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Global higher integrability for very weak solutions to nonlinear subelliptic equations

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Abstract

In this paper we consider the following nonlinear subelliptic Dirichlet problem:

$$\begin{cases} X^*A(x,u,Xu) + B(x,u,Xu) = 0, & x \in \Omega, \\ u - u_0 \in W_{X,0}^{1,r}(\Omega), \end{cases}$$

where $X = \{X_1, \dots, X_m\}$ is a system of smooth vector fields defined in $\mathbf{R}^n$ with globally Lipschitz coefficients satisfying Hörmander's condition, and we prove the global higher integrability for the very weak solutions.

MSC: Primary 35H20; secondary 35J60

Keywords: nonlinear subelliptic equations; very weak solutions; global higher integrability

1 Introduction and main result

The theory of very weak solutions was introduced in the work of Iwaniec and Sbordone [1]. Iwaniec and Sbordone realized that the usual Sobolev assumption for weak solutions to p-harmonic equation can be relaxed to a slightly weaker Sobolev space and proved that very weak solutions are actually classical weak solutions by using the nonlinear Hodge decomposition to construct suitable test functions. Based on Whitney's extension theorem and theory of A_p weights, Lewis [2] showed a completely different proof and obtained the same result to certain elliptic systems. After [1] and [2], many authors have devoted their energy to the study of the regularity of such solutions; see for example [3–5] and the references therein. We mention here that Xie and Fang [5] obtained the global higher integrability for very weak solutions to a class of nonlinear elliptic systems with Lipschitz boundary condition by using Hodge decomposition to construct a suitable test function. Recently the authors in [6] proved the global regularity result for a second-order degenerate elliptic systems of p-Laplacian type in the Euclidean setting.

In 2005, Zatorska-Goldstein [7] showed the local higher integrability of very weak solutions to the nonlinear subelliptic equations

$$X^*A(x,u,Xu) + B(x,u,Xu) = 0, \quad x \in \Omega, \tag{1.1}$$

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where $\Omega \subset \mathbf{R}^n$ is a bounded domain and $X = \{X_1, \dots, X_m\}$ ($m \leq n$) is a system of smooth vector fields in $\mathbf{R}^n$ with globally Lipschitz coefficients satisfying the Hörmander's condition and $X^* = (X_1^*, \dots, X_m^*)$ is a family of operators formal adjoint to X_j in L^2.

In this work we are concerned with the boundary value problem for (1.1) with the boundary condition $u - u_0 \in W^{1,r}_{X,0}(\Omega)$, i.e.,

$$\begin{cases} X^*A(x, u, Xu) + B(x, u, Xu) = 0, & x \in \Omega, \\ u - u_0 \in W^{1,r}_{X,0}(\Omega), \end{cases} \tag{1.2}$$

and establish the global higher integrability for very weak solutions. We assume that the functions $A = (A_1, \dots, A_m) : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ and $B : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ are both Carathéodory functions satisfying

$$\left| A(x, u, \xi) \right| \leq \alpha \left(|u|^{p-1} + |\xi|^{p-1} \right), \tag{1.3}$$

$$\left| B(x, u, \xi) \right| \leq \alpha \left(|u|^{p-1} + |\xi|^{p-1} \right), \tag{1.4}$$

$$\left\langle A(x, u, \xi) - A(x, v, \zeta), \xi - \zeta \right\rangle \geq \beta |\xi - \zeta|^2 \left(|\xi| + |\zeta| \right)^{p-2}, \tag{1.5}$$

for a.e. $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $\xi \in \mathbf{R}^m$. Here $p \geq 2$, α, β are positive constants.

A function $u \in W^{1,r}_X(\Omega)$ ($r < p$) is called a *very weak solution* to (1.1) if

$$\int_\Omega A(x, u, Xu) \cdot X\varphi \, dx + \int_\Omega B(x, u, Xu)\varphi \, dx = 0 \tag{1.6}$$

holds for all $\varphi \in C_0^\infty(\Omega)$.

In the above definition, the very weak means the integrable exponent is strictly lower than the natural exponent p and if $r = p$, this is the classical definition of weak solution to (1.1).

To get our result, some regularity assumption introduced in [8] should be imposed on Ω. Let us first recall the notion of uniform (X, p)-fatness which can be found in [9]: A set $E \subset \mathbf{R}^n$ is called *uniformly (X, p)-fat* if there exist constants $C_0, R_0 > 0$ such that

$$\mathrm{cap}_p \left(E \cap \bar{B}(x, R), B(x, 2R) \right) \geq C_0 \, \mathrm{cap}_p \left(\bar{B}(x, R), B(x, 2R) \right)$$

for all $x \in \partial E$ and $0 < R < R_0$, where cap_p is the variational p-capacity defined in Section 2.

We consider the following hypotheses on Ω:

(H_1) there exists a constant $C_1 \geq 1$ such that, for all $x \in \Omega$,

$$|B_{\rho(x)}| \leq C_1 \left| B_{\rho(x)} \cap \left(\mathbf{R}^n \setminus \Omega \right) \right|,$$

where $\rho(x) = 2 \, \mathrm{dist}(x, \mathbf{R}^n \setminus \Omega)$;

(H_2) the complement $\mathbf{R}^n \setminus \Omega$ of Ω is uniformly (X, p)-fat.

Under the hypotheses stated above, we prove the following.

Theorem 1.1 *Assume that $u_0 \in W^{1,s}_X(\Omega)$, $s > p$. Then there exists a $\delta > 0$ such that if $u \in W^{1,p-\delta}_X(\Omega)$ is a very weak solution to the Dirichlet problem* (1.2), *we have $u \in W^{1,p+\tilde{\delta}}_X(\Omega)$ for some $\tilde{\delta} > 0$.*

The key technical tool in proving Theorem 1.1 is a Sobolev type inequality with a capacity term. With it we can prove a reverse Hölder inequality for the generalized gradient Xu of a very weak solution, which allows us to get the global higher integrability of Xu. This paper is organized as follows. In Section 2 we collect some known results on Carnot-Carathéodory spaces and prove a Sobolev type inequality characterized by capacity. Section 3 is devoted to the proof of Theorem 1.1.

2 Some known results and a Sobolev type inequality

Let $\{X_1, \ldots, X_m\}$ be a system of C^∞-smooth vector fields in $\mathbf{R}^n (n \geq 3)$ satisfying Hörmander's condition (see [10]):

$$\text{rank}\big(\text{Lie}\{X_1, \ldots, X_m\}\big) = n.$$

The generalized gradient is denoted by $Xu = (X_1u, \ldots, X_mu)$ and its length is given by

$$\big|Xu(x)\big| = \left(\sum_{j=1}^{m}\big|X_ju(x)\big|^2\right)^{\frac{1}{2}}.$$

An absolutely continuous curve $\gamma : [a, b] \to \mathbf{R}^n$ is said to be *admissible* with respect to the system $\{X_1, \ldots, X_m\}$, if there exist functions $c_i(t), a \leq t \leq b$, satisfying

$$\sum_{i=1}^{m} c_i(t)^2 \leq 1 \quad \text{and} \quad \gamma'(t) = \sum_{i=1}^{m} c_i(t)X_i\big(\gamma(t)\big).$$

The Carnot-Carathéodory distance $d(x, y)$ generated by $\{X_1, \ldots, X_m\}$ is defined as the infimum of those $T > 0$ for which there exists an admissible path $\gamma : [0, T] \to \mathbf{R}^n$ with $\gamma(0) = x$, $\gamma(T) = y$.

By the accessibility theorem of Chow [11], the distance d is a metric and therefore $(\mathbf{R}^n, d)$ is a metric space which is called the *Carnot-Carathéodory* space associated with the system $\{X_1, \ldots, X_m\}$. The ball is denoted by

$$B(x_0, R) = \big\{x \in \mathbf{R}^n : d(x, x_0) < R\big\}.$$

For $\sigma > 0$ and $B = B(x_0, R)$, we will write σB to indicate $B(x_0, \sigma R)$ and diam Ω the diameter of Ω with respect to d.

It was proved in [12] that the identity map is a homeomorphism of $(\mathbf{R}^n, d)$ into $\mathbf{R}^n$ with the usual Euclidean metric, and every set which is bounded with respect to the Euclidean metric is also bounded with respect to d. Moreover, by a result of Garofalo and Nhieu [13], Proposition 2.11, if the given vector fields have globally Lipschitz coefficients in addition, then a subset of $\mathbf{R}^n$ is bounded with respect to d if and only if it is bounded with respect to the Euclidean metric.

Hereafter we assume that the vector fields $X_1, \ldots, X_m$ satisfy the Hörmander condition and have globally Lipschitz coefficients.

Lemma 2.1 ([12, 14]) *For every bounded open set $\Omega \subset \mathbf{R}^n$ there exists $C_d \geq 1$ such that*

$$\big|B(x, 2R)\big| \leq C_d\big|B(x, R)\big| \tag{2.1}$$

for any $x \in \Omega$ and $0 < R \leq 5\,\text{diam}\,\Omega$.

Here, $|B(x,R)|$ denotes the Lebesgue measure of $B(x,R)$. The best constant C_d in (2.1) is called the doubling constant, the measure such that (2.1) holds is called a doubling measure and the homogeneous dimension relative to Ω is $Q = \log_2 C_d$.

Given $1 \le p < \infty$, we define the Sobolev space $W_X^{1,p}(\Omega)$ by

$$W_X^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : X_j u \in L^p(\Omega), j = 1, 2, \ldots, m \right\},$$

endowed with the norm

$$\|u\|_{W_X^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}.$$

Here, $X_j u$ is the distributional derivative of $u \in L_{\text{loc}}^1(\Omega)$ given by the identity

$$\langle X_j u, \varphi \rangle = \int_\Omega u X_j^* \varphi \, dx, \quad \varphi \in C_0^\infty(\Omega).$$

The space $W_X^{1,p}(\Omega)$ is a Banach space which admits $C^\infty(\Omega) \cap W_X^{1,p}(\Omega)$ as its dense subset. The completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{W_X^{1,p}(\Omega)}$ is denoted by $W_{X,0}^{1,p}(\Omega)$. The following Sobolev-Poincaré inequalities can be found in [14] and [15]:

Lemma 2.2 *Let Q be the homogeneous dimension relative to Ω, $B = B(x_0, R) \subset \Omega$, $0 < R < \text{diam}\,\Omega$, $1 \le p < \infty$. There exists a constant $C > 0$ such that, for every $u \in W_X^{1,p}(B)$,*

$$\left(\fint_B |u - u_B|^{\kappa p} \, dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_B |Xu|^p \, dx \right)^{\frac{1}{p}},$$

where $u_B = \fint_B u \, dx = \frac{1}{|B|} \int_B u \, dx$, and $1 \le \kappa \le Q/(Q-p)$, if $1 \le p < Q$; $1 \le \kappa < \infty$, if $p \ge Q$. Moreover, for any $u \in W_{X,0}^{1,p}(B)$,

$$\left(\fint_B |u|^{\kappa p} \, dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_B |Xu|^p \, dx \right)^{\frac{1}{p}}.$$

Next we recall a Gehring lemma on the metric measure space (Y, d, μ), where d is a metric and μ is a doubling measure.

Lemma 2.3 ([7]) *Let $q \in [q_0, 2Q]$, $q_0 > 1$ is fixed. Assume that functions f, g are nonnegative and $g \in L_{\text{loc}}^q(Y, \mu)$, $f \in L_{\text{loc}}^{r_0}(Y, \mu)$, for some $r_0 > q$. If there exist constants $b > 1$ and θ such that for every ball $B \subset \sigma B \subset Y$ the following inequality holds:*

$$\fint_B g^q \, d\mu \le b \left[\left(\fint_{\sigma B} g \, d\mu \right)^q + \fint_{\sigma B} f^q \, d\mu \right] + \theta \fint_{\sigma B} g^q \, d\mu,$$

then there exist nonnegative constants $\theta_0 = \theta_0(q_0, Q, C_d, \sigma)$ and $\varepsilon_0 = \varepsilon_0(b, q_0, Q, C_d, \sigma)$ such that if $0 < \theta < \theta_0$ then $g \in L_{\text{loc}}^p(Y, \mu)$ for $p \in [q, q + \varepsilon_0)$.

For the Hardy-Littlewood maximal functions

$$Mf(x) = \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)| \, dy$$

and

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)\cap\Omega} |f(y)|\, dy,$$

we will use the following properties proved in [14] and [15].

Lemma 2.4 *If $f \in L^p(\Omega)$, $1 < p \leq \infty$, then $M_\Omega f \in L^p(\Omega)$ and there exists a constant $C = C(C_d, p) > 0$ such that*

$$\|M_\Omega f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Lemma 2.5 *If $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$, $1 < p < \infty$, then there exists $C > 0$ such that, for a.e. $x, y \in \Omega$,*

$$|u(x) - u(y)| \leq Cd(x,y)\big(M_\Omega|Xu|(x) + M_\Omega|Xu|(y)\big).$$

Moreover, for any $B = B(x_0, R) \subset \Omega$ and $u \in W^{1,p}_X(B)$, we have

$$|u(x) - u_B| \leq CRM_B|Xu|(x), \quad a.e.\ x \in B. \tag{2.2}$$

It is worth noting that from Lemma 2.5 and Lemma 2.2 we can infer that, for a.e. $x \in B$ and $u \in W^{1,p}_{X,0}(B)$,

$$|u(x)| \leq CRM_B|Xu|(x). \tag{2.3}$$

Let $\omega(x) \geq 0$ be a locally integrable function, we say that $\omega \in A_p$, $1 < p < \infty$, if there exists some positive constant A such that

$$\sup_{B\subset\mathbf{R}^n} \left(\fint_B \omega\, dx\right)\left(\fint_B \omega^{\frac{1}{1-p}}\, dx\right)^{p-1} \leq A < \infty.$$

Lemma 2.6 *Assume $\omega \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is nonnegative and $1 < p < \infty$. Then $\omega \in A_p$ if and only if there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^n} |Mf|^p \omega\, dx \leq C \int_{\mathbf{R}^n} |f|^p \omega\, dx,$$

for all $f \in L^p(\omega(x)\, dx)$.

The (X,p)-*capacity* of a compact set $K \subset \Omega$ in Ω is defined by

$$\mathrm{cap}_p(K, \Omega) = \inf\left\{\int_\Omega |Xu|^p\, dx : u \in C_0^\infty(\Omega), u = 1 \text{ on } K\right\}$$

and for an arbitrary set $E \subset \Omega$, the (X,p)-capacity of E is

$$\mathrm{cap}_p(E, \Omega) = \inf_{\substack{G\subset\Omega\,\mathrm{open} \\ E\subset G}} \sup_{\substack{K\subset G \\ K\,\mathrm{compact}}} \mathrm{cap}_p(K, \Omega).$$

We will use the following two-sided estimate of (X,p)-capacity in [16]: For $x \in \Omega$ and $0 < R < \operatorname{diam} \Omega$, there exist $C_1, C_2 > 0$ such that

$$C_1 \frac{|B(x,R)|}{R^p} \leq \operatorname{cap}_p\big(\bar{B}(x,R), B(x,2R)\big) \leq C_2 \frac{|B(x,R)|}{R^p}. \tag{2.4}$$

Lemma 2.7 ([9]) *If* $\mathbf{R}^n \setminus \Omega$ *is uniformly* (X,p)-*fat, then there exists* $1 < q < p$ *such that* $\mathbf{R}^n \setminus \Omega$ *is also uniformly* (X,q)-*fat.*

The uniform (X,q)-fatness also implies uniform (X,p)-fatness for all $p \geq q$, which is a simple consequence of Hölder's and Young's inequality.

At the end of this section we prove a Sobolev type inequality characterized by capacity. A similar inequality in the Euclidean setting can be found in [8].

Lemma 2.8 *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded open set with the homogeneous dimension* Q, $1 < q < \infty$ *and* $0 < R < \operatorname{diam} \Omega$. *For any* $x \in \Omega$, *denote* $B = B(x,R)$ *and* $N(\varphi) = \{y \in \bar{B} : \varphi(y) = 0\}$. *Then there exists a constant* $C = C(Q,q) > 0$ *such that, for all* $\varphi \in C^\infty(2B) \cap W_X^{1,q}(2B)$,

$$\left(\fint_{2B} |\varphi|^{\kappa q} \, dx \right)^{\frac{1}{\kappa q}} \leq C \left(\frac{1}{\operatorname{cap}_q(N(\varphi), 2B)} \int_{2B} |X\varphi|^q \, dx \right)^{\frac{1}{q}}, \tag{2.5}$$

where $1 \leq \kappa \leq Q/(Q-q)$ *if* $1 \leq q < Q$ *and* $1 \leq \kappa < \infty$ *if* $q \geq Q$.

Proof We always assume $\varphi_{2B} \neq 0$; otherwise, (2.5) follows immediately from Lemma 2.2 and (2.4). Let $\eta \in C_0^\infty(2B), 0 \leq \eta \leq 1$ such that $\eta = 1$ on $\bar{B}$ and $|X\eta| \leq \frac{c}{R}$. Denoting $v = \eta(\varphi_{2B} - \varphi)/\varphi_{2B}$, then $v \in C_0^\infty(2B)$ and $v = 1$ in $N(\varphi)$. It follows from Lemma 2.2 that

$$\operatorname{cap}_q\big(N(\varphi), 2B\big) \leq \int_{2B} |Xv|^q \, dx$$

$$\leq |\varphi_{2B}|^{-q} \int_{2B} |X\eta|^q |\varphi - \varphi_{2B}|^q \, dx + |\varphi_{2B}|^{-q} \int_{2B} |X\varphi|^q \, dx$$

$$\leq C |\varphi_{2B}|^{-q} \int_{2B} |X\varphi|^q \, dx,$$

and then

$$|\varphi_{2B}| \leq C \left(\frac{1}{\operatorname{cap}_q(N(\varphi), 2B)} \int_{2B} |X\varphi|^q \, dx \right)^{\frac{1}{q}}. \tag{2.6}$$

Then Lemma 2.2 and (2.6) lead to

$$\left(\fint_{2B} |\varphi|^{\kappa q} \, dx \right)^{\frac{1}{\kappa q}} \leq \left(\fint_{2B} |\varphi - \varphi_{2B}|^{\kappa q} \, dx \right)^{\frac{1}{\kappa q}} + |\varphi_{2B}|$$

$$\leq CR \left(\fint_{2B} |X\varphi|^q \, dx \right)^{\frac{1}{q}} + C \left(\frac{1}{\operatorname{cap}_q(N(\varphi), 2B)} \int_{2B} |X\varphi|^q \, dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\frac{1}{\operatorname{cap}_q(N(\varphi), 2B)} \int_{2B} |X\varphi|^q \, dx \right)^{\frac{1}{q}},$$

where in the last step we used the estimate

$$\operatorname{cap}_q\big(N(\varphi), 2B\big) \leq \operatorname{cap}_q(\bar{B}, 2B) \leq C|B|R^{-q}.$$

The proof is complete. □

3 Proof of Theorem 1.1

Assume that the function $u \in W_X^{1,p-\delta}(\Omega)(\delta < \frac{1}{2})$ is a very weak solution to the Dirichlet problem (1.2). Choose a ball B_0 such that $\overline{\Omega} \subset \frac{1}{2}B_0$ and let B be a ball of radius R with $3B \subset B_0$ for fixed $0 < R < 1$. There are two cases: (i) $3B \subset \Omega$ or (ii) $3B \backslash \Omega \neq \emptyset$. In the case (i), the following estimate has been proved in [7]:

$$\fint_{\frac{B}{2}} |Xu|^{p-\delta}\, dx \leq \theta \fint_{3B} |Xu|^{p-\delta}\, dx + b\left[\fint_{3B} |u|^{p-\delta} + \left(\fint_{3B} |Xu|^t\, dx\right)^{\frac{p-\delta}{t}}\right], \tag{3.1}$$

where θ small enough, $b > 1$, $\max\{1, (p-\delta)_*\} < t < p - \delta$.

When $3B \backslash \Omega \neq \emptyset$, a similar inequality (see (3.31) below) will be achieved.

Step 1. Let η be a smooth cut-off function on $2B$, i.e. $\eta \in C_0^\infty(2B)$ such that

$$0 \leq \eta \leq 1, \qquad \eta = 1 \quad \text{on } B \quad \text{and} \quad |X\eta| \leq c/R.$$

Define $\hat{u} = \eta(u - u_0)$ and

$$E_\mu = \big\{x \in \mathbf{R}^n : M|X\hat{u}|(x) \leq \mu\big\}, \quad \text{for } \mu > 0. \tag{3.2}$$

We conclude from Lemma 2.5 and the assumption (H_1) that $\hat{u}$ is Lipschitz continuous on $E_\mu \cup (\mathbf{R}^n \setminus \Omega)$.

Indeed, if $x, y \in E_\mu \cap \Omega$, then Lemma 2.5 implies $|\hat{u}(x) - \hat{u}(y)| \leq c\mu d(x, y)$; if $x, y \in \mathbf{R}^n \setminus \Omega$, then $\hat{u}(x) = \hat{u}(y) = 0$. We set $B_{\rho_x} = B(x, \rho_x)$ with $\rho_x = 2\operatorname{dist}(x, \mathbf{R}^n \setminus \Omega)$ for the case $x \in E_\mu \cap \Omega$ and $y \in \mathbf{R}^n \setminus \Omega$. Since $\hat{u}$ is zero on $\mathbf{R}^n \setminus \Omega$, it follows that

$$\int_{B_{\rho_x} \cap (\mathbf{R}^n \setminus \Omega)} |\hat{u} - \hat{u}_{B_{\rho_x}}|\, dz = \int_{B_{\rho_x} \cap (\mathbf{R}^n \setminus \Omega)} |\hat{u}_{B_{\rho_x}}|\, dz$$

$$= \big|B_{\rho_x} \cap (\mathbf{R}^n \setminus \Omega)\big| |\hat{u}_{B_{\rho_x}}|$$

and then, from assumption (H_1) and Lemma 2.2,

$$|\hat{u}_{B_{\rho_x}}| \leq C_1 \frac{|B_{\rho_x} \cap (\mathbf{R}^n \setminus \Omega)|}{|B_{\rho_x}|} |\hat{u}_{B_{\rho_x}}|$$

$$= \frac{C_1}{|B_{\rho_x}|} \int_{B_{\rho_x} \cap (\mathbf{R}^n \setminus \Omega)} |\hat{u} - \hat{u}_{B_{\rho_x}}|\, dz$$

$$\leq C_1 \fint_{B_{\rho_x}} |\hat{u} - \hat{u}_{B_{\rho_x}}|\, dz \leq cC_1\rho_x \fint_{B_{\rho_x}} |X\hat{u}|\, dz$$

$$\leq cC_1\rho_x M|X\hat{u}|(x) \leq cC_1\mu\rho_x.$$

Therefore, we have by (2.2) and (3.2)

$$
\begin{aligned}
\left|\hat{u}(x) - \hat{u}(y)\right| = \left|\hat{u}(x)\right| \\
&\le \left|\hat{u}(x) - \hat{u}_{B_{\rho_x}}\right| + |\hat{u}_{B_{\rho_x}}| \\
&\le c\rho_x M|X\hat{u}|(x) + cC_1\mu\rho_x \\
&\le cC_1\mu\rho_x \\
&\le cC_1\mu d(x, y).
\end{aligned}
$$

It follows that $\hat{u}$ is a Lipschitz function on $E_\mu \cup (\mathbf{R}^n \setminus \Omega)$ with the Lipschitz constant $cC_1\mu$.

As in [7], we can use the Kirszbraun theorem (see e.g. [17]) to extend $\hat{u}$ to a Lipschitz function v_μ defined on $\mathbf{R}^n$ with the same Lipschitz constant. Moreover, there exists μ_0 such that, for every $\mu \ge \mu_0$, supp $v_\mu \subset 3B \cap \Omega$.

In fact, let $D = 2B \cap \Omega$ and $x \in \mathbf{R}^n \setminus (3B \cap \Omega)$, we have by Lemma 2.1 that

$$
M|X\hat{u}|(x) = \sup_{B' \ni x, B' \cap 2B \neq \emptyset} \fint_{B'} |X\hat{u}|(y)\, dy \le \frac{C_d}{|2B|} \int_D |X\hat{u}|(y)\, dy,
$$

where $|B'| > |B|$, C_d is the doubling constant. Setting

$$
\mu_0 = \frac{C_d}{|2B|} \int_D |X\hat{u}|(y)\, dy,
$$

then $M|X\hat{u}|(x) \le \mu$, $\mu \ge \mu_0$, which implies $v_\mu(x) = \hat{u}(x) = 0$ for $x \in \mathbf{R}^n \setminus (3B \cap \Omega)$. So we can take the function v_μ as a test function in (1.6).

Let $\mu \ge \mu_0$ and take v_μ as a test function in (1.6) to have

$$
\int_{3B \cap \Omega} A(x, u, Xu) \cdot Xv_\mu\, dx + \int_{3B \cap \Omega} B(x, u, Xu)v_\mu\, dx = 0.
$$

Noting that $v_\mu = \hat{u}$ on $(3B \cap \Omega) \cap E_\mu$ and that supp $\hat{u} \subset D$, we have by the structure conditions on $A(x, u, \xi)$ and $B(x, u, \xi)$

$$
\begin{aligned}
\int_{D \cap E_\mu} A(x, u, Xu) \cdot X\hat{u}\, dx &+ \int_{D \cap E_\mu} B(x, u, Xu)\hat{u}\, dx \\
&\le \int_{(3B \cap \Omega) \setminus E_\mu} \left|A(x, u, Xu)\right| |Xv_\mu|\, dx + \int_{(3B \cap \Omega) \setminus E_\mu} \left|B(x, u, Xu)\right| |v_\mu|\, dx \\
&\le c\mu \int_{(3B \cap \Omega) \setminus E_\mu} \left(|u|^{p-1} + |Xu|^{p-1}\right) dx,
\end{aligned}
\tag{3.3}
$$

where in the last inequality we use the fact that $|Xv_\mu| \le c\mu$, $|v_\mu| \le cR\mu$ (see [7]).

Multiplying both sides of (3.3) by $\mu^{-(1+\delta)}$ and integrating over (μ_0, ∞), we get

$$
\begin{aligned}
L := \int_{\mu_0}^{\infty} \int_{D \cap E_\mu} \mu^{-(1+\delta)} \left(A(x, u, Xu) \cdot X\hat{u} + B(x, u, Xu)\hat{u}\right) dx\, d\mu \\
&\le c \int_{\mu_0}^{\infty} \int_{(3B \cap \Omega) \setminus E_\mu} \mu^{-\delta} \left(|u|^{p-1} + |Xu|^{p-1}\right) dx\, d\mu := P.
\end{aligned}
\tag{3.4}
$$

Interchanging the order of integration and applying (3.2), we have

$$
\begin{aligned}
P &= c \int_{3B} \int_{\mu_0}^{M|X\hat{u}|} \mu^{-\delta} \left(|u|^{p-1} + |Xu|^{p-1} \right) d\mu \, dx \\
&\leq \frac{c}{1-\delta} \int_{(3B\cap\Omega)\setminus E_{\mu_0}} \left(M|X\hat{u}| \right)^{1-\delta} \left(|u|^{p-1} + |Xu|^{p-1} \right) dx \\
&\leq c \int_{3B\cap\Omega} \left(|u|^{p-\delta} + |Xu|^{p-\delta} \right) dx + c \int_{3B\cap\Omega} \left(M|X\hat{u}| \right)^{p-\delta} dx.
\end{aligned}
\tag{3.5}
$$

Using Lemma 2.4 and Lemma 2.8, we have

$$
\begin{aligned}
c &\int_{3B\cap\Omega} \left(M|X\hat{u}| \right)^{p-\delta} dx \\
&\leq c \int_{D} |X\hat{u}|^{p-\delta} dx \\
&\leq c \int_{D} |Xu - Xu_0|^{p-\delta} dx + \frac{c}{R^{p-\delta}} \int_{2B} |u - u_0|^{p-\delta} dx \\
&\leq c \int_{D} |Xu - Xu_0|^{p-\delta} dx + \frac{c|2B|}{R^{p-\delta}} \left(\frac{1}{\mathrm{cap}_{p-\delta}(N(u-u_0), 2B)} \int_{2B} |Xu - Xu_0|^{p-\delta} dx \right),
\end{aligned}
$$

where $N(u-u_0) = \{x \in \bar{B} : u(x) = u_0(x)\}$. Since $u-u_0$ vanishes outside Ω, we have $\mathbf{R}^n \setminus \Omega \subset \{u-u_0 = 0\}$. On the other hand, by Lemma 2.7 and assumption (H_2), there exists δ_0 such that if $0 < \delta < \delta_0$, $\mathbf{R}^n \setminus \Omega$ is uniformly $(X, p-\delta)$-fat, and hence

$$
\begin{aligned}
\mathrm{cap}_{p-\delta}\left(N(u-u_0), 2B \right) &\geq \mathrm{cap}_{p-\delta}\left(\bar{B} \cap \left(\mathbf{R}^n \setminus \Omega \right), 2B \right) \\
&\geq c \, \mathrm{cap}_{p-\delta}(\bar{B}, 2B) \geq c|B|R^{-(p-\delta)}.
\end{aligned}
\tag{3.6}
$$

From (3.6) and the doubling condition, we derive

$$
\begin{aligned}
c \int_{3B\cap\Omega} \left(M|X\hat{u}| \right)^{p-\delta} dx &\leq c \int_{D} |X\hat{u}|^{p-\delta} dx \\
&\leq c \int_{D} |Xu|^{p-\delta} dx + c \int_{D} |Xu_0|^{p-\delta} dx,
\end{aligned}
\tag{3.7}
$$

and then (3.5) becomes

$$
P \leq c \int_{3B\cap\Omega} |u|^{p-\delta} dx + c \int_{3B\cap\Omega} |Xu_0|^{p-\delta} dx + c \int_{3B\cap\Omega} |Xu|^{p-\delta} dx.
\tag{3.8}
$$

As regards the estimation of L, by changing the order of integration, we have

$$
\begin{aligned}
L &= \int_{\mu_0}^{\infty} \int_{D} \mu^{-(1+\delta)} \left(A(x,u,Xu) \cdot X\hat{u} + B(x,u,Xu)\hat{u} \right) \chi_{\{M|X\hat{u}|(x)\leq\mu\}} \, dx \, d\mu \\
&= \int_{D\setminus E_{\mu_0}} \int_{M|X\hat{u}|}^{\infty} \mu^{-(1+\delta)} \left(A(x,u,Xu) \cdot X\hat{u} + B(x,u,Xu)\hat{u} \right) dx \, d\mu \\
&\quad + \int_{D\cap E_{\mu_0}} \int_{\mu_0}^{\infty} \mu^{-(1+\delta)} \left(A(x,u,Xu) \cdot X\hat{u} + B(x,u,Xu)\hat{u} \right) dx \, d\mu
\end{aligned}
$$

$$= \frac{1}{\delta} \int_{D \setminus E_{\mu_0}} \big(M|X\hat{u}|\big)^{-\delta} \big(A(x, u, Xu) \cdot X\hat{u} + B(x, u, Xu)\hat{u}\big) \, dx$$

$$+ \frac{1}{\delta} \int_{D \cap E_{\mu_0}} \mu_0^{-\delta} \big(A(x, u, Xu) \cdot X\hat{u} + B(x, u, Xu)\hat{u}\big) \, dx.$$

Since $D \setminus E_{\mu_0} = D \setminus (D \cap E_{\mu_0})$, (1.3) and (1.4) imply

$$L = \frac{1}{\delta} \int_D \big(M|X\hat{u}|\big)^{-\delta} A(x, u, Xu) \cdot X\hat{u} \, dx - \frac{1}{\delta} \int_{D \cap E_{\mu_0}} \big(M|X\hat{u}|\big)^{-\delta} A(x, u, Xu) \cdot X\hat{u} \, dx$$

$$+ \frac{1}{\delta} \int_{D \setminus E_{\mu_0}} \big(M|X\hat{u}|\big)^{-\delta} B(x, u, Xu)\hat{u} \, dx$$

$$+ \frac{1}{\delta} \int_{D \cap E_{\mu_0}} \mu_0^{-\delta} \big(A(x, u, Xu) \cdot X\hat{u} + B(x, u, Xu)\hat{u}\big) \, dx$$

$$\geq \frac{1}{\delta} \int_D \big(M|X\hat{u}|\big)^{-\delta} A(x, u, Xu) \cdot X\hat{u} \, dx$$

$$- \frac{2\alpha}{\delta} \int_{D \cap E_{\mu_0}} \big(M|X\hat{u}|\big)^{-\delta} \big(|u|^{p-1} + |Xu|^{p-1}\big)|X\hat{u}| \, dx$$

$$- \frac{\alpha}{\delta} \int_D \big(M|X\hat{u}|\big)^{-\delta} \big(|u|^{p-1} + |Xu|^{p-1}\big)|\hat{u}| \, dx$$

$$:= \frac{1}{\delta}(I_1 - 2\alpha I_2 - \alpha I_3). \tag{3.9}$$

Step 2. Next, we will estimate I_i $(i = 1, 2, 3)$ one by one.

Now for estimation of I_1. To this end, define the sets

$$D_1 = \big\{ x \in D \setminus B : M|X\hat{u}| \leq \delta\big(M_D|Xu - Xu_0|\big) \big\},$$

$$D_2 = \big\{ x \in D \setminus B : M|X\hat{u}| > \delta\big(M_D|Xu - Xu_0|\big) \big\}$$

and $B_\Omega = B \cap \Omega$. Thus

$$I_1 = \int_{B_\Omega \cup D_2} \big(M|X\hat{u}|\big)^{-\delta} \big(A(x, u, Xu) - A(x, u_0, Xu_0)\big) \cdot \eta X(u - u_0) \, dx$$

$$+ \int_{B_\Omega \cup D_2} \big(M|X\hat{u}|\big)^{-\delta} A(x, u_0, Xu_0) \cdot \eta(Xu - Xu_0) \, dx$$

$$+ \int_{D_2} \big(M|X\hat{u}|\big)^{-\delta} A(x, u, Xu) \cdot X\eta(u - u_0) \, dx$$

$$+ \int_{D_1} \big(M|X\hat{u}|\big)^{-\delta} A(x, u, Xu) \cdot X\hat{u} \, dx.$$

Since $(M|X\hat{u}|)^{-\delta} \leq |X\hat{u}|^{-\delta}$ a.e., it follows from (1.5) and (1.3) that

$$I_1 \geq \beta \int_{B_\Omega} \big(M|X\hat{u}|\big)^{-\delta} |Xu - Xu_0|^p \, dx$$

$$- \alpha \bigg(\int_{B_\Omega} |X\hat{u}|^{-\delta} \big(|u_0|^{p-1} + |Xu_0|^{p-1}\big)$$

$$\times |Xu - Xu_0| \, dx + \int_{D_2} \big(M|X\hat{u}|\big)^{-\delta} \big(|u_0|^{p-1} + |Xu_0|^{p-1}\big)|Xu - Xu_0| \, dx$$

$$+ \int_{D_2} \left(M|X\hat{u}| \right)^{-\delta} \left(|u|^{p-1} + |Xu|^{p-1} \right) \left| X\eta(u - u_0) \right| dx$$

$$+ \int_{D_1} \left(M|X\hat{u}| \right)^{-\delta} \left(|u|^{p-1} + |Xu|^{p-1} \right) |X\hat{u}| \, dx \Bigg)$$

$$:= I_{11} - \alpha(I_{12} + I_{13} + I_{14} + I_{15}). \tag{3.10}$$

Since the function $(M|X\hat{u}|)^{-\delta}$ is an A_p-weight, we obtain from Lemma 2.6 that

$$I_{11} \geq c\beta \int_{B_\Omega} \left(M|X\hat{u}| \right)^{-\delta} \left(M_{B_\Omega} |Xu - Xu_0| \right)^p dx.$$

By the doubling condition and Lemma 2.8 we see that, for $x \in \frac{B}{2} \cap \Omega$,

$$M|X\hat{u}|(x) \leq \sup_{B' \ni x, B' \subset B} \fint_{B'} |X\hat{u}| \, dy + \sup_{B' \ni x, B' \cap \partial B \neq \emptyset} \fint_{B'} |X\hat{u}| \, dy$$

$$\leq M_{B_\Omega} |X(u - u_0)| + \frac{c}{R} \left(\fint_{2B} |u - u_0|^{s'} dx \right)^{\frac{1}{s'}} + c \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s'} dx \right)^{\frac{1}{s'}}$$

$$\leq M_{B_\Omega} |X(u - u_0)| + \frac{c}{R} \left(\frac{1}{\mathrm{cap}_{s'}(N(u - u_0), 2B)} \int_{2B} |X(u - u_0)|^{s'} dx \right)^{\frac{1}{s'}}$$

$$+ c \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s'} dx \right)^{\frac{1}{s'}}$$

$$\leq M_{B_\Omega} |X(u - u_0)| + c \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s'} dx \right)^{\frac{1}{s'}}, \tag{3.11}$$

where $\max\{1, (p - \delta)_*\} < s' < p - \delta$ is such that $\mathbf{R}^n \setminus \Omega$ is uniformly (X, s')-fat and the last inequality comes from an argument similar to (3.6).

To continue, we define

$$G = \left\{ x \in \frac{B}{2} \cap \Omega : M_{B_\Omega} |X(u - u_0)| \geq c \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s'} dx \right)^{\frac{1}{s'}} \right\}.$$

So from (3.11) we see that $M|X\hat{u}| \leq c M_{B_\Omega} |X(u - u_0)|$ on G, and then

$$I_{11} \geq c \int_G \left(M_{B_\Omega} |Xu - Xu_0| \right)^{-\delta} \left(M_{B_\Omega} |Xu - Xu_0| \right)^p dx$$

$$\geq c \int_{\frac{B}{2} \cap \Omega} |Xu - Xu_0|^{p-\delta} dx - c|B| \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s'} dx \right)^{\frac{p-\delta}{s'}}$$

$$\geq c \int_{\frac{B}{2} \cap \Omega} |Xu|^{p-\delta} dx - c \int_D |Xu_0|^{p-\delta} dx - c|B| \left(\frac{1}{|2B|} \int_D |Xu|^{s'} dx \right)^{\frac{p-\delta}{s'}}. \tag{3.12}$$

Using the fact $X\hat{u} = X(u - u_0)$ on B and Young's inequality, we have

$$I_{12} \leq c \int_D \left(|u_0|^{p-\delta} + |Xu_0|^{p-\delta} \right) dx + c\varepsilon \int_D |Xu - Xu_0|^{p-\delta} dx$$

$$\leq c \int_D |u_0|^{p-\delta} dx + c \int_D |Xu_0|^{p-\delta} dx + c\varepsilon \int_D |Xu|^{p-\delta} dx. \tag{3.13}$$

Next from the definition of D_2 and Lemma 2.4, we see

$$I_{13} \leq c\delta^{-\delta} \int_D \left(M_D|Xu - Xu_0|\right)^{1-\delta} \left(|u_0|^{p-1} + |Xu_0|^{p-1}\right) dx$$

$$\leq c \int_D \left(|u_0|^{p-\delta} + |Xu_0|^{p-\delta}\right) dx + c\varepsilon \int_D |Xu - Xu_0|^{p-\delta} dx$$

$$\leq c \int_D |u_0|^{p-\delta} dx + c \int_D |Xu_0|^{p-\delta} dx + c\varepsilon \int_D |Xu|^{p-\delta} dx. \tag{3.14}$$

For I_{14}, we have by using $|X\eta(u - u_0)| \leq |X\hat{u}| + |Xu - Xu_0|$

$$I_{14} \leq c \int_{D_2} \left(M|X\hat{u}|\right)^{-\delta} \left(|u|^{p-1} + |Xu - Xu_0|^{p-1} + |Xu_0|^{p-1}\right) \left|X\eta(u - u_0)\right| dx$$

$$\leq c \int_{D_2} \left(M|X\hat{u}|\right)^{-\delta} \left(|u|^{p-1} + |Xu_0|^{p-1}\right) \left(|X\hat{u}| + |Xu - Xu_0|\right) dx$$

$$\quad + c \int_{D_2} \left(M|X\hat{u}|\right)^{-\delta} |Xu - Xu_0|^{p-1} |X\eta| |u - u_0| dx$$

$$\leq c \int_{D_2} \left(M|X\hat{u}|\right)^{-\delta} \left(|u|^{p-1} + |Xu_0|^{p-1}\right) |X\hat{u}| dx$$

$$\quad + c \int_{D_2} \left(M|X\hat{u}|\right)^{-\delta} \left(|u|^{p-1} + |Xu_0|^{p-1}\right) |Xu - Xu_0| dx$$

$$\quad + \frac{c}{R} \int_{D_2} \left(M|X\hat{u}|\right)^{-\delta} |Xu - Xu_0|^{p-1} |u - u_0| dx$$

$$:= K_1 + K_2 + K_3.$$

Using Young's inequality and (3.7), we get

$$K_1 \leq c \int_D \left(|u|^{p-1} + |Xu_0|^{p-1}\right) |X\hat{u}|^{1-\delta} dx$$

$$\leq c \int_D |u|^{p-\delta} dx + c \int_D |Xu_0|^{p-\delta} dx + c\varepsilon \int_D |Xu|^{p-\delta} dx. \tag{3.15}$$

By the definition of D_2 and noting that $|X(u - u_0)| \leq M_D|X(u - u_0)|$ a.e. D,

$$K_2 \leq c\delta^{-\delta} \int_D |Xu - Xu_0|^{1-\delta} \left(|u|^{p-1} + |Xu_0|^{p-1}\right) dx$$

$$\leq c \int_D |u|^{p-\delta} dx + c \int_D |Xu_0|^{p-\delta} dx + c\varepsilon \int_D |Xu|^{p-\delta} dx. \tag{3.16}$$

Finally, by Young's inequality,

$$K_3 \leq \frac{c\delta^{-\delta}}{R} \int_{D_2} \left(M_D|Xu - Xu_0|\right)^{-\delta} |Xu - Xu_0|^{p-1} |u - u_0| dx$$

$$\leq \frac{c\delta^{-\delta}}{R} \int_D |Xu - Xu_0|^{p-1-\delta} |u - u_0| dx$$

$$\leq c\varepsilon \int_D |Xu - Xu_0|^{p-\delta} dx + c \int_D \left|\frac{u - u_0}{R}\right|^{p-\delta} dx. \tag{3.17}$$

In order to estimate the second component of the right-hand side, we let $s'' = (p - \delta)(1 - \vartheta)$, where $0 < \vartheta < \frac{p-\delta}{p-\delta+Q}$ if $p - \delta \le Q$ and $0 < \vartheta < \min\{\frac{p-\delta-Q}{p-\delta}, \frac{1}{2}\}$ if $p - \delta > Q$. Denote

$$\kappa = \begin{cases} \frac{Q}{Q-s''}, & s'' < Q, \\ 2, & s'' > Q, \end{cases}$$

then $\kappa s'' \ge p - \delta$. Using Lemma 2.7 and Lemma 2.8, we derive

$$\left(\fint_{2B} \left| \frac{u - u_0}{R} \right|^{p-\delta} dx \right)^{\frac{1}{p-\delta}} \le cR^{-1} \left(\fint_{2B} |u - u_0|^{\kappa s''} dx \right)^{\frac{1}{\kappa s''}}$$

$$\le cR^{-1} \left(\frac{1}{\mathrm{cap}_{s''}(N(u - u_0), 2B)} \int_{2B} \left| X(u - u_0) \right|^{s''} dx \right)^{\frac{1}{s''}}$$

$$\le c \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s''} \right)^{\frac{1}{s''}},$$

where the proof of the last inequality is similar to (3.6). Therefore,

$$c \int_{2B} \left| \frac{u - u_0}{R} \right|^{p-\delta} dx \le c|2B| \left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s''} dx \right)^{\frac{p-\delta}{s''}}. \tag{3.18}$$

Inserting (3.18) into (3.17), we have

$$K_3 \le c\varepsilon \int_D |Xu|^{p-\delta} dx + c \int_D |Xu_0|^{p-\delta} dx$$

$$+ c|2B| \left(\frac{1}{|2B|} \int_D |Xu|^{s''} dx \right)^{\frac{p-\delta}{s''}}. \tag{3.19}$$

A combination of (3.15), (3.16) and (3.19) implies

$$I_{14} \le c \int_D \left(|u|^{p-\delta} + |Xu_0|^{p-\delta} \right) dx$$

$$+ c\varepsilon \int_D |Xu|^{p-\delta} dx + c|2B| \left(\frac{1}{|2B|} \int_D |Xu|^{s''} dx \right)^{\frac{p-\delta}{s''}}. \tag{3.20}$$

The definition of D_1 and Lemma 2.4 give

$$I_{15} \le c \int_{D_1} \left(M|X\hat{u}| \right)^{1-\delta} \left(|u|^{p-1} + |Xu|^{p-1} \right) dx$$

$$\le c\delta^{1-\delta} \int_D \left(M_D |Xu - Xu_0| \right)^{1-\delta} \left(|u|^{p-1} + |Xu|^{p-1} \right) dx$$

$$\le c\delta^{1-\delta} \left[\int_D |Xu - Xu_0|^{p-\delta} dx + \int_D \left(|u|^{p-\delta} + |Xu|^{p-\delta} \right) dx \right]$$

$$\le c \int_D \left(|u|^{p-\delta} + |Xu_0|^{p-\delta} \right) dx + c\delta \int_D |Xu|^{p-\delta} dx. \tag{3.21}$$

The previous estimates show that

$$I_1 \geq c \int_{\frac{B}{2} \cap \Omega} |Xu|^{p-\delta}\, dx - c \int_D \left(|u|^{p-\delta} + |u_0|^{p-\delta} + |Xu_0|^{p-\delta}\right) dx$$

$$- c(\varepsilon + \delta) \int_D |Xu|^{p-\delta}\, dx - c|2B| \left(\frac{1}{|2B|} \int_D |Xu|^t\, dx\right)^{\frac{p-\delta}{t}}, \tag{3.22}$$

where $t = \max\{s', s''\} < p - \delta$.

Now we address the estimation of I_2. Using (3.7), we have

$$I_2 \leq \int_D |u|^{p-1}|X\hat{u}|^{1-\delta}\, dx + \int_{D \cap E_{\mu_0}} \left(M|X\hat{u}|\right)^{-\delta} |Xu|^{p-1}|X\hat{u}|\, dx$$

$$\leq c \int_D |u|^{p-\delta}\, dx + c\varepsilon \int_D |X\hat{u}|^{p-\delta}\, dx + \int_{D \cap E_{\mu_0}} |Xu|^{p-1}\left(M|X\hat{u}|\right)^{1-\delta}\, dx$$

$$\leq c \int_D \left(|u|^{p-\delta} + |Xu_0|^{p-\delta}\right) dx$$

$$+ c\varepsilon \int_D |Xu|^{p-\delta}\, dx + \int_{D \cap E_{\mu_0}} |Xu|^{p-1}\left(M|X\hat{u}|\right)^{1-\delta}\, dx. \tag{3.23}$$

To estimate the last integral in (3.23), let $0 < \tau < \frac{1}{2}$ and $x \in D \cap E_{\mu_0}$. If $|Xu| \geq \tau^{-1}\mu_0$, then $M|X\hat{u}| \leq \mu_0 \leq \tau|Xu|$ and

$$|Xu|^{p-1}\left(M|X\hat{u}|\right)^{1-\delta} \leq |Xu|^{p-1}\left(\tau|Xu|\right)^{1-\delta} = \tau^{1-\delta}|Xu|^{p-\delta}; \tag{3.24}$$

if $|Xu| < \tau^{-1}\mu_0$, then

$$|Xu|^{p-1}\left(M|X\hat{u}|\right)^{1-\delta} \leq \left(\tau^{-1}\mu_0\right)^{p-1}\mu_0^{1-\delta} \leq \tau^{1-p}\mu_0^{p-\delta}. \tag{3.25}$$

By (3.24) and (3.25), we deduce that, for any $x \in D \cap E_{\mu_0}$,

$$|Xu|^{p-1}\left(M|X\hat{u}|\right)^{1-\delta} \leq c\left(\tau^{1-\delta}|Xu|^{p-\delta} + \tau^{1-p}\mu_0^{p-\delta}\right). \tag{3.26}$$

For the second term in (3.26), we first observe from the proof of (3.11) that

$$\frac{1}{R}\left(\fint_{2B} |u - u_0|^{s'}\, dx\right)^{\frac{1}{s'}} \leq c\left(\frac{1}{|2B|} \int_D |Xu - Xu_0|^{s'}\, dx\right)^{\frac{1}{s'}}.$$

Noticing $\mu_0 = \frac{c}{|2B|} \int_D |X\hat{u}|\, dx$, we have from Hölder's inequality

$$\tau^{1-p}\mu_0^{p-\delta} \leq c\tau^{1-p}\left(\frac{1}{|2B|} \int_D \left|X\eta(u - u_0) + \eta X(u - u_0)\right| dx\right)^{p-\delta}$$

$$\leq c\tau^{1-p}\left(\frac{1}{R}\left(\fint_{2B} |u - u_0|^{s'}\, dx\right)^{\frac{1}{s'}}\right)^{p-\delta}$$

$$+ c\tau^{1-p}\left(\frac{1}{|2B|} \int_D \left|X(u - u_0)\right|^{s'} dx\right)^{\frac{p-\delta}{s'}}$$

$$\le c\tau^{1-p}\left(\frac{1}{|2B|}\int_D \left|X(u-u_0)\right|^{s'} dx\right)^{\frac{p-\delta}{s'}}$$

$$\le c\tau^{1-p}\left(\frac{1}{|2B|}\int_D |Xu_0|^{p-\delta} dx\right) + c\tau^{1-p}\left(\frac{1}{|2B|}\int_D |Xu|^{s'} dx\right)^{\frac{p-\delta}{s'}}. \tag{3.27}$$

By (3.26) and (3.27), it follows that

$$\int_{D\cap E_{\mu_0}} |Xu|^{p-1}\left(M|X\hat{u}|\right)^{1-\delta} dx$$

$$\le c\tau^{1-\delta}\int_D |Xu|^{p-\delta} dx + c\int_D |Xu_0|^{p-\delta} dx + c\tau^{1-p}|2B|\left(\frac{1}{|2B|}\int_D |Xu|^{s'} dx\right)^{\frac{p-\delta}{s'}}. \tag{3.28}$$

Taking (3.28) into (3.23), we have

$$I_2 \le c\int_D |u|^{p-\delta} dx + c\int_D |Xu_0|^{p-\delta} dx$$

$$+ c\left(\varepsilon + \tau^{1-\delta}\right)\int_D |Xu|^{p-\delta} dx + c\tau^{1-p}|2B|\left(\frac{1}{|2B|}\int_D |Xu|^{s'} dx\right)^{\frac{p-\delta}{s'}}. \tag{3.29}$$

For the estimation of I_3: From (2.3), Lemma 2.8 and a similar process to the proof of (3.18), we have

$$I_3 \le c\int_D \left(|u|^{p-1} + |Xu|^{p-1}\right)|\hat{u}|^{1-\delta} dx$$

$$\le \int_D |u|^{p-\delta} + c\varepsilon\int_D |Xu|^{p-\delta} dx + c\int_D |u-u_0|^{p-\delta} dx$$

$$\le c\int_D \left(|u|^{p-\delta} + |Xu_0|^{p-\delta}\right) dx$$

$$+ c\varepsilon\int_D |Xu|^{p-\delta} dx + c|2B|\left(\frac{1}{|2B|}\int_D |Xu|^{t} dx\right)^{\frac{p-\delta}{t}}, \tag{3.30}$$

where $t = \max\{s', s''\} < p - \delta$.

Step 3. Taking into account (3.4), (3.8), substituting (3.22), (3.29) and (3.30) into (3.9), and letting $\varepsilon = \tau^{1-\delta}$, it follows that

$$\int_{\frac{B}{2}\cap\Omega} |Xu|^{p-\delta} dx$$

$$\le c\int_{3B\cap\Omega} \left(|u|^{p-\delta} + |u_0|^{p-\delta} + |Xu_0|^{p-\delta}\right) dx$$

$$+ c\left(\delta + \tau^{1-\delta}\right)\int_{3B\cap\Omega} |Xu|^{p-\delta} dx + c\tau^{1-p}|2B|\left(\frac{1}{|2B|}\int_{3B\cap\Omega} |Xu|^{t} dx\right)^{\frac{p-\delta}{t}}. \tag{3.31}$$

To sum up the cases $3B \subset \Omega$ and $3B\backslash\Omega \neq \emptyset$, we let

$$g(x) = \begin{cases} |Xu|^{t}, & x \in \Omega, \\ 0, & x \in \mathbf{R}^{n}\backslash\Omega, \end{cases}$$

and

$$f(x) = \begin{cases} (|u - u_0| + |u_0| + |Xu_0|)^t, & x \in \Omega, \\ 0, & x \in \mathbf{R}^n \backslash \Omega. \end{cases}$$

Thus we have from (3.1) and (3.31)

$$\fint_{\frac{B}{2}} g^q \, dx \le b \left[\left(\fint_{3B} g \, dx \right)^q + \fint_{3B} f^q \, dx \right] + \theta \fint_{3B} g^q \, dx,$$

where $q = \frac{p-\delta}{t}$, $\theta = c(\delta + \tau^{1-\delta})$ and $b = c\tau^{1-p}$. Choosing τ, δ small enough, we see by Lemma 2.3 that there exists $t_1 = p - \delta + \varepsilon_0$, for some $\varepsilon_0 > 0$, such that $|Xu| \in L^{t_1}(\Omega)$.

Furthermore, we will show that there exists $t_2 > r = p - \delta$ such that $u \in L^{t_2}(\Omega)$. Since $u - u_0 \in W_{X,0}^{1,r}(\Omega)$, we obtain from Lemma 2.2 that, for $r < Q$, $r^* = Qr/(Q - r)$,

$$\left(\int_\Omega |u - u_0|^{r^*} \, dx \right)^{\frac{1}{r^*}} \le C(\Omega) \left(\int_\Omega |X(u - u_0)|^r \, dx \right)^{\frac{1}{r}} < \infty.$$

Taking $t_2 = \min\{s, r^*\} > r$, we have

$$\left(\int_\Omega |u|^{t_2} \, dx \right)^{\frac{1}{t_2}} \le \left(\int_\Omega |u - u_0|^{t_2} \, dx \right)^{\frac{1}{t_2}} + \left(\int_\Omega |u_0|^{t_2} \, dx \right)^{\frac{1}{t_2}}$$

$$\le C \left(\int_\Omega |u - u_0|^{r^*} \, dx \right)^{\frac{1}{r^*}} + \left(\int_\Omega |u_0|^{t_2} \, dx \right)^{\frac{1}{t_2}}$$

and then $u \in L^{t_2}(\Omega)$ by $u_0 \in L^s(\Omega)$. If $r \ge Q$ then we can apply the above reasoning for any $r^* < \infty$ to obtain $u \in L^{t_2}(\Omega)$.

We set $\tilde{p} = \min\{t_1, t_2\} > p - \delta$ and $u \in W_X^{1,\tilde{p}}(\Omega)$. Repeating the preceding reasoning, we know that there exists $\tilde{\delta} > 0$ such that $u \in W_X^{1,p+\delta}(\Omega)$ and the proof is complete.

4 Conclusions

In this paper, we obtained the global higher integrability for very weak solutions to the Dirichlet problem for a nonlinear subelliptic equation on Carnot-Carathéodory spaces which implies that such solutions are classical weak solutions. It is a generalization of the corresponding result in the classical Euclidean setting.

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References

1. Iwaniec, T, Sbordone, C: Weak minima of variational integrals. J. Reine Angew. Math. **454**, 143-161 (1994)
2. Lewis, JL: On very weak solutions of certain elliptic systems. Commun. Partial Differ. Equ. **18**(9-10), 1515-1537 (1993)
3. Giannetti, F, Passarelli di Napoli, A: On very weak solutions of degenerate p-harmonic equations. Nonlinear Differ. Equ. Appl. **14**(5-6), 739-751 (2007)
4. Kinnunen, J, Lewis, JL: Very weak solutions of parabolic systems of p-Laplacian type. Ark. Mat. **40**(1), 105-132 (2002)
5. Xie, S, Fang, A: Global higher integrability for the gradient of very weak solutions of a class of nonlinear elliptic systems. Nonlinear Anal. **53**(7-8), 1127-1147 (2003)
6. Fattorusso, L, Molica Bisci, G, Tarsia, A: A global regularity result for some degenerate elliptic systems. Nonlinear Anal. **125**, 54-66 (2015)
7. Zatorska-Goldstein, A: Very weak solutions of nonlinear subelliptic equations. Ann. Acad. Sci. Fenn., Math. **30**(2), 407-436 (2005)
8. Kilpeläinen, T, Koskela, P: Global integrability of the gradients of solutions to partial differential equations. Nonlinear Anal. **23**(7), 899-909 (1994)
9. Danielli, D, Garofalo, N, Phuc, NC: Inequalities of Hardy-Sobolev type in Carnot-Carathéodory spaces. In: Sobolev Spaces in Mathematics. I: Sobolev Type Inequalities, pp. 117-151. Springer, New York (2009)
10. Hörmander, L: Hypoelliptic second order differential equations. Acta Math. **119**, 147-171 (1967)
11. Chow, WL: Über systeme von linearen partiellen differentialgleichungen erster ordnung. Math. Ann. **117**, 98-105 (1939)
12. Nagel, A, Stein, EM, Wainger, S: Balls and metrics defined by vector fields. I: basic properties. Acta Math. **155**, 103-147 (1985)
13. Garofalo, N, Nhieu, DM: Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces. J. Anal. Math. **74**, 67-97 (1998)
14. Hajłasz, P, Koskela, P: Sobolev met Poincaré. Mem. Am. Math. Soc. **688**, 1-101 (2000)
15. Lu, G: Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications. Rev. Mat. Iberoam. **8**(3), 367-439 (1992)
16. Danielli, D: Regularity at the boundary for solutions of nonlinear subelliptic equations. Indiana Univ. Math. J. **44**(1), 269-286 (1995)
17. Federer, H: Geometric Measure Theory. Die Grundlehren der Mathematischen Wissenschaften, vol. 153. Springer, New York (1969)