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Study on a kind of neutral Rayleigh equation with singularity

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Abstract

In this paper, we consider a kind of neutral Rayleigh equation with singularity,

$$(x(t) - cx(t-\delta))'' + f(t, x'(t)) + g(t, x(t)) = e(t),$$

where g has a singularity at $x = 0$. By applications of coincidence degree theory, we find that the existence of positive periodic solution for this equation.

MSC: 34C25; 34K13; 34K40

Keywords: periodic solution; neutral operator; Rayleigh equation; singularity

1 Introduction

More recently, some classical tools have been used to study periodic solution for Rayleigh equation in the literature, including coincidence degree theory [1–4], the method of upper and lower solutions [5], the Manásevich-Mawhin continuation theorem [6–8], and the time map continuation theorem [9–11].

From then on, the study of the existence of positive periodic solutions for Rayleigh equations with singularity has attracted many researchers' attention [12, 13]. In 2015, Wang and Ma [12] investigated the following singular Rayleigh equation:

$$x'' + f(t, x') + g(x) = p(t),$$

where g had a singularity at the origin, *i.e.*, $\lim_{x\to+\infty} g(x) = +\infty$. By applications of the limit properties of time map, the authors found that the existence of periodic solution for this equation. Afterwards, by using Manásevich-Mawhin continuation theorem, Lu, Zhong and Chen [13] discussed the existence of periodic solution for the following two kinds of p-Laplacian singular Rayleigh equations:

$$(|x'|^{p-2}x')' + f(x') - g_1^*(x) + g_2^*(x) = h(t)$$

and

$$(|x'|^{p-2}x')' + f(x') + g_1^*(x) - g_2^*(x) = h(t),$$

where $g_1, g_2 : (0, +\infty) \to \mathbb{R}$ were continuous and $g_1(x)$ was unbounded as $x \to 0^+$.

In the above papers, the authors investigated several kinds of Rayleigh equations or singular Rayleigh equations. However, the study of the neutral Rayleigh equation with singularity is relatively rare. Motivated by [12, 13], we consider the neutral Rayleigh equation with singularity

$$\bigl(x(t) - cx(t-\delta)\bigr)'' + f\bigl(t, x'(t)\bigr) + g\bigl(t, x(t)\bigr) = e(t), \tag{1.1}$$

where $|c| \neq 1$, δ is a constant, $e \in C[0,T]$ and $\int_0^T e(t)\,dt = 0$; f is continuous functions defined on $\mathbb{R}^2$ and periodic in t with $f(t,\cdot) = f(t+T,\cdot)$, and $f(t,0) = 0$; $g : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is an L^2-Carathéodory function and $g(t,\cdot) = g(t+T,\cdot)$, $g(t,x) = g_0(x) + g_1(t,x)$, here $g_1 : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, $g_1(t,\cdot) = g_1(t+T,\cdot)$; $g_0 \in C((0,\infty);\mathbb{R})$ has a strong singularity at the origin such that

$$\int_0^1 g_0(s)\,ds = -\infty. \tag{1.2}$$

By application of coincidence degree theory, we find the existence of positive periodic solutions of (1.1). Our results improve and extend the results in [12, 13].

2 Preparation

In this section, we give some lemmas, which will be used in this paper.

Lemma 2.1 (see [14]) *If* $|c| \neq 1$, *then the operator* $(Ax)(t) := x(t) - cx(t-\delta)$ *has a continuous inverse* A^{-1} *on the space*

$$C_T := \bigl\{x | x \in (\mathbb{R},\mathbb{R}), x(t+T) - x(t) \equiv 0, \forall t \in \mathbb{R}\bigr\},$$

and satisfying

$$\bigl|\bigl(A^{-1}x\bigr)(t)\bigr| \le \frac{\|x\|}{|1-|c||},$$

where $\|x\| = \max_{t\in[0,T]} |x(t)|, \forall x \in C_T$.

Lemma 2.2 (Gaines and Mawhin [1]) *Suppose that* X *and* Y *are two Banach spaces, and* $L : D(L) \subset X \to Y$ *is a Fredholm operator with index zero. Let* $\Omega \subset X$ *be an open bounded set and* $N : \overline{\Omega} \to Y$ *be L-compact on* $\overline{\Omega}$. *Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has a solution in* $\overline{\Omega} \cap D(L)$.

Set

$$X := \bigl\{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+T) - x(t) \equiv 0, \forall t \in \mathbb{R}\bigr\},$$

$$Y := \bigl\{y \in C(\mathbb{R},\mathbb{R}) : y(t+T) - y(t) \equiv 0, \forall t \in \mathbb{R}\bigr\},$$

with the norm

$$\|x\|_X = \max\{\|x\|, \|x'\|\}, \qquad \|y\|_Y = \|y\|.$$

Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \{x \in X : x'' \in C(\mathbb{R}, \mathbb{R})\} \subset X \to Y$$

by

$$(Lx)(t) = (Ax)''(t)$$

and $N : X \to Y$ by

$$(Nx)(t) = -f(t, x'(t)) - g(t, x(t)) + e(t). \tag{2.1}$$

Then (1.1) can be converted to the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}, \qquad \operatorname{Im} L = \left\{y \in Y : \int_0^T y(s)\,ds = 0\right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}$ be defined by

$$Px = (Ax)(0); \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Let K denote the inverse of $L|_{\operatorname{Ker} p \cap D(L)}$. It is easy to see that $\operatorname{Ker} L = \operatorname{Im} Q = \mathbb{R}$ and

$$[Ky](t) = \int_0^T G(t, s)y(s)\,ds,$$

where

$$G(t, s) = \begin{cases} \frac{-s(T-t)}{T}, & 0 \le s < t \le T; \\ \frac{-t(T-s)}{T}, & 0 \le t \le s \le T. \end{cases} \tag{2.2}$$

From (2.1) and (2.2), it is clear that QN and $K(I - Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I - Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\bar{\Omega}$.

3 Positive periodic solution for (1.1)

For the sake of convenience, we list the following assumptions, which will be used repeatedly in the sequel:

(H_1) there exists a positive constant K such that $|f(t,u)| \leq K$, for $(t,u) \in \mathbb{R} \times \mathbb{R}$;
(H_2) there exist positive constants α and β such that $|f(t,u)| \leq \alpha|u| + \beta$, for $(t,u) \in \mathbb{R} \times \mathbb{R}$;
(H_3) $f(t,u) \geq 0$, for $(t,u) \in \mathbb{R} \times \mathbb{R}$;
(H_4) there exists a positive constant D such that $g(t,x) > K$, for $x > D$;
(H_5) there exists a positive constant D_1 such that $g(t,x) > \|e\|$ for $x > D_1$;
(H_6) there exist positive constants m, n such that

$$g(t,x) \leq mx + n, \quad \text{for all } x > 0.$$

Now we give our main results on periodic solutions for (1.1).

Theorem 3.1 *Assume that conditions* (H_1), (H_4), (H_6) *hold. Then* (1.1) *has at least one solution with period T if $mT^2 < \pi|1 - |c||$.*

Proof By construction (1.1) has an T-periodic solution if and only if the operator equation

$$Lx = Nx$$

has an T-periodic solution. To use Lemma 2.1, we embed this operator equation into an equation family with a parameter $\lambda \in (0,1)$,

$$Lx = \lambda Nx,$$

which is equivalent to the following equation:

$$\big((Ax)(t)\big)'' + \lambda f\big(t,x'(t)\big) + \lambda g\big(t,x(t)\big) = \lambda e(t), \tag{3.1}$$

where $(Ax)(t) = x(t) - cx(t-\delta)$ in Section 2.

We first claim that there is a point $\xi \in (0,T)$ such that

$$0 < x(\xi) \leq D. \tag{3.2}$$

Integrating both sides of (3.1) over $[0,T]$, we have

$$\int_0^T \big[f\big(t,x'(t)\big) + g\big(t,x(t)\big)\big]\,dt = 0. \tag{3.3}$$

This shows that there at least exists a point $\xi \in (0,T)$ such that

$$f\big(\xi,x'(\xi)\big) + g\big(\xi,x(\xi)\big) = 0,$$

then by (H_1), we have

$$g\big(\xi,x(\xi)\big) = \big|-f\big(\xi,x'(\xi)\big)\big| \leq K,$$

and in view to (H_4) we get $x(\xi) \le D$. Since $x(t)$ is periodic with periodic T and $x(t) > 0$, for $t \in [0, T]$. Then $0 < x(\xi) \le D$. (3.2) is proved. Therefore, we have

$$\begin{aligned}
\|x\| &= \max_{t\in[0,T]} \left|x(t)\right| = \max_{t\in[\xi,\xi+T]} \left|x(t)\right| \\
&= \frac{1}{2} \max_{t\in[\xi,\xi+T]} \left(\left|x(t)\right| + \left|x(t-T)\right|\right) \\
&= \frac{1}{2} \max_{t\in[\xi,\xi+T]} \left(\left|x(\xi) + \int_\xi^T x'(s)\,ds\right| + \left|x(\xi) - \int_{t-T}^{\xi} x'(s)\,ds\right|\right) \\
&\le D + \frac{1}{2}\left(\int_\xi^t \left|x'(s)\right| ds + \int_{t-T}^{\xi} \left|x'(s)\right| ds\right) \\
&\le D + \frac{1}{2}\int_0^T \left|x'(s)\right| ds. \qquad (3.4)
\end{aligned}$$

For $|c| \neq 1$, by applying Lemma 2.1, we have

$$\begin{aligned}
\left\|x'\right\| &= \max_{t\in[0,T]} \left|A^{-1}Ax'(t)\right| \\
&\le \frac{\max_{t\in[0,T]} |Ax'(t)|}{|1-|c||} \\
&= \frac{|(Ax)'(t)|}{|1-|c||}, \qquad (3.5)
\end{aligned}$$

since $(Ax)'(t) = (x(t) - cx(t-\delta))' = x'(t) - cx'(t-\delta) = (Ax')(t)$ (see [15, 16]).

On the other hand, from $\int_0^T (Ax)'(t)\,dt = 0$, there exists a point $t_2 \in (0, T)$ such that $(Ax)'(t_2) = 0$, which together with the integration of (3.1) on interval $[0, T]$ yields

$$\begin{aligned}
2\left|(Ax)'(t)\right| &\le 2\left((Ax)'(t_2) + \frac{1}{2}\int_0^T \left|(Ax)''(t)\right| dt\right) \\
&\le \lambda \int_0^T \left|-f\left(t,x'(t)\right) - g\left(t,x(t)\right) + e(t)\right| dt \\
&\le \int_0^T \left|f\left(t,x'(t)\right)\right| dt + \int_0^T \left|g\left(t,x(t)\right)\right| dt + \int_0^T \left|e(t)\right| dt. \qquad (3.6)
\end{aligned}$$

Write

$$I_+ = \left\{t \in [0,T] : g\left(t,x(t)\right) \ge 0\right\}; \qquad I_- = \left\{t \in [0,T] : g\left(t,x(t)\right) \le 0\right\}.$$

Then we get from (H_1), (H_6) and (3.3)

$$\begin{aligned}
\int_0^T \left|g\left(t,x(t)\right)\right| dt &= \int_{I_+} g\left(t,x(t)\right) dt - \int_{I_-} g\left(t,x(t)\right) dt \\
&= 2\int_{I_+} g\left(t,x(t)\right) dt + \int_0^T f\left(t,x'(t)\right) dt \\
&\le 2\int_0^T \left(mx(t) + n\right) dt + \int_0^T \left|f\left(t,x'(t)\right)\right| dt \\
&\le 2m\int_0^T \left|x(t)\right| dt + 2nT + KT. \qquad (3.7)
\end{aligned}$$

Substituting (3.7) into (3.6), and from (H_1), we have

$$2\left|(Ax)'(t)\right| \le 2m\int_0^T \left|x(t)\right| dt + 2nT + 2KT + \|e\|T$$
$$\le 2mT^{\frac{1}{2}}\left(\int_0^T \left|x(t)\right| dt\right)^{\frac{1}{2}} + N_1, \tag{3.8}$$

where $N_1 = 2T(n+K) + \|e\|T$. In view of an inequality (found in [17], Lemma 2.3) and (3.1), we have

$$\left(\int_0^T \left|x(t)\right|^2 dt\right)^{\frac{1}{2}} \le \left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^p dt\right)^{\frac{1}{2}} + DT^{\frac{1}{2}}. \tag{3.9}$$

Substituting (3.9) into (3.8), we have

$$2\left|(Ax)'(t)\right| \le 2mT^{\frac{1}{2}}\left(\left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^p dt\right)^{\frac{1}{2}} + DT^{\frac{1}{2}}\right) + N_1. \tag{3.10}$$

Substituting (3.10) into (3.5), we have

$$\|x'\| \le \frac{mT^{\frac{1}{2}}((\frac{T}{\pi})(\int_0^T |x'(t)|^2\,dt)^{\frac{1}{2}} + DT^{\frac{1}{2}}) + \frac{N_1}{2}}{|1-|c||}$$
$$\le \frac{mT(\frac{T}{\pi})\|x'\| + mTD + \frac{N_1}{2}}{|1-|c||}.$$

Since $\frac{mT^2}{\pi|1-|c||} < 1$, it is easy to see that there exists a positive constant M_2 such that

$$\|x'\| \le M_2. \tag{3.11}$$

Substituting (3.11) into (3.4), we have

$$\|x\| \le D + \frac{1}{2}\int_0^T \left|x'(t)\right| dt \le D + \frac{1}{2}TM_2 := M_1. \tag{3.12}$$

Next, it follows from (3.1) that

$$(Ax)''(t) + \lambda f\left(t, x'(t)\right) + \lambda\left(g_0\left(x(t)\right) + g_1\left(t, x(t)\right)\right) = \lambda e(t). \tag{3.13}$$

Multiplying both sides of (3.13) by $x'(t)$, we get

$$(Ax)''(t)x'(t) + \lambda f\left(t, x'(t)\right)x'(t) + \lambda g_0\left(x(t)\right)x'(t) + \lambda g_1\left(t, x(t)\right)x'(t) = \lambda e(t)x'(t). \tag{3.14}$$

Let $\tau \in [0,T]$, for any $\tau \le t \le T$, we integrate (3.14) on $[\tau,t]$ and get

$$\begin{aligned}\lambda \int_{x(\tau)}^{x(t)} g_0(u)\,du &= \lambda \int_\tau^t g_0\big(x(s)\big)x'(s)\,ds \\ &= -\int_\tau^t (Ax)''(s)x'(s)\,ds - \lambda \int_\tau^t f\big(t,x'(s)\big)x'(s)\,ds \\ &\quad - \lambda \int_\tau^t g_1\big(s,x(s)\big)x'(s)\,ds + \lambda \int_\tau^t e(s)x'(s)\,ds. \end{aligned} \tag{3.15}$$

By (3.1), (3.7), (3.12) and (H_1), we have

$$\begin{aligned}\left|\int_\tau^t (Ax)''(s)x'(s)\,ds\right| &\le \int_\tau^t \big|(Ax)''(s)\big|\big|x'(s)\big|\,ds \\ &\le \|x'\| \int_0^T \big|(Ax)''(s)\big|\,ds \\ &\le \lambda M_2\left(\int_0^T \big|f\big(t,x'(s)\big)\big|\,ds + \int_0^T \big|g\big(s,x(s)\big)\big|\,ds + \int_0^T \big|e(s)\big|\,ds\right) \\ &\le \lambda M_2\big(2mTM_1 + 2nT + 2KT + T\|e\|\big).\end{aligned}$$

We have

$$\begin{aligned}&\left|\int_\tau^t f\big(t,x(s)\big)x'(s)\,ds\right| \le \|x'\| \int_0^T \big|f\big(t,x(s)\big)\big|\,ds \le M_2KT. \\ &\left|\int_\tau^t g_1\big(s,x(s)\big)x'(s)\,ds\right| \le \|x'\| \int_0^T \big|g_1\big(t,x(t)\big)\big|\,dt \le M_2\sqrt{T}|g_{M_1}|_2,\end{aligned}$$

where $g_{M_1} = \max_{0\le x\le M_1} |g_1(t,x)| \in L^2(0,T)$.

$$\left|\int_\tau^t e(s)x'(s)\,dt\right| \le M_2T\|e\|.$$

From these inequalities we can derive from (3.15) that

$$\left|\int_{x(\tau)}^{x(t)} g_0(u)\,du\right| \le M_2\big(2mTM_1 + 2nT + \sqrt{T}|g_{M_1}|_2 + 3KT + 2T\|e\|\big).$$

In view of the strong force condition (1.2), we know that there exists a constant $M_3 > 0$ such that

$$x(t) \ge M_3, \quad \forall t \in [\tau,T]. \tag{3.16}$$

The case $t \in [0,\tau]$ can be treated similarly.

From (3.11), (3.12) and (3.16), we let

$$\Omega = \big\{x : E_1 \le x(t) \le E_2, \|x'\| \le E_3, \forall t \in [0,T]\big\},$$

where $0 < E_1 < M_3$, $E_2 > \max\{M_1, D\}$, $E_3 > M_2$. Then condition (1) of Lemma 2.1 is satisfied. If $x \in \partial\Omega \cap \operatorname{Ker} L$, then $x(t) = E_1$ or (E_2). In this case

$$QNx = \frac{1}{T}\int_0^T g(t, E_1)\,dt := -\bar{g}(E_1),$$

or

$$QNx = \frac{1}{T}\int_0^T g(t, E_2)\,dt := -\bar{g}(E_2),$$

since $f(t, 0) = 0$. According to the condition (H_4), we get $QNx \neq 0$, which implies $Nx \notin \operatorname{Im} L$ for $x \in \partial\Omega \cap \operatorname{Ker} L$. Hence, condition (2) of Lemma 2.1 holds. To check condition (3) of Lemma 2.1, we define an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L = R$, $J(u) = u$. It is noted that if $x \in \Omega \cap \operatorname{Ker} L$, then $x(t) = c$ with $E_1 < c < E_2$,

$$JQNx = -\int_0^T g(t, c)\,dt.$$

From (H_4), we can derive

$$\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) = -1.$$

So condition (3) of Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that the equation $Lx = Nx$ has a solution x on $\bar{\Omega} \cap D(L)$, *i.e.*, (1.1) has at least one positive T-periodic solution $x(t)$. □

Theorem 3.2 *Assume that conditions* (H_2), (H_3), (H_5)-(H_6) *hold. Then* (1.1) *has at least one positive solution with period* T *if* $\frac{mT(\frac{T}{\pi})+\alpha T}{|1-|c||} < 1$.

Proof We will follow the same strategy and notations as the proof of Theorem 3.1. Now, we consider $\|x'\| \leq M_2$.

We first claim that there is a constant $\xi^* \in (0, T)$ such that

$$0 < x(\xi^*) \leq D_1. \tag{3.17}$$

In view of $\int_0^T (Ax)'(t)\,dt = 0$, we know that there exist two constants $t_3, t_4 \in [0, \omega]$ such that $(Ax)'(t_3) \geq 0$, $(Ax)'(t_4) \leq 0$. Let $\xi^* \in (0, T)$ be a global maximum point of $(Ax)'(t)$. Clearly, we have

$$(Ax)'(\xi^*) \geq 0, \qquad (Ax)''(\xi^*) = 0.$$

From (H_3), we know $f(\xi^*, x'(\xi^*)) \geq 0$. Therefore, we see that

$$g(\xi^*, x(\xi^*)) - e(\xi^*) = -f(\xi^*, x'(\xi^*)) \leq 0,$$

i.e.

$$g(\xi^*, x'(\xi^*)) \leq e(\xi^*) \leq \|e\|.$$

From (H_5), we have

$$x(\xi) \le D_1.$$

Since $x(t) > 0$, hence, we can get $0 < x(\xi^*) \le D_1$. This proves (3.17).

Similarly, from (3.4), we have

$$\left|x(t)\right| \le D_1 + \frac{1}{2}\int_0^T \left|x'(t)\right| dt. \tag{3.18}$$

From (3.7), (H_2) and (H_6), we have

$$\begin{aligned}
\int_0^T \left|g\big(t,x(t)\big)\right| dt &= \int_{I_+} g\big(t,x(t)\big)\,dt - \int_{I_-} g\big(t,x(t)\big)\,dt \\
&= 2\int_{I_+} g\big(t,x(t)\big)\,dt + \int_0^T f\big(t,x'(t)\big)\,dt \\
&\le 2\int_{I_+} \big(mx(t)+n\big)\,dt + \int_0^T \left|f\big(t,x'(t)\big)\right| dt \\
&\le 2m\int_0^T \left|x(t)\right| dt + 2nT + \alpha\int_0^T \left|x'(t)\right| dt + \beta T.
\end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.6), and from (H_2), we have

$$\begin{aligned}
2\left|(Ax)'(t)\right| &\le 2m\int_0^T \left|x(t)\right| dt + 2nT + 2\alpha\int_0^T \left|x'(t)\right| dt + 2\beta T + \|e\|T \\
&\le 2mT^{\frac{1}{2}}\left(\int_0^T \left|x(t)\right|^2 dt\right)^{\frac{1}{2}} + 2\alpha\left\|x'\right\|T + N_2,
\end{aligned} \tag{3.20}$$

where $N_2 = 2T(n+\beta) + \|e\|T$. Substituting (3.9) into (3.20), we have

$$\begin{aligned}
2\left|(Ax)'(t)\right| &\le 2mT^{\frac{1}{2}}\left(\left(\frac{T}{\pi}\right)\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + D_1T^{\frac{1}{2}}\right) + 2\alpha\left\|x'\right\|T + N_2 \\
&\le 2mT\left(\frac{T}{\pi}\right)\left\|x'\right\| + 2\alpha\left\|x'\right\|T + 2mD_1T + N_2 \\
&= \left(2mT\left(\frac{T}{\pi}\right) + 2\alpha T\right)\left\|x'\right\| + 2mD_1T + N_2.
\end{aligned} \tag{3.21}$$

Similarly, for $|c| \ne 1$, we can get

$$\left\|x'\right\| \le \frac{(mT(\frac{T}{\pi}) + \alpha T)\|x'\|}{|1-|c||} + \frac{mD_1T + \frac{N_2}{2}}{|1-|c||}.$$

Since $\frac{mT(\frac{T}{\pi})+\alpha T}{|1-|c||} < 1$, it is easy to see that there exists a positive constant M_2 such that

$$\left\|x'\right\| \le M_2.$$

The proof left is as same as Theorem 3.1. □

We illustrate our results with some examples.

Example 3.1 Consider the following neutral Rayleigh equation with singularity:

$$\left(x(t)-\frac{1}{10}x(t-\delta)\right)''+\cos^2(2t)\sin x'(t)+\frac{1}{6\pi}\big(\sin(4t)+5\big)x(t)-\frac{1}{u^\mu}=\cos^2(2t), \quad (3.22)$$

where $\mu \geq 1$ and δ is a constant.

It is clear that $T=\frac{\pi}{2}$, $c=\frac{1}{10}$, $e(t)=\cos^2(2t)$, $f(t,u)=\cos^2(2t)\sin u$, $g(t,x)=\frac{1}{6\pi}\times(\sin(4t)+5)x(t)-\frac{1}{x^\mu(t)}$. Choose $K=1$, $D=2$, $m=\frac{1}{\pi}$, it is obvious that (H_1), (H_4) and (H_6) hold. Next, we consider

$$\begin{aligned}\frac{mT^2}{\pi|1-|c||}&=\frac{\frac{1}{\pi}\times(\frac{\pi}{2})^2}{\pi|1-\frac{1}{10}|}\\&=\frac{5}{18}<1.\end{aligned}$$

Therefore, by Theorem 3.1, (3.22) has at least one $\frac{\pi}{2}$-periodic solution.

Example 3.2 Consider the following a kind of neutral Rayleigh equation:

$$\big(x(t)-100x(t-\delta)\big)''+\frac{1}{5\pi}\big(\sin^2 t+4\big)x'(t)+\big(\cos^2 t+4\big)x(t)-\frac{1}{x^\mu}=\sin(2t), \quad (3.23)$$

where $\mu \geq 1$ and δ is a constant.

It is clear that $T=\pi$, $c=100$, $e(t)=\sin(2t)$, $f(t,u)=\frac{1}{5\pi}(\sin^2 t+4)u(t)$, $g(t,x)=(\cos^2 t+4)x(t)-\frac{1}{x^\mu(t)}$. Choose $m=5$, $D_1=3$, $a=\frac{1}{\pi}$, it is obvious that (H_1), (H_2), (H_5) and (H_6) hold. Next, we consider

$$\begin{aligned}\frac{mT(\frac{T}{\pi})+\alpha T}{|1-|c||}&=\frac{5\times\pi(\frac{\pi}{\pi})+\frac{1}{\pi}\times\pi}{1-100}\\&=\frac{5\pi+1}{99}<1.\end{aligned}$$

So, (3.23) has at least one nonconstant π-periodic solution by application of Theorem 3.2.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YX and ZBC worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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