next, we shall show the complete continuity of the operator T. Let $u_n, \bar{u} \in P_{mn}$, with $\|u_n - \bar{u}\| \to 0$ ($n \to \infty$); then $\lim_{n\to\infty} u_n(t) = \bar{u}(t)$, $t \in J$. Let

$$\begin{aligned}(T_1u)(t) &= f\big(t,u(t)\big), \quad \text{for any } t\in J_+, u\in P_{mn}, \\ (T_2u)(t) &= \int_0^1 H(t,s)u(s)\,\mathrm{d}s, \quad \text{for any } t\in J_+, u\in L^1(J).\end{aligned}$$

By (H_1),

$$f\big(t,u_n(t)\big) \to f\big(t,\bar{u}(t)\big) \quad (n\to+\infty), \text{for any } t\in J_+. \tag{16}$$

Similar to (12), for $u_n, \bar{u} \in P_{mn}$, one has

$$f\big(t,u_n(t)\big) \leq \widehat{a}(t)\widehat{g}_{mn}(t) + \widehat{b}(t), \qquad f\big(t,\bar{u}(t)\big) \leq \widehat{a}(t)\widehat{g}_{mn}(t) + \widehat{b}(t), \quad \forall t\in J_+.$$

Then one gets

$$\big|f\big(t,u_n(t)\big) - f\big(t,\bar{u}(t)\big)\big| \leq 2\widehat{a}(t)\widehat{g}_{mn}(t) + 2\widehat{b}(t) \triangleq \sigma(t) \in L^1(J). \tag{17}$$

The Lebesgue dominated convergence theorem together with (16) and (17) generates

$$\lim_{n\to\infty}\int_0^1 \big|(T_1u_n)(t) - \big(T_1\bar{u}(t)\big)\big|\,\mathrm{d}t = 0.$$

That is to say $T_1 : P_{mn} \to L^1(J)$ is continuous. Furthermore, the complete continuity of the operator $T_2 : L^1(J) \to C(J)$ can easily be verified by the Arzela-Ascoli theorem and a standard discussion. Hence, by the property of compound operators we see that $T = T_2 \circ T_1 : P_{mn} \to C(J)$ is completely continuous. □

Lemma 4 ([23]) *Let E be a Banach space, $P \subset E$ a cone in E. For $r > 0$, define $P_r = \{u \in P : \|u\| \leq r\}$. Assume that $T : P_r \to P$ is a compact map such that $Tu \neq u$ for $u \in \partial P_r = \{u \in P : \|u\| = r\}$.*

(i) *If $\|u\| \leq \|Tu\|$, $\forall u \in \partial P_r$, then*

$$i(T, P_r, P) = 0.$$

(ii) *If $\|u\| \geq \|Tu\|$, $\forall u \in \partial P_r$, then*

$$i(T, P_r, P) = 1.$$

3 Main results

Theorem 1 *Let conditions* (H$_1$)-(H$_6$) *be satisfied. Furthermore, there exists a constant* $\widehat{r} > 0$ *satisfying*

$$\widehat{a}^*_{\widehat{r}} + \widehat{b}^* < \widehat{r}, \tag{18}$$

here, $\widehat{a}^*_{\widehat{r}}$ *and* $\widehat{b}^*$ *are given in* (H$_4$). *Then the BVP* (1) *admits at least two positive solutions* x^* *and* x^{**} *such that* $0 < \|x^*\| < \widehat{r} < \|x^{**}\|$.

Proof From Lemma 3, we know that for any $n > m > 0$, the operator T maps P_{mn} into P and is completely continuous. Next, we are in a position to show that T has two distinct positive fixed points x^*, x^{**} such that $0 < \|x^*\| < \widehat{r} < \|x^{**}\|$.

From (H$_5$) we know there exists a constant $r_1 > 0$ satisfying

$$f(t,u) > \left(\gamma_{\frac{1}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{c}(s)\,\mathrm{d}s\right)^{-1} \widehat{c}(t)u, \quad \forall t \in J_+, u \ge r_1. \tag{19}$$

Let $0 < \gamma_{\frac{1}{4}} = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\{\psi_1(t), \psi_2(t)\} < 1$. Choose

$$r_2 > \max\left\{\frac{r_1}{\gamma_{\frac{1}{4}}}, \widehat{r}\right\}. \tag{20}$$

For $u \in \partial P_{r_2}$, considering the definition of cone P, we have

$$u(t) \ge \gamma(t)r_2 \ge \gamma_{\frac{1}{4}} r_2 > r_1, \quad \forall t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{21}$$

So, we get from (19), (20), (21) that

$$\begin{aligned} (Tu)\left(\frac{1}{2}\right) &= \int_0^1 H\left(\frac{1}{2}, s\right) f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &> \left(\gamma_{\frac{1}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{c}(s)\,\mathrm{d}s\right)^{-1} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{c}(s)u(s)\,\mathrm{d}s \\ &\ge \left(\gamma_{\frac{1}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{c}(s)\,\mathrm{d}s\right)^{-1} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{c}(s)\,\mathrm{d}s \cdot \gamma_{\frac{1}{4}} r_2 = r_2, \end{aligned} \tag{22}$$

i.e., $\|Tu\| > \|u\|$, $u \in \partial P_{r_2}$. Therefore, by Lemma 4,

$$i(T, P_{r_2}, P) = 0. \tag{23}$$

By condition (H$_6$), there exists a constant $r_3 > 0$ satisfying

$$f(t,u) > \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{d}(s)\,\mathrm{d}s\right)^{-1} \widehat{d}(t)\widehat{r}, \quad \forall t \in J_+, 0 < u < r_3. \tag{24}$$

Choose

$$0 < r_4 < \min\{r_3, \widehat{r}\}. \tag{25}$$

For $u \in \partial P_{r_4}$, we have

$$r_3 > r_4 = \|u\| \geq \gamma_{\frac{1}{4}} r_4 > 0, \quad \forall t \in J_+.$$

So, we get from (24) and (25) that

$$\begin{aligned}(Tu)\left(\frac{1}{2}\right) &= \int_0^1 H\left(\frac{1}{2}, s\right) f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &> \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{d}(s)\,\mathrm{d}s\right)^{-1} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{d}(s)\widehat{r}\,\mathrm{d}s \\ &\geq \left(\int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{d}(s)\,\mathrm{d}s\right)^{-1} \int_{\frac{1}{4}}^{\frac{3}{4}} H\left(\frac{1}{2}, s\right)\widehat{d}(s)\,\mathrm{d}s \cdot \widehat{r} = \widehat{r} > r_4, \end{aligned} \tag{26}$$

i.e., $\|Tu\| > \|u\|$, $u \in \partial P_{r_4}$. Therefore, by Lemma 4,

$$i(T, P_{r_4}, P) = 0. \tag{27}$$

In a similar manner, for $u \in \partial P_{\widehat{r}}$, by ($H_4$), Lemma 2 and (14), we get

$$\begin{aligned}(Tu)(t) &= \int_0^1 H(t,s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &\leq \int_0^1 \mathcal{H}(s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &\leq \int_0^1 \mathcal{H}(s)\bigl[\widehat{a}(s)\widehat{g}_{\widehat{r}}(s) + \widehat{b}(s)\bigr]\,\mathrm{d}s \\ &\leq \widehat{a}_{\widehat{r}}^* + \widehat{b}^* < \widehat{r}, \quad \forall t \in J, \end{aligned} \tag{28}$$

i.e., $\|Tu\| < \|u\|$, $u \in \partial P_{\widehat{r}}$. Therefore, by Lemma 4, we get

$$i(T, P_{\widehat{r}}, P) = 1. \tag{29}$$

Now, the additivity of the fixed point index together with (23), (27), (29) implies that

$$i(T, P_{r_2} \setminus \mathring{P}_{\widehat{r}}, P) = -1$$

and

$$i(T, P_{\widehat{r}} \setminus \mathring{P}_{r_4}, P) = 1.$$

Hence, T has two fixed points x^* and x^{**} which belong to $P_{\widehat{r}} \setminus \mathring{P}_{r_4}$ and $P_{r_2} \setminus \mathring{P}_{\widehat{r}}$, respectively, such that $0 < r_4 < \|x^*\| < \widehat{r} < \|x^{**}\| \leq r_2$. □

4 An example

Example 1 Consider the following second-order singular integral BVPs:

$$\begin{cases} u''(t) + u'(t) - 2u(t) + \frac{1}{560\sqrt[8]{(1-t)}}\left(u^2 + \frac{1}{3\sqrt[4]{u}}\right) + \frac{1}{420\sqrt[3]{t(1-t)}} = 0, & 0 < t < 1, \\ u(0) = \int_0^1 su(s)\,\mathrm{d}s, \qquad u(1) = \int_0^1 s^2 u(s)\,\mathrm{d}s. \end{cases} \tag{30}$$

Conclusion BVP (30) admits at least two positive solutions x^* and x^{**} satisfying $0 < \|x^*\| < 1 < \|x^{**}\|$.

Proof Clearly, BVP (30) has the form (1), in which $a(t) \equiv 1$, $b(t) \equiv -2$, $f(t,u) = \frac{1}{560\sqrt[8]{(1-t)}} \times (u^2 + \frac{1}{3\sqrt[4]{u}}) + \frac{1}{420\sqrt[3]{t(1-t)}}$, $g(s) = s$, $h(s) = s^2$. Obviously, $f(t,u)$ permits singularities at $t = 0,1$ and $u = 0$.

Let ψ_1 and ψ_2 satisfy

$$\begin{cases} \psi_1''(t) + \psi_1'(t) - 2\psi_1(t) = 0, \\ \psi_1(0) = 0, \qquad \psi_1(1) = 1, \end{cases} \qquad \begin{cases} \psi_2''(t) + \psi_2'(t) - 2\psi_2(t) = 0, \\ \psi_2(0) = 1, \qquad \psi_2(1) = 0. \end{cases}$$

By a simple computation, we get

$$\begin{aligned}
&\psi_1(t) = \frac{1}{e - e^{-2}}\left(e^t - e^{-2t}\right), \qquad \psi_2(t) = \frac{1}{e - e^{-2}}\left(e^{1-2t} - e^{t-2}\right), \\
&\rho = \frac{3}{e - e^{-2}}, \qquad p(t) = e^t, \\
&k_1 = 1 - \frac{-\frac{3}{4}e^{-1} + \frac{1}{4}e - e^{-2}}{e - e^{-2}} = 0.8961, \qquad k_2 = \frac{\frac{3}{4}e^{-2} + \frac{3}{4}}{e - e^{-2}} = 0.3297, \\
&k_3 = \frac{-\frac{9}{4}e^{-1} + \frac{1}{4}e + 2e^{-2}}{e - e^{-2}} = 0.0474, \qquad k_4 = 1 - \frac{e + \frac{5}{4}e^{-2} - \frac{9}{4}}{e - e^{-2}} = 0.7532, \\
&k = k_1k_4 - k_2k_3 = 0.6593 > 0, \\
&G(t,s) = \frac{1}{3(e - e^{-2})} \begin{cases} (e^t - e^{-2t})(e^{1-2s} - e^{s-2}), & 0 \le t \le s \le 1, \\ (e^s - e^{-2s})(e^{1-2t} - e^{t-2}), & 0 \le s \le t \le 1. \end{cases}
\end{aligned}$$

It is not difficult to see that $0 < G(t,s) < 2s$ and

$$\mathcal{H}(s) = G(s,s)p(s) + \frac{k_3 + k_4}{k}\int_0^1 G(\tau,s)p(s)g(\tau)\,\mathrm{d}\tau + \frac{k_1 + k_2}{k}\int_0^1 G(\tau,s)p(s)h(\tau)\,\mathrm{d}\tau < 28s.$$

For any given $r > 0$, we can see that (H_4) is valid for $\widehat{a}(t) = \frac{1}{560\sqrt[8]{(1-t)}}$, $g(u) = u^2 + \frac{1}{3\sqrt[4]{u}}$, $b(t) = \frac{1}{420\sqrt[3]{t(1-t)}}$, and it follows from $0 \le \frac{3}{2(e-e^{-2})^2}t(1-t)^2 \le \gamma(t) \le 1$ that

$$\begin{aligned}
\widehat{a}_r^* &= \int_0^1 \mathcal{H}(t)\widehat{a}(t)\widehat{g}_r(t)\,\mathrm{d}t \\
&< \int_0^1 28t\frac{1}{560\sqrt[8]{(1-t)}}\left(r^2 + \frac{e - e^{-2}}{3\sqrt[4]{\frac{3}{2(e-e^{-2})^2}t(1-t)^2 r}}\right)\mathrm{d}t < +\infty,
\end{aligned} \tag{31}$$

$\widehat{b}^* = \int_0^1 \mathcal{H}(t)\widehat{b}(t)\,\mathrm{d}t < 0.0684$. Clearly, (H_5) and (H_6) hold for $\widehat{c}(t) = \widehat{d}(t) = \frac{1}{560\sqrt[8]{(1-t)}}$, and $c^* = d^* = 0.0020$. Take $\widehat{r} = 1$; we have by (31)

$$\begin{aligned}
\widehat{a}_{\widehat{r}}^* + \widehat{b}^* &< \int_0^1 t\frac{1}{20\sqrt[8]{(1-t)}}\left(1 + \frac{e - e^{-2}}{3\sqrt[4]{\frac{3}{2(e-e^{-2})^2}t(1-t)^2}}\right)\mathrm{d}t + 0.0684 \\
&= 0.0305 + 0.0773 + 0.0684 = 0.1762 < 1 = \widehat{r}.
\end{aligned}$$

Consequently, (18) holds and our conclusion can be deduced from Theorem 1. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QZ and XZ obtained the results in an equally joint work. All the authors read and approved the final manuscript.

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