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Existence of ground state solutions for a class of quasilinear elliptic systems in Orlicz-Sobolev spaces

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Abstract
In this paper, we investigate the following nonlinear and non-homogeneous elliptic system:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) + V_1(x)a_1(|u|)u = F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(a_2(|\nabla v|)\nabla v) + V_2(x)a_2(|v|)v = F_v(x,u,v) & \text{in } \mathbb{R}^N, \\ (u,v) \in W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N), \end{cases}$$

where $\phi_i(t) = a_i(|t|)t (i = 1,2)$ are two increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, functions $V_i(i = 1,2)$ and F are 1-periodic in x, and F satisfies some (ϕ_1,ϕ_2)-superlinear Orlicz-Sobolev conditions. By using a variant mountain pass lemma, we obtain that the system has a ground state.

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Keywords: Orlicz-Sobolev spaces; quasilinear; critical point; ground state

1 Introduction

In this paper, we consider the following nonlinear and non-homogeneous elliptic system in Orlicz-Sobolev spaces:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) + V_1(x)a_1(|u|)u = F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(a_2(|\nabla v|)\nabla v) + V_2(x)a_2(|v|)v = F_v(x,u,v) & \text{in } \mathbb{R}^N, \\ (u,v) \in W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a_i(i = 1,2) : (0,+\infty) \to \mathbb{R}$ are two functions satisfying:

(ϕ_1) *$\phi_i(i = 1,2) : \mathbb{R} \to \mathbb{R}$ defined by*

$$\phi_i(t) = \begin{cases} a_i(|t|)t & \textit{for } t \neq 0, \\ 0 & \textit{for } t = 0, \end{cases} \tag{1.2}$$

are two increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$;

(ϕ_2)

$$1 < l_i := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \le \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} =: m_i < \min\{N, l_i^*\},$$

where

$$\Phi_i(t) := \int_0^t \phi_i(s)\,ds, \quad t \in [0,\infty) \quad \textit{and} \quad l_i^* := \frac{l_i N}{N - l_i},$$

$V_i (i = 1,2)$ satisfy

(V_1) $V_i(i = 1,2) \in C(\mathbb{R}^N, \mathbb{R})$ *are 1-periodic in* $x_1, \dots, x_N$ *(called 1-periodic in* x *for short);*

(V_2) *there exist two constants* $\alpha_1, \alpha_2 > 0$ *such that*

$$\alpha_1 \le \min\{V_1(x), V_2(x)\} \le \max\{V_1(x), V_2(x)\} \le \alpha_2 \quad \textit{for all } x \in \mathbb{R}^N,$$

and F satisfies

(F_1) $F \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R})$ *is 1-periodic in* x, $F(x,0,0) = 0$ *for all* $x \in \mathbb{R}^N$.

Set $a_2 = a_1$, $v = u$, $V_2 = V_1$ and $F(x,u,v) = F(x,v,u)$. Then system (1.1) reduces to the following quasilinear elliptic equation:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) + V_1(x)a_1(|u|)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,\Phi_1}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

When $a_1(|t|)t = |t|^{p-2}t (p > 1)$, equation (1.3) reduces to the following well-known p-Laplacian equation:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V_1(x)|u|^{p-2}u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases} \tag{1.4}$$

To investigate the solutions of p-Laplacian equations like (1.4), the variational method has become one of useful tools over the past several decades (see [1] and the references therein). In most of the references, to ensure the boundedness of the Palais-Smale ((PS) for short) sequence of the energy functional, the following growth condition due to Ambrosetti-Rabinowitz [1] was always assumed for the nonlinearity f:

(AR) there exists $\mu > p$ *such that*

$$0 < \mu F(x,u) \le uf(x,u) \quad \textit{for all } u \ne 0,$$

where, and in the sequel, $F(x,u) = \int_0^u f(x,s)\,ds$. (AR) implies that there exist two positive constants c_1, c_2 such that

$$F(x,u) \ge c_1|u|^\mu - c_2 \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

which shows that (AR) is a p-superlinear growth condition. Based on the fact that the (PS) condition can be replaced by the weaker Cerami condition for some deformation theorems which are the footstone for minimax methods, some new p-superlinear growth conditions were established in order to weaken (AR). For example, in [2], for the case $p = 2$, Ding and Szulkin replaced (AR) with conditions:

(f_1) $\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^p} = +\infty$ *uniformly in* $x \in \mathbb{R}^N$;

(f_2) $\mathcal{F}(x,u) > 0$ *for all* $u \neq 0$, *and* $|f(x,u)|^\tau \le c_3\mathcal{F}(x,u)|u|^\tau$ *for some* $c_3 > 0, \tau > \max\{1, \frac{N}{2}\}$ *and all* (x,u) *with* $|u|$ *large enough, where* $\mathcal{F}(x,u) = f(x,u)u - 2F(x,u)$.
They proved that (f_1) and (f_2) hold if the nonlinearity f satisfies (AR) and a subcritical growth condition that $|f(x,u)| \le c_4(|u| + |u|^{q-1})$ for some $c_4 > 0, q \in (2, 2^*)$ and all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, where $2^* = \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = \infty$ if $N = 1$ or $N = 2$. Some conditions similar to (f_2) were also introduced in [3] for the case $p = 2$ and in [4] for the case $p > 1$. Moreover, in [5], Liu proved the existence of ground state for equation (1.4) when the nonlinearity f satisfies (f_1), the following p-superlinear growth condition:

(f_3) *there exists* $\theta \ge 1$ *such that* $\theta\mathcal{F}(x,u) \ge \mathcal{F}(x,su)$ *for all* $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ *and* $s \in [0,1]$, *where* $\mathcal{F}(x,u) = f(x,u)u - pF(x,u)$,

and some reasonable assumptions. Instead of minimizing the energy functional on the Nehari manifold, they obtained that (1.4) has a nontrivial solution by a mountain pass type argument, and then, by using a technique of Jeanjean and Tanaka in [6], they obtained that (1.4) has a ground state. (f_1) and (f_3) are different from (AR). Indeed, in [5], an example which satisfies (f_1) and (f_3) but does not satisfy (AR) was given, that is,

$$f(x,u) = |t|^{p-2}\log\bigl(1 + |t|\bigr),$$

and in [3], an example which satisfies (AR) but does not satisfy (f_3) was also given when $p = 2$, that is,

$$f(x.u) = 3|u|^2 \int_0^u |t|^{1+\sin t} t\,dt + |u|^{4+\sin u}u. \tag{1.5}$$

Under assumption (ϕ_1), equations like (1.3) may be allowed to possess more complicated nonlinear or non-homogeneous operator ϕ_1, which can be used to model many phenomena (see [7, 8]). Based on these interesting facts, this type of equations has caused great interest among scholars in recent years. In Clément et al. [9], the authors firstly studied the existence of nontrivial solution for the following equation in Orlicz-Sobolev spaces by the variational method:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and f satisfies the following (AR) type condition for ϕ-Laplacian operator and some reasonable assumptions:

$(\mathrm{AR})^*$ *there exist* $\mu > \limsup_{t\to+\infty} \frac{t\phi_1(t)}{\Phi_1(t)}$ *and* $R_1 > 0$ *such that*

$$0 \le \mu F(x,u) \le uf(x,u) \quad \textit{for all } (x,u) \in \overline{\Omega} \times \mathbb{R} \textit{ with } |u| \ge R_1.$$

From then on, the variational method has been used widely to study the existence and multiplicity of solutions for this type of elliptic equations, and some growth conditions for the nonlinearity f in the case of p-Laplacian type were extended to the case of ϕ-Laplacian type (for example, see [8, 10, 11]). However, there are very few results regarding the existence of ground state for equations like (1.3). In [12], by using the mountain pass lemma and the Nehari manifold method, Alves and Silva proved the existence of nonnegative ground state for the following ϕ-Laplacian equation with autonomous nonlinearity f:

$$\begin{cases} -\operatorname{div}(a_1(|\nabla u|)\nabla u) + V_1(\epsilon x)a_1(|u|)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,\Phi_1}(\mathbb{R}^N), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying an (AR) type condition and some reasonable assumptions, ϵ is a positive parameter, and $V_1 : \mathbb{R}^N \to \mathbb{R}$ is a continuous function belonging to the autonomous case $V_1(x) \equiv \mu$ (see Theorem 3.4 in [12]) or the nonautonomous case

$$V_\infty := \liminf_{|x|\to\infty} V_1(x) > V_0 := \inf_{\mathbb{R}^N} V_1(x) > 0 \quad \text{(see Theorem 4.11 in [12]).}$$

For the systems like (1.1), on the whole space $\mathbb{R}^N$, to the best of our knowledge, there is no paper to study the existence and multiplicity of solutions by the variational method, except for [13]. In [13], we investigated system (1.1) with $V_i(x)(i = 1,2) : \mathbb{R}^N \to \mathbb{R}$ satisfying (V_2) and $a_i(i = 1,2) : (0,+\infty) \to \mathbb{R}$ satisfying (ϕ_1) and (ϕ_2). By using the least action principle, we obtained that system (1.1) has at least one nontrivial solution if $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function, $F(x,0,0) = 0$ and satisfies

(1) *there exist constants* $p_i \in [m_i, l_i^*)(i = 1,2)$, $\max\{\frac{1}{p_1}, \frac{1}{p_2}\} \le q_1 < q_2 < \cdots < q_k < \min\{\frac{l_1}{p_1}, \frac{l_2}{p_2}\}$, *and functions* $a_{1j}, a_{2j}, a_{3j}, a_{4j} \in L^{\frac{1}{1-q_j}}(\mathbb{R}^N, [0,+\infty))(j = 1,2,\ldots,k)$ *such that*

$$\bigl|F_u(x,u,v)\bigr| \le \sum_{j=1}^k a_{1j}(x)|u|^{p_1q_j-1} + \sum_{j=1}^k a_{2j}(x)|v|^{\frac{p_2(p_1q_j-1)}{p_1}},$$

$$\bigl|F_v(x,u,v)\bigr| \le \sum_{j=1}^k a_{3j}(x)|u|^{\frac{p_1(p_2q_j-1)}{p_2}} + \sum_{j=1}^k a_{4j}(x)|v|^{p_2q_j-1}$$

for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$;

(2) *there exist an open set* $\Omega \subset \mathbb{R}^N$ *with* $|\Omega| > 0$ *and constants* $\alpha_0 \in [1,l_1)$, $\beta_0 \in [1,l_2)$, $\delta > 0$, $c > 0$ *and* $\iota,\kappa \in \mathbb{R}$ *with* $\iota^2 + \kappa^2 \neq 0$ *such that*

$$F(x,\iota t,\kappa t) \ge c\bigl(|\iota t|^{\alpha_0} + |\kappa t|^{\beta_0}\bigr) \quad \textit{for all } (x,t) \in \Omega \times [0,\delta].$$

Moreover, when F satisfies an additional symmetric condition, by using the genus theory, we also obtained that system (1.1) has infinitely many solutions.

On the whole space $\mathbb{R}^N$, the main difficulty for this type of elliptic equations and systems without the (AR) type conditions is the lack of compactness of the Sobolev embedding, which is crucial to ensure the boundedness of (PS) or Cerami sequence. To overcome this difficulty, the usual way is to reconstruct the compactness of the Sobolev embedding, which can be done by assuming that V_1 and f possess the radially symmetric structure (that is, V_1 and f depend on $|x|$) and then choosing a radially symmetric function subspace as the working space (see [8, 14, 15]) or by assuming that V_1 is coercive and then choosing a subspace depending on V_1 as the working space (see [3, 4, 16, 17]). Then radial and nonradical solutions can be obtained, respectively. When V_1 is bounded and V_1, f are without the radially symmetric structure, the compactness of the Sobolev embedding will be lost. For this situation, to ensure the boundedness of (PS) or Cerami sequence, a useful way is to assume that V_1 and f satisfy some specific periodicity conditions (see [5, 8, 18]), and another useful way is to assume that the nonlinearity satisfies a sublinear growth condition such that the energy functional is coercive (see [13]).

In this paper, we study the existence of ground state for system (1.1) under the assumption that $V_i (i = 1, 2)$ and F are 1-periodic in x. Motivated by [5], we also obtain that system (1.1) has a nontrivial solution by a variant mountain pass lemma, and then by using a technique of Jeanjean and Tanaka in [6], we obtain the existence of ground state. We manage to extend the p-superlinear growth conditions (AR) and (f_1) with (f_2) for p-Laplacian equations to (ϕ_1, ϕ_2)-superlinear growth conditions in the Orlicz-Sobolev space (called (ϕ_1, ϕ_2)-superlinear Orlicz-Sobolev conditions for short) for (ϕ_1, ϕ_2)-Laplacian systems, respectively (see (F_3)-(F_5) in Section 3). Since the system case is different from the scalar case, we will come across some new difficulties, and more computing skills are needed in the process of our proofs. We point out that our results are different from those in [12] and [5] even if system (1.1) reduces to equations (1.3) and (1.4).

This paper is organized as follows. In Section 2, we recall some preliminary knowledge on Orlicz and Orlicz-Sobolev spaces. In Section 3, we give our main results and complete the proofs. In Section 4, we present some examples to illustrate our results.

2 Preliminaries

In this section, we introduce some fundamental notions and important properties about Orlicz and Orlicz-Sobolev spaces. We refer the reader for more details to the books [19, 20] and the references therein.

First of all, we recall the notion of N-function. Let $\phi : [0, +\infty) \to [0, +\infty)$ be a right-continuous, monotone increasing function with

(1) $\phi(0) = 0$;

(2) $\lim_{t\to+\infty} \phi(t) = +\infty$;

(3) $\phi(t) > 0$ whenever $t > 0$.

Then the function defined on $[0, +\infty)$ by $\Phi(t) = \int_0^t \phi(s)\, ds$ is called an N-function. It is obvious that $\Phi(0) = 0$ and Φ is strictly increasing and convex in $[0, +\infty)$.

An N-function Φ satisfies a Δ_2-condition globally (or near infinity) if

$$\sup_{t>0} \frac{\Phi(2t)}{\Phi(t)} < +\infty \quad \left(\text{or } \limsup_{t\to+\infty} \frac{\Phi(2t)}{\Phi(t)} < +\infty\right),$$

which implies that there exists a constant $K > 0$ such that $\Phi(2t) \le K\Phi(t)$ for all $t \ge 0$ (or $t \ge t_0 > 0$). Φ satisfies a Δ_2-condition globally (or near infinity) if and only if for any given $c \ge 1$, there exists a constant $K_c > 0$ such that $\Phi(ct) \le K_c\Phi(t)$ for all $t \ge 0$ (or $t \ge t_0 > 0$).

For the N-function Φ, the complement of Φ is given by

$$\widetilde{\Phi}(t) = \max_{s\ge0} \{ts - \Phi(s)\} \quad \text{for } t \ge 0.$$

$\widetilde{\Phi}$ is also an N-function and $\widetilde{\widetilde{\Phi}} = \Phi$. In addition, we have Young's inequality, that is,

$$st \leq \Phi(s) + \widetilde{\Phi}(t) \quad \text{for all } s, t \geq 0, \tag{2.1}$$

and the following inequality (see [21], Lemma A.2):

$$\widetilde{\Phi}\bigl(\phi(t)\bigr) \leq \Phi(2t) \quad \text{for all } t \geq 0. \tag{2.2}$$

Now we recall the Orlicz space $L^{\Phi}(\Omega)$ associated with Φ. When Φ satisfies the Δ_2-condition globally, the Orlicz space $L^{\Phi}(\Omega)$ is the vectorial space of measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\int_{\Omega} \Phi\bigl(|u|\bigr)\, dx < +\infty,$$

where $\Omega \subset \mathbb{R}^N$ is an open set. $L^{\Phi}(\Omega)$ is a Banach space endowed with Luxemburg norm

$$\|u\|_{\Phi} := \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\} \quad \text{for } u \in L^{\Phi}(\Omega).$$

Particularly, when $\Phi(t) = |t|^p (1 < p < +\infty)$, the corresponding Orlicz space $L^{\Phi}(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ and the corresponding Luxemburg norm $\|u\|_{\Phi}$ is equal to the classical $L^p(\Omega)$ norm, that is,

$$\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} \bigl|u(x)\bigr|^p\, dx\right)^{\frac{1}{p}} \quad \text{for } u \in L^p(\Omega).$$

When $\Omega = \mathbb{R}^N$, we denote $\|u\|_{L^p(\mathbb{R}^N)}$ by $\|u\|_p$.

The fact that Φ satisfies the Δ_2-condition globally implies that

$$u_n \to u \quad \text{in } L^{\Phi}(\Omega) \quad \Longleftrightarrow \quad \int_{\Omega} \Phi\bigl(|u_n - u|\bigr)\, dx \to 0. \tag{2.3}$$

By the above Young's inequality (2.1), the following generalized Hölder's inequality

$$\left|\int_{\Omega} uv\, dx\right| \leq 2\|u\|_{\Phi}\|v\|_{\widetilde{\Phi}} \quad \text{for all } u \in L^{\Phi}(\Omega) \text{ and all } v \in L^{\widetilde{\Phi}}(\Omega) \tag{2.4}$$

can be obtained (see [19, 20]).

Define

$$W^{1,\Phi}(\Omega) := \left\{u \in L^{\Phi}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), i = 1, \ldots, N\right\}$$

with the norm

$$\|u\|_{1,\Phi} := \|u\|_{\Phi} + \|\nabla u\|_{\Phi}.$$

Then $W^{1,\Phi}(\Omega)$ is a Banach space called an Orlicz-Sobolev space. Denote the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ by $W_0^{1,\Phi}(\Omega)$. Then, by some basic properties in Orlicz-Sobolev spaces, we obtain that $W_0^{1,\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.

Next, we recall some inequalities. For more details, we refer the reader to the references [19, 21].

Lemma 2.1 (see [19, 21]) *If Φ is an N-function, then the following conditions are equivalent*:

(1)

$$1 \le l = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \le \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} = m < +\infty; \tag{2.5}$$

(2) *let* $\zeta_0(t) = \min\{t^l, t^m\}$, $\zeta_1(t) = \max\{t^l, t^m\}$ *for* $t \ge 0$. Φ *satisfies*

$$\zeta_0(t)\Phi(\rho) \le \Phi(\rho t) \le \zeta_1(t)\Phi(\rho) \quad \textit{for all } \rho, t \ge 0;$$

(3) Φ *satisfies a* Δ_2*-condition globally.*

Lemma 2.2 (see [21]) *If Φ is an N-function and* (2.5) *holds, then Φ satisfies*

$$\zeta_0(\|u\|_\Phi) \le \int_{\mathbb{R}^N} \Phi(|u|)\,dx \le \zeta_1(\|u\|_\Phi) \quad \textit{for all } u \in L^\Phi(\mathbb{R}^N).$$

Lemma 2.3 (see [21]) *If Φ is an N-function and* (2.5) *holds with $l > 1$. Let $\widetilde{\Phi}$ be the complement of Φ and* $\zeta_2(t) = \min\{t^{\widetilde{l}}, t^{\widetilde{m}}\}$, $\zeta_3(t) = \max\{t^{\widetilde{l}}, t^{\widetilde{m}}\}$ *for* $t \ge 0$, *where* $\widetilde{l} := \frac{l}{l-1}$ *and* $\widetilde{m} := \frac{m}{m-1}$. *Then $\widetilde{\Phi}$ satisfies*

(1)

$$\widetilde{m} = \inf_{t>0} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} \le \sup_{t>0} \frac{t\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} = \widetilde{l};$$

(2)

$$\zeta_2(t)\widetilde{\Phi}(\rho) \le \widetilde{\Phi}(\rho t) \le \zeta_3(t)\widetilde{\Phi}(\rho) \quad \textit{for all } \rho, t \ge 0;$$

(3)

$$\zeta_2(\|u\|_{\widetilde{\Phi}}) \le \int_{\mathbb{R}^N} \widetilde{\Phi}(|u|)\,dx \le \zeta_3(\|u\|_{\widetilde{\Phi}}) \quad \textit{for all } u \in L^{\widetilde{\Phi}}(\mathbb{R}^N).$$

If

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds < +\infty \quad \text{and} \quad \int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds = +\infty, \tag{2.6}$$

then the Sobolev conjugate N-function, function Φ_* of Φ, is given in [19] by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds \quad \textit{for } t \ge 0.$$

Lemma 2.4 (see [21]) *If Φ is an N-function and* (2.5) *holds with* $l, m \in (1, N)$, *then* (2.6) *holds. Let* $\zeta_4(t) = \min\{t^{l^*}, t^{m^*}\}$, $\zeta_5(t) = \max\{t^{l^*}, t^{m^*}\}$ *for* $t \ge 0$, *where* $l^* := \frac{lN}{N-l}$, $m^* := \frac{mN}{N-m}$. *Then Φ_* satisfies*

(1)

$$l^* = \inf_{t>0} \frac{t\Phi'_*(t)}{\Phi_*(t)} \le \sup_{t>0} \frac{t\Phi'_*(t)}{\Phi_*(t)} = m^*;$$

(2)

$$\zeta_4(t)\Phi_*(\rho) \le \Phi_*(\rho t) \le \zeta_5(t)\Phi_*(\rho) \quad \textit{for all } \rho, t \ge 0;$$

(3)

$$\zeta_4\big(\|u\|_{\Phi_*}\big) \le \int_{\mathbb{R}^N} \Phi_*\big(|u|\big)\,dx \le \zeta_5\big(\|u\|_{\Phi_*}\big) \quad \textit{for all } u \in L^{\Phi_*}\big(\mathbb{R}^N\big).$$

The following important embedding proposition involving the Orlicz-Sobolev spaces will be used frequently in our proofs.

Lemma 2.5 (see [19, 20]) *If* Φ *is an* N*-function and* (2.5) *holds with* $l, m \in (1, N)$, *then the embedding*

$$W^{1,\Phi}\big(\mathbb{R}^N\big) \hookrightarrow L^{\Psi}\big(\mathbb{R}^N\big)$$

is continuous for any N*-function* Ψ *satisfying*

$$\limsup_{t\to 0^+} \frac{\Psi(t)}{\Phi(t)} < \infty \quad \textit{and} \quad \limsup_{t\to +\infty} \frac{\Psi(t)}{\Phi_*(t)} < \infty.$$

Therefore, there exists a constant C_Ψ *such that*

$$\|u\|_\Psi \le C_\Psi \|u\|_{1,\Phi} \quad \textit{for all } u \in W^{1,\Phi}\big(\mathbb{R}^N\big). \tag{2.7}$$

If the space $\mathbb{R}^N$ *is replaced by a bounded domain* $D \subset \mathbb{R}^N$ *and* Ψ *increases essentially more slowly than* Φ_* *near infinity, that is,*

$$\lim_{t\to +\infty} \frac{\Psi(ct)}{\Phi_*(t)} = 0$$

for any constant $c > 0$, *then the embedding* $W^{1,\Phi}(D) \hookrightarrow L^\Psi(D)$ *is compact.*

Remark 2.6 By Lemmas 2.1, 2.3 and 2.4, (ϕ_1)-(ϕ_2) imply that $\Phi_i (i = 1, 2)$, $\widetilde{\Phi}_i (i = 1, 2)$, $\Phi_{i*} (i = 1, 2)$ and $\widetilde{\Phi}_{i*} (i = 1, 2)$ are N-functions that satisfy the Δ_2-condition globally, where and in the sequel $\widetilde{\Phi}_i$ denotes the complement of $\Phi_i (i = 1, 2)$, Φ_{i*} denotes the Sobolev conjugate N-function function of $\Phi_i (i = 1, 2)$ and $\widetilde{\Phi}_{i*}$ denotes the complement of $\Phi_{i*} (i = 1, 2)$. Moreover, the fact that $\Phi_i (i = 1, 2)$ and $\widetilde{\Phi}_i (i = 1, 2)$ satisfy the Δ_2-condition globally implies that $L^{\Phi_i}(\mathbb{R}^N) (i = 1, 2)$ and $W^{1,\Phi_i}(\mathbb{R}^N) (i = 1, 2)$ are separable and reflexive Banach spaces (see [19, 20]).

Remark 2.7 Under assumptions (ϕ_1) and (ϕ_2), Lemmas 2.1, 2.4 and 2.5 imply that the embeddings

$$W^{1,\Phi_i}\big(\mathbb{R}^N\big) \hookrightarrow L^{\Phi_i}\big(\mathbb{R}^N\big), \qquad W^{1,\Phi_i}\big(\mathbb{R}^N\big) \hookrightarrow L^{\Phi_{i*}}\big(\mathbb{R}^N\big), \quad i = 1, 2, \tag{2.8}$$

are continuous and the embeddings

$$W^{1,\Phi_i}(B_r) \hookrightarrow L^{\Phi_i}(B_r), \quad i = 1,2, \tag{2.9}$$

are compact, where and in the sequel $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ for $r > 0$.

3 Main results and proofs

Theorem 3.1 *Assume that* (ϕ_1), (ϕ_2), (V_1), (V_2), (F_1) *and the following conditions hold*:

(F_2)

$$\lim_{|(u,v)|\to 0} \frac{F_u(x,u,v)}{\phi_1(|u|) + \widetilde{\Phi}_1^{-1}(\Phi_2(|v|))} = 0, \qquad \lim_{|(u,v)|\to 0} \frac{F_v(x,u,v)}{\widetilde{\Phi}_2^{-1}(\Phi_1(|u|)) + \phi_2(|v|)} = 0,$$
$$\lim_{|(u,v)|\to \infty} \frac{F_u(x,u,v)}{\Phi_{1*}'(|u|) + \widetilde{\Phi}_{1*}^{-1}(\Phi_{2*}(|v|))} = 0, \qquad \lim_{|(u,v)|\to \infty} \frac{F_v(x,u,v)}{\widetilde{\Phi}_{2*}^{-1}(\Phi_{1*}(|u|)) + \Phi_{2*}'(|v|)} = 0,$$

uniformly in $x \in \mathbb{R}^N$, *where and in the sequel* $\widetilde{\Phi}_i^{-1}$ *denotes the inverse of* $\widetilde{\Phi}_i (i = 1,2)$, Φ_{i*}' *denotes the derivative of* $\Phi_{i*} (i = 1,2)$ *and* $\widetilde{\Phi}_{i*}^{-1}$ *denotes the inverse of* $\widetilde{\Phi}_{i*} (i = 1,2)$;

(F_3) *there exist* $\mu_i > m_i (i = 1,2)$ *such that*

$$0 < F(x,u,v) \le \frac{1}{\mu_1} uF_u(x,u,v) + \frac{1}{\mu_2} vF_v(x,u,v) \quad \textit{for all } (u,v) \ne (0,0).$$

Then system (1.1) *has a ground state, that is, a nontrivial solution* (u_0,v_0) *such that*

$$I(u_0,v_0) = \inf\bigl\{I(u,v) : (u,v) \in W \setminus \bigl\{(\mathbf{0},\mathbf{0})\bigr\} \textit{ and } I'(u,v) = 0\bigr\},$$

where $W = W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N)$ *and*

$$\begin{aligned} I(u,v) = {} & \int_{\mathbb{R}^N} \Phi_1\bigl(|\nabla u|\bigr)\,dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1\bigl(|u|\bigr)\,dx \\ & + \int_{\mathbb{R}^N} \Phi_2\bigl(|\nabla v|\bigr)\,dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2\bigl(|v|\bigr)\,dx - \int_{\mathbb{R}^N} F(x,u,v)\,dx. \end{aligned}$$

Theorem 3.2 *Assume that* (ϕ_1), (ϕ_2), (V_1), (V_2), (F_1), (F_2) *and the following conditions hold*:

(ϕ_3)

$$\limsup_{t\to 0} \frac{|t|^{l_i}}{\Phi_i(|t|)} < \infty, \quad i = 1,2;$$

(F_4)

$$\lim_{|(u,v)|\to\infty} \frac{F(x,u,v)}{\Phi_1(|u|) + \Phi_2(|v|)} = +\infty$$

uniformly in $x \in \mathbb{R}^N$;

(F_5) $\overline{F}(x,u,v) > 0$ *for all* $(u,v) \neq (0,0)$ *and there exists* $k > \max\{\frac{N}{l_1}, \frac{N}{l_2}\}$ *such that*

$$\limsup_{|(u,v)|\to\infty} \left(\frac{F(x,u,v)}{|u|^{l_1} + |v|^{l_2}}\right)^k \frac{1}{\overline{F}(x,u,v)} < \infty,$$

where

$$\overline{F}(x,u,v) = \frac{1}{m_1} uF_u(x,u,v) + \frac{1}{m_2} vF_v(x,u,v) - F(x,u,v).$$

Then system (1.1) *has a ground state.*

By Lemmas 2.1 and 2.4, it is easy to check that the following conditions $(F_2)'$ and $(F_4)'$ imply (F_2) and (F_4), respectively.

$(F_2)'$

$$\lim_{|(u,v)|\to 0} \frac{F_u(x,u,v)}{|u|^{m_1-1} + |v|^{\frac{m_2(m_1-1)}{m_1}}} = 0, \qquad \lim_{|(u,v)|\to 0} \frac{F_v(x,u,v)}{|u|^{\frac{m_1(m_2-1)}{m_2}} + |v|^{m_2-1}} = 0,$$

$$\lim_{|(u,v)|\to\infty} \frac{F_u(x,u,v)}{|u|^{l_1^*-1} + |v|^{\frac{l_2^*(l_1^*-1)}{l_1^*}}} = 0,$$

$$\lim_{|(u,v)|\to\infty} \frac{F_v(x,u,v)}{|u|^{\frac{l_1^*(l_2^*-1)}{l_2^*}} + |v|^{l_2^*-1}} = 0, \quad \textit{uniformly in } x \in \mathbb{R}^N;$$

$(F_4)'$

$$\lim_{|(u,v)|\to\infty} \frac{F(x,u,v)}{|u|^{m_1} + |v|^{m_2}} = +\infty, \quad \textit{uniformly in } x \in \mathbb{R}^N.$$

Thus, we have the following corollary.

Corollary 3.3 *In Theorems* 3.1 *and* 3.2, *if conditions* (F_2) *and* (F_4) *are replaced by* $(F_2)'$ *and* $(F_4)'$, *respectively, then the conclusions still hold.*

Remark 3.4 We point out that Theorems 3.1 and 3.2 are complementary, which is based on the fact that there are functions satisfying (F_4) and (F_5) but not satisfying (F_3) (see Example 4.2 in Section 4) and there are also functions $\phi_i (i = 1,2)$ defined by (1.2) satisfying (ϕ_1) and (ϕ_2) but not satisfying (ϕ_3) (see Case 4 in Section 4).

When system (1.1) reduces to equation (1.3), we present the following results which correspond to Theorems 3.1 and 3.2.

Corollary 3.5 *Assume that functions* a_1, V_1 *and* f *satisfy* (ϕ_1)-(ϕ_2), (V_1)-(V_2) *and*

$(f_1)^*$ $f \in C(\mathbb{R}^N, \mathbb{R})$ *is* 1*-periodic in* x,
$(f_2)^*$

$$\lim_{|u|\to 0} \frac{f(x,u)}{\phi_1(|u|)} = 0, \qquad \lim_{|u|\to\infty} \frac{f(x,u)}{\Phi'_{1*}(|u|)} = 0$$

uniformly in $x \in \mathbb{R}^N$;

$(f_3)^*$ *there exists* $\mu > m_1$ *such that*

$$0 < \mu F(x,u) \le uf(x,u) \quad \text{for all } u \neq 0.$$

Then equation (1.3) *has a ground state in* $W^{1,\Phi_1}(\mathbb{R}^N)$.

Corollary 3.6 *Assume that functions* a_1, V_1 *and* f *satisfy* (ϕ_1)-(ϕ_3), (V_1)-(V_2), $(f_1)^*$-$(f_2)^*$ *and*

$(f_4)^*$

$$\lim_{|u|\to\infty} \frac{F(x,u)}{\Phi_1(|u|)} = +\infty$$

uniformly in $x \in \mathbb{R}^N$;

$(f_5)^*$ $\overline{F}(x,u) > 0$ *for all* $u \neq 0$ *and there exists* $k > \frac{N}{l_1}$ *such that*

$$\limsup_{|(u,v)|\to\infty} \left(\frac{F(x,u)}{|u|^{l_1}}\right)^k \frac{1}{\overline{F}(x,u)} < \infty,$$

where

$$\overline{F}(x,u) = uf(x,u) - m_1 F(x,u,v).$$

Then equation (1.3) *has a ground state in* $W^{1,\Phi_1}(\mathbb{R}^N)$.

Remark 3.7 It is easy to see that our results are different from Theorem 3.4 and Theorem 4.11 in [12].

Remark 3.8 For the nonlinearity f, our subcritical growth condition in the Orlicz-Sobolev space

$$\lim_{|u|\to\infty} \frac{f(x,u)}{\Phi'_{1*}(|u|)} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N \tag{3.1}$$

in $(f_2)^*$ is weaker than the following one which is usually assumed in many papers in order to consider ϕ-Laplacian problems (for example, see [9–12]):

(*SC*) *there exist a constant* $C > 0$ *and an* N*-function defined by* $\Psi(t) := \int_0^t \psi(s)\,ds, t \in [0,+\infty)$ *satisfying*

$$m_1 < l_\Psi := \inf_{t>0} \frac{t\psi(t)}{\Psi(t)} \le \sup_{t>0} \frac{t\psi(t)}{\Psi(t)} =: m_\Psi < l_1^*$$

or increasing essentially more slowly than Φ_{1*} *near infinity, such that*

$$\limsup_{|u|\to\infty} \left|\frac{f(x,u)}{\psi(u)}\right| < \infty, \quad \text{uniformly in } x \in \mathbb{R}^N.$$

Condition (3.1) *was introduced by Alves et al.* [8] *for the autonomous nonlinearity f in the Orlicz-Sobolev space. When* $a_1(|t|)t = |t|^{p-2}t(p > 1)$, (3.1) *reduces to*

$$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p^*-1}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N, \tag{3.2}$$

which was first introduced by Liu and Wang [22] instead of the usual subcritical growth condition, that is, *there exist constants* $C > 0$ *and* $q \in (p, p^*)$ *such that*

$$\left|f(x,u)\right| \le C\left(|u|^{p-1} + |u|^{q-1}\right) \quad \textit{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.3}$$

Remark 3.9 A condition similar to $(f_5)^*$ was introduced by Carvalho et al. [11] for the ϕ-Laplacian equation in the bounded domain $\Omega \subset \mathbb{R}^N$. In this paper, because we consider problems on the whole space $\mathbb{R}^N$ where the Sobolev spaces lack compactness of the Sobolev embedding, we claim $\overline{F}(x,u) > 0$ for all $u \neq 0$ in $(f_5)^*$.

When $a_1(|t|)t = |t|^{p-2}t(1 < p < N)$, it is obvious that (ϕ_1)-(ϕ_3) hold, and then we also present the corresponding results for equation (1.4).

Corollary 3.10 *Assume that* $N > p$ *and functions* V_1 *and* f *satisfy* (V_1)-(V_2), $(f_1)^*$, (AR) *and*

$(f_2)'$

$$\lim_{|u|\to 0} \frac{f(x,u)}{|u|^{p-1}} = 0, \qquad \lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p^*-1}} = 0$$

uniformly in $x \in \mathbb{R}^N$.

Then equation (1.4) *has a ground state in* $W^{1,p}(\mathbb{R}^N)$.

Corollary 3.11 *Assume that* $N > p$ *and functions* V_1 *and* f *satisfy* (V_1)-(V_2), $(f_1)^*$, $(f_2)'$ *and*

$(f_4)'$

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^p} = +\infty$$

uniformly in $x \in \mathbb{R}^N$;

$(f_5)'$ $\overline{F}(x,u) > 0$ *for all* $u \neq 0$ *and there exists* $k > \frac{N}{p}$ *such that*

$$\limsup_{|(u,v)|\to\infty} \left(\frac{F(x,u)}{|u|^p}\right)^k \frac{1}{\overline{F}(x,u)} < \infty,$$

where

$$\overline{F}(x,u) = uf(x,u) - pF(x,u,v).$$

Then equation (1.4) *has a ground state in* $W^{1,p}(\mathbb{R}^N)$.

Remark 3.12 If the subcritical growth condition (3.2) in $(f_2)'$ is replaced by (3.3), Corollary 3.10 becomes a corollary of Corollary 3.11 based on the fact that (AR) and (3.3) imply $(f_4)'$ and $(f_5)'$ (see [3, 4]). However, we are not sure whether (AR) and (3.2) imply $(f_4)'$ and $(f_5)'$ so that we do not know whether Corollary 3.10 is a corollary of Corollary 3.11. It is remarkable that our Corollaries 3.10 and 3.11 are different from Theorem 1.1 in [5] because there are examples satisfying (AR) and $(f_5)'$ but not satisfying (f_3) (see example (1.5) for $p = 2$).

Next, we start to present our proofs. By (ϕ_1) and (ϕ_2), we define the space $W := W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N)$ with the norm

$$\|(u,v)\| = \|u\|_{1,\Phi_1} + \|v\|_{1,\Phi_2} = \|\nabla u\|_{\Phi_1} + \|u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} + \|v\|_{\Phi_2}.$$

Then W is a separable and reflexive Banach space by Remark 2.6.

On W, define a functional I by

$$\begin{aligned} I(u,v) :=& \int_{\mathbb{R}^N} \Phi_1\big(|\nabla u|\big)\,dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1\big(|u|\big)\,dx \\ &+ \int_{\mathbb{R}^N} \Phi_2\big(|\nabla v|\big)\,dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2\big(|v|\big)\,dx - \int_{\mathbb{R}^N} F(x,u,v)\,dx. \end{aligned} \tag{3.4}$$

Standard arguments show that I is well defined and of class $C^1(W,\mathbb{R})$ and

$$\begin{aligned} \big\langle I'(u,v),(\tilde{u},\tilde{v})\big\rangle =& \int_{\mathbb{R}^N} a_1\big(|\nabla u|\big)\nabla u\nabla\tilde{u}\,dx + \int_{\mathbb{R}^N} V_1(x)a_1\big(|u|\big)u\tilde{u}\,dx \\ &+ \int_{\mathbb{R}^N} a_2\big(|\nabla v|\big)\nabla v\nabla\tilde{v}\,dx + \int_{\mathbb{R}^N} V_2(x)a_2\big(|v|\big)v\tilde{v}\,dx \\ &- \int_{\mathbb{R}^N} F_u(x,u,v)\tilde{u}\,dx - \int_{\mathbb{R}^N} F_v(x,u,v)\tilde{v}\,dx \end{aligned} \tag{3.5}$$

for all $(\tilde{u},\tilde{v}) \in W$. For the sake of completeness, we give the proof in the Appendix. Thus, the critical points of I in W are weak solutions of system (1.1). Denote by $I_i (i = 1,2) : W \to \mathbb{R}$ the functionals

$$\begin{aligned} I_1(u,v) =& \int_{\mathbb{R}^N} \Phi_1\big(|\nabla u|\big)\,dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1\big(|u|\big)\,dx + \int_{\mathbb{R}^N} \Phi_2\big(|\nabla v|\big)\,dx \\ &+ \int_{\mathbb{R}^N} V_2(x)\Phi_2\big(|v|\big)\,dx \end{aligned} \tag{3.6}$$

and

$$I_2(u,v) = \int_{\mathbb{R}^N} F(x,u,v)\,dx. \tag{3.7}$$

Then

$$I(u,v) = I_1(u,v) - I_2(u,v).$$

Lemma 3.13 *If* (F_1) *and* (F_2) *hold, then there exist positive constants* C_i $(i = 1, 2, 3)$ *such that*

$$\big|F_u(x,u,v)\big| \le C_1\big(\phi_1\big(|u|\big) + \widetilde{\Phi}_1^{-1}\big(\Phi_2\big(|v|\big)\big) + \Phi_{1*}'\big(|u|\big) + \widetilde{\Phi}_{1*}^{-1}\big(\Phi_{2*}\big(|v|\big)\big)\big), \tag{3.8}$$

$$\big|F_v(x,u,v)\big| \le C_2\big(\widetilde{\Phi}_2^{-1}\big(\Phi_1\big(|u|\big)\big) + \phi_2\big(|v|\big) + \widetilde{\Phi}_{2*}^{-1}\big(\Phi_{1*}\big(|u|\big)\big) + \Phi_{2*}'\big(|v|\big)\big), \tag{3.9}$$

$$\big|F(x,u,v)\big| \le C_3\big(\Phi_1\big(|u|\big) + \Phi_2\big(|v|\big) + \Phi_{1*}\big(|u|\big) + \Phi_{1*}\big(|v|\big)\big) \tag{3.10}$$

for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, *and*

$$\lim_{|(u,v)|\to 0} \frac{F(x,u,v)}{\Phi_1(|u|) + \Phi_2(|v|)} = 0, \qquad \lim_{|(u,v)|\to \infty} \frac{F(x,u,v)}{\Phi_{1*}(|u|) + \Phi_{2*}(|v|)} = 0 \tag{3.11}$$

uniformly in $x \in \mathbb{R}^N$.

Proof The proof can be easily completed by virtue of Young's inequality (2.1) and the fact

$$F(x,u,v) = \int_0^u F_s(x,s,v)\,ds + \int_0^v F_t(x,0,t)\,dt + F(x,0,0) \quad \text{for all } (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}.$$

We omit the details. □

Notation C_a denotes a positive constant which depends on the real number a.

Lemma 3.14 *Assume that* (ϕ_1), (ϕ_2), (V_2), (F_1) *and* (F_2) *hold. Then there exist two positive constants* ρ, η *such that* $I(u,v) \ge \eta$ *for all* $(u,v) \in W$ *with* $\|(u,v)\| = \rho$.

Proof By (3.11), for any given $\varepsilon \in (0, \alpha_1)$, there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned}\big|F(x,u,v)\big| &\le \varepsilon\big(\Phi_1\big(|u|\big) + \Phi_2\big(|v|\big)\big) + C_\varepsilon\big(\Phi_{1*}\big(|u|\big) + \Phi_{2*}\big(|v|\big)\big)\\ &\qquad \text{for all } (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}.\end{aligned}$$

Then, by (3.4), (V_2), Lemma 2.2, (3) in Lemma 2.4, (2.8) and (2.7), when $\|(u,v)\| = \|u\|_{1,\Phi_1} + \|v\|_{1,\Phi_2} = \|\nabla u\|_{\Phi_1} + \|u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} + \|v\|_{\Phi_2} \le 1$, we have

$$\begin{aligned}
I(u,v) \ge{}& \int_{\mathbb{R}^N} \Phi_1\big(|\nabla u|\big)\,dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1\big(|u|\big)\,dx \\
&+ \int_{\mathbb{R}^N} \Phi_2\big(|\nabla v|\big)\,dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2\big(|v|\big)\,dx - \int_{\mathbb{R}^N} \big|F(x,u,v)\big|\,dx \\
\ge{}& \int_{\mathbb{R}^N} \Phi_1\big(|\nabla u|\big)\,dx + \alpha_1 \int_{\mathbb{R}^N} \Phi_1\big(|u|\big)\,dx + \int_{\mathbb{R}^N} \Phi_2\big(|\nabla v|\big)\,dx + \alpha_1 \int_{\mathbb{R}^N} \Phi_2\big(|v|\big)\,dx \\
&- \varepsilon \int_{\mathbb{R}^N} \Phi_1\big(|u|\big)\,dx - \varepsilon \int_{\mathbb{R}^N} \Phi_2\big(|v|\big)\,dx \\
&- C_\varepsilon \int_{\mathbb{R}^N} \Phi_{1*}\big(|u|\big)\,dx - C_\varepsilon \int_{\mathbb{R}^N} \Phi_{2*}\big(|v|\big)\,dx \\
\ge{}& \|\nabla u\|_{\Phi_1}^{m_1} + (\alpha_1 - \varepsilon)\|u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2} + (\alpha_1 - \varepsilon)\|v\|_{\Phi_2}^{m_2} \\
&- C_\varepsilon \max\big\{\|u\|_{\Phi_{1*}}^{l_1^*}, \|u\|_{\Phi_{1*}}^{m_1^*}\big\} - C_\varepsilon \max\big\{\|v\|_{\Phi_{2*}}^{l_2^*}, \|v\|_{\Phi_{2*}}^{m_2^*}\big\}
\end{aligned}$$

$$\geq \min\{1,\alpha_1-\varepsilon\}C_{m_1}\|u\|_{1,\Phi_1}^{m_1}+\min\{1,\alpha_1-\varepsilon\}C_{m_2}\|v\|_{1,\Phi_2}^{m_2}$$
$$-C_\varepsilon C_{\Phi_{1*}}^{l_1^*}\|u\|_{1,\Phi_1}^{l_1^*}-C_\varepsilon C_{\Phi_{1*}}^{m_1^*}\|u\|_{1,\Phi_1}^{m_1^*}-C_\varepsilon C_{\Phi_{2*}}^{l_2^*}\|v\|_{1,\Phi_2}^{l_2^*}-C_\varepsilon C_{\Phi_{2*}}^{m_2^*}\|v\|_{1,\Phi_2}^{m_2^*}.$$

Note that $m_i < l_i^* \leq m_i^* (i = 1, 2)$. It is easy to see that the foregoing inequality implies that there exist positive constants ρ and η small enough such that $I(u,v) \geq \eta$ for all $(u,v) \in W$ with $\|(u,v)\| = \rho$. □

Lemma 3.15 *Assume that* (ϕ_1), (ϕ_2), (V_2), (F_1) *and* (F_3) (*or* (F_4)) *hold. Then there exists* $(u_0,v_0) \in W$ *such that* $I(tu_0,tv_0) \to -\infty$ *as* $t \to +\infty$.

Proof First, we prove that under assumptions (F_1) and (F_3) (or (F_4)), for any given constant $M > \alpha_2$, there exists a constant $C_M > 0$ such that

$$F(x,u,0) \geq M\Phi_1(|u|) - C_M \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.12}$$

In fact, it is obvious by (F_1) and (F_4). Let $v = 0$ in (F_3). Then (F_3) reduces to

$$0 < F(x,u,0) \leq \frac{1}{\mu_1} uF_u(x,u,0) \quad \text{for all } u \neq 0,$$

where $\mu_1 > m_1$, which implies that $F(x,u,0) \geq C(|u|^{\mu_1} - 1)$ for some $C > 0$ and all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Moreover, it follows from (2) in Lemma 2.1 that $\Phi_1(|u|) \leq \Phi_1(1)\max\{|u|^{l_1}, |u|^{m_1}\}$ for all $u \in \mathbb{R}$. Since $\mu_1 > m_1$, then for any given constant $M > \alpha_2$, there exists a constant $C_M > 0$ such that (3.12) holds.

Now, choose $u_0 \in C_0^\infty(B_r) \setminus \{\mathbf{0}\}$ with $0 \leq u_0(x) \leq 1$, where $r > 0$. Then $(u_0, 0) \in W$, and by (3.4), (V_2), (F_1), (3.12) and (2) in Lemma 2.1, when $t > 0$, we have

$$\begin{aligned}
I(tu_0,\mathbf{0}) &= \int_{\mathbb{R}^N}\Phi_1\big(|t\nabla u_0|\big)\,dx + \int_{\mathbb{R}^N}V_1(x)\Phi_1\big(|tu_0|\big)\,dx - \int_{\mathbb{R}^N}F(x,tu_0,0)\,dx \\
&= \int_{\mathbb{R}^N}\Phi_1\big(|t\nabla u_0|\big)\,dx + \int_{B_r}V_1(x)\Phi_1\big(|tu_0|\big)\,dx - \int_{B_r}F(x,tu_0,0)\,dx \\
&\leq \int_{\mathbb{R}^N}\Phi_1\big(|t\nabla u_0|\big)\,dx + \alpha_2\int_{B_r}\Phi_1\big(|tu_0|\big)\,dx - M\int_{B_r}\Phi_1\big(|tu_0|\big) + C_M|B_r| \\
&\leq \Phi_1(t)\int_{\mathbb{R}^N}\max\big\{|\nabla u_0|^{l_1}, |\nabla u_0|^{m_1}\big\}\,dx \\
&\quad - \Phi_1(t)(M-\alpha_2)\int_{B_r}\min\big\{|u_0|^{l_1}, |u_0|^{m_1}\big\}\,dx + C_M|B_r| \\
&\leq \Phi_1(t)\big[\big\||\nabla u_0|\big\|_{l_1}^{l_1} + \big\||\nabla u_0|\big\|_{m_1}^{m_1} - (M-\alpha_2)\|u_0\|_{m_1}^{m_1}\big] + C_M|B_r|.
\end{aligned}$$

Since $\lim_{t\to+\infty}\Phi_1(t) = +\infty$, we can choose $M > \frac{\||\nabla u_0|\|_{l_1}^{l_1}+\||\nabla u_0|\|_{m_1}^{m_1}}{\|u_0\|_{m_1}^{m_1}} + \alpha_2$ such that $I(tu_0,\mathbf{0}) \to -\infty$ as $t \to +\infty$. □

Lemmas 3.14, 3.15 and the fact $I(\mathbf{0},\mathbf{0}) = 0$ show that I has a mountain pass geometry, that is, setting

$$\Gamma = \big\{\gamma \in C\big([0,1], W\big) : \gamma(0) = \mathbf{0} \text{ and } I\big(\gamma(1)\big) < 0\big\},$$

we have $\Gamma \neq \emptyset$. By a special version of the mountain pass lemma (see [23]), for the mountain pass level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big), \tag{3.13}$$

there exists a $(C)_c$-sequence $\{(u_n, v_n)\}$ of I in W. Moreover, Lemma 3.14 implies that $c > 0$. We recall that $(C)_c$-sequence $\{u_n, v_n\}$ of I in W means

$$I(u_n, v_n) \to c \quad \text{and} \quad \big(1 + \big\|(u_n, v_n)\big\|\big)\big\|I'(u_n, v_n)\big\|_{W^*} \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Lemma 3.16 *Assume that* (ϕ_1), (ϕ_2), (V_2), (F_1)-(F_3) *hold. Then any* $(C)_c$*-sequence of* I *in* W *is bounded for all* $c \geq 0$.

Proof Let $\{(u_n, v_n)\}$ be a $(C)_c$-sequence of I in W for $c \geq 0$. By (3.14), we have

$$I(u_n, v_n) \to c \quad \text{as } n \to \infty \tag{3.15}$$

and

$$\big\|I'(u_n, v_n)\big\|_{W^*}\big\|(u_n, v_n)\big\| = \big\|I'(u_n, v_n)\big\|_{W^*}\big(\|u_n\|_{1,\Phi_1} + \|v_n\|_{1,\Phi_2}\big) \to 0 \quad \text{as } n \to \infty,$$

which implies

$$\left|\left\langle I'(u_n, v_n), \left(\frac{1}{\mu_1}u_n, \frac{1}{\mu_2}v_n\right)\right\rangle\right| \leq \big\|I'(u_n, v_n)\big\|_{W^*}\left(\frac{1}{\mu_1}\|u_n\|_{1,\Phi_1} + \frac{1}{\mu_2}\|v_n\|_{1,\Phi_2}\right) \to 0 \\ \text{as } n \to \infty. \tag{3.16}$$

Then, by (3.15), (3.16), (3.4), (3.5), (ϕ_2), (V_2), (F_3) and Lemma 2.2, for n large, we have

$$\begin{aligned}
c + 1 &\geq I(u_n, v_n) - \left\langle I'(u_n, v_n), \left(\frac{1}{\mu_1}u_n, \frac{1}{\mu_2}v_n\right)\right\rangle \\
&= \int_{\mathbb{R}^N}\left(\Phi_1\big(|\nabla u_n|\big) - \frac{1}{\mu_1}a_1\big(|\nabla u_n|\big)|\nabla u_n|^2\right)dx \\
&\quad + \int_{\mathbb{R}^N} V_1(x)\left(\Phi_1\big(|u_n|\big) - \frac{1}{\mu_1}a_1\big(|u_n|\big)|u_n|^2\right)dx \\
&\quad + \int_{\mathbb{R}^N}\left(\Phi_2\big(|\nabla v_n|\big) - \frac{1}{\mu_2}a_2\big(|\nabla v_n|\big)|\nabla v_n|^2\right)dx \\
&\quad + \int_{\mathbb{R}^N} V_2(x)\left(\Phi_2\big(|v_n|\big) - \frac{1}{\mu_2}a_2\big(|v_n|\big)|v_n|^2\right)dx \\
&\quad + \int_{\mathbb{R}^N}\left(\frac{1}{\mu_1}u_nF_u(x, u_n, v_n) + \frac{1}{\mu_2}v_nF_v(x, u_n, v_n) - F(x, u_n, v_n)\right)dx \\
&\geq \left(1 - \frac{m_1}{\mu_1}\right)\int_{\mathbb{R}^N}\Phi_1\big(|\nabla u_n|\big)\,dx + \left(1 - \frac{m_1}{\mu_1}\right)\alpha_1\int_{\mathbb{R}^N}\Phi_1\big(|u_n|\big)\,dx \\
&\quad + \left(1 - \frac{m_2}{\mu_2}\right)\int_{\mathbb{R}^N}\Phi_2\big(|\nabla v_n|\big)\,dx + \left(1 - \frac{m_2}{\mu_2}\right)\alpha_1\int_{\mathbb{R}^N}\Phi_2\big(|v_n|\big)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(1-\frac{m_1}{\mu_1}\right)\min\left\{\|\nabla u_n\|_{\Phi_1}^{l_1},\|\nabla u_n\|_{\Phi_1}^{m_1}\right\}+\left(1-\frac{m_1}{\mu_1}\right)\alpha_1\min\left\{\|u_n\|_{\Phi_1}^{l_1},\|u_n\|_{\Phi_1}^{m_1}\right\}\\
&\quad+\left(1-\frac{m_2}{\mu_2}\right)\min\left\{\|\nabla v_n\|_{\Phi_2}^{l_2},\|\nabla v_n\|_{\Phi_2}^{m_2}\right\}+\left(1-\frac{m_2}{\mu_2}\right)\alpha_1\min\left\{\|v_n\|_{\Phi_2}^{l_2},\|v_n\|_{\Phi_2}^{m_2}\right\},
\end{aligned}
$$

which implies that $\|(u_n,v_n)\|=\|\nabla u_n\|_{\Phi_1}+\|u_n\|_{\Phi_1}+\|\nabla v_n\|_{\Phi_2}+\|v_n\|_{\Phi_2}\leq C$ for some $C>0$, that is, $\{(u_n,v_n)\}$ is bounded in W. □

Lemma 3.17 *Assume that* (ϕ_1)-(ϕ_3), (V_1), (V_2), (F_1), (F_2), (F_4) *and* (F_5) *hold. Then any* $(C)_c$*-sequence of* I *in* W *is bounded for all* $c\geq 0$.

Proof Let $\{(u_n,v_n)\}$ be a $(C)_c$-sequence of I in W for $c\geq 0$. By (3.14), we have

$$
I(u_n,v_n)=I_1(u_n,v_n)-I_2(u_n,v_n)\to c \quad\text{and}\quad \left|\left\langle I'(u_n,v_n),\left(\frac{1}{m_1}u_n,\frac{1}{m_2}v_n\right)\right\rangle\right|\to 0 \tag{3.17}
$$

as $n\to\infty$.

Then, by (3.4), (3.5), (ϕ_2) and (V_2), for n large, we have

$$
\begin{aligned}
c+1&\geq I(u_n,v_n)-\left\langle I'(u_n,v_n),\left(\frac{1}{m_1}u_n,\frac{1}{m_2}v_n\right)\right\rangle\\
&=\int_{\mathbb{R}^N}\left(\Phi_1(|\nabla u_n|)-\frac{1}{m_1}a_1(|\nabla u_n|)|\nabla u_n|^2\right)dx\\
&\quad+\int_{\mathbb{R}^N}V_1(x)\left(\Phi_1(|u_n|)-\frac{1}{m_1}a_1(|u_n|)|u_n|^2\right)dx\\
&\quad+\int_{\mathbb{R}^N}\left(\Phi_2(|\nabla v_n|)-\frac{1}{m_2}a_2(|\nabla v_n|)|\nabla v_n|^2\right)dx\\
&\quad+\int_{\mathbb{R}^N}V_2(x)\left(\Phi_2(|v_n|)-\frac{1}{m_2}a_2(|v_n|)|v_n|^2\right)dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{1}{m_1}u_nF_u(x,u_n,v_n)+\frac{1}{m_2}v_nF_v(x,u_n,v_n)-F(x,u_n,v_n)\right)dx\\
&\geq\int_{\mathbb{R}^N}\overline{F}(x,u_n,v_n)\,dx.
\end{aligned}
\tag{3.18}
$$

To prove the boundedness of $\{(u_n,v_n)\}$, arguing by contradiction, we suppose that there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, such that $\|(u_n,v_n)\|=\|u_n\|_{1,\Phi_1}+\|v_n\|_{1,\Phi_2}\to\infty$. Next, we discuss the problem in three cases.

Case 1. Suppose that $\|u_n\|_{1,\Phi_1}\to\infty$ and $\|v_n\|_{1,\Phi_2}\to\infty$. Let $\tilde{u}_n=\frac{u_n}{\|u_n\|_{1,\Phi_1}}$ and $\tilde{v}_n=\frac{v_n}{\|v_n\|_{1,\Phi_2}}$. Then $\{\tilde{u}_n\}$ and $\{\tilde{v}_n\}$ are bounded in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. We claim that

$$
\lambda_1:=\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_2(y)}\left(\Phi_1(|\tilde{u}_n|)+\Phi_2(|\tilde{v}_n|)\right)dx=0.
$$

Indeed, if $\lambda_1\neq 0$, there exist a constant $\delta>0$, a subsequence of $\{(\tilde{u}_n,\tilde{v}_n)\}$, still denoted by $\{(\tilde{u}_n,\tilde{v}_n)\}$, and a sequence $\{z_n\}\in\mathbb{Z}^N$ such that

$$
\int_{B_2(z_n)}\left(\Phi_1(|\tilde{u}_n|)+\Phi_2(|\tilde{v}_n|)\right)dx>\delta \quad\text{for all } n\in\mathbb{N}. \tag{3.19}
$$

Let $\bar{u}_n = \tilde{u}_n(\cdot + z_n)$ and $\bar{v}_n = \tilde{v}_n(\cdot + z_n)$. Then $\|\bar{u}_n\|_{1,\Phi_1} = \|\tilde{u}_n\|_{1,\Phi_1}$ and $\|\bar{v}_n\|_{1,\Phi_2} = \|\tilde{v}_n\|_{1,\Phi_2}$, that is, $\{\bar{u}_n\}$ and $\{\bar{v}_n\}$ are bounded in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. Passing to a subsequence of $\{(\bar{u}_n, \bar{v}_n)\}$, still denoted by $\{(\bar{u}_n, \bar{v}_n)\}$, by Remark 2.7, there exists $(\bar{u}, \bar{v}) \in W$ such that

- ⋆ $\bar{u}_n \rightharpoonup \bar{u}$ in $W^{1,\Phi_1}(\mathbb{R}^N)$, $\bar{u}_n \to \bar{u}$ in $L^{\Phi_1}(B_2)$ and $\bar{u}_n(x) \to \bar{u}(x)$ a.e. in B_2;
- ⋆ $\bar{v}_n \rightharpoonup \bar{v}$ in $W^{1,\Phi_2}(\mathbb{R}^N)$, $\bar{v}_n \to \bar{v}$ in $L^{\Phi_2}(B_2)$ and $\bar{v}_n(x) \to \bar{v}(x)$ a.e. in B_2.

Since

$$\int_{B_2} \bigl(\Phi_1\bigl(|\bar{u}_n|\bigr) + \Phi_2\bigl(|\bar{v}_n|\bigr)\bigr)\,dx = \int_{B_2(z_n)} \bigl(\Phi_1\bigl(|\tilde{u}_n|\bigr) + \Phi_2\bigl(|\tilde{v}_n|\bigr)\bigr)\,dx,$$

then, by (3.19), ⋆ and (2.3), we obtain that $\bar{u} \neq \mathbf{0}$ in $L^{\Phi_1}(B_2)$ or $\bar{v} \neq \mathbf{0}$ in $L^{\Phi_2}(B_2)$. Without loss of generality, we can assume that $\bar{u} \neq \mathbf{0}$ in $L^{\Phi_1}(B_2)$, that is, $[\bar{u} \neq 0] := \{x \in B_2 : \bar{u}(x) \neq 0\}$ has nonzero Lebesgue measure. Let $u_n^* = u_n(\cdot + z_n)$ and $v_n^* = v_n(\cdot + z_n)$. Then $\|(u_n^*, v_n^*)\| = \|(u_n, v_n)\|$, and it follows from that fact that $V_i (i = 1, 2)$ and F are 1-periodic in x that

$$I\bigl(u_n^*, v_n^*\bigr) = I(u_n, v_n) \quad \text{and} \quad \bigl\|I'\bigl(u_n^*, v_n^*\bigr)\bigr\|_{W^*} = \bigl\|I'(u_n, v_n)\bigr\|_{W^*} \quad \text{for all } n \in \mathbb{N},$$

that is, $\{(u_n^*, v_n^*)\}$ is also a $(C)_c$-sequence of I. Then, by (3.18), for n large, we have

$$\int_{\mathbb{R}^N} \overline{F}\bigl(x, u_n^*, v_n^*\bigr)\,dx \le c + 1. \tag{3.20}$$

However, by (2) in Lemma 2.1, (F_4) and (F_5) imply

$$\lim_{|(u,v)|\to\infty} \overline{F}(x, u, v) = +\infty \quad \text{uniformly in } x \in \mathbb{R}^N, \tag{3.21}$$

and by ⋆, $\bar{u}_n = \tilde{u}_n(\cdot + z_n) = \frac{u_n(\cdot + z_n)}{\|u_n\|_{1,\Phi_1}} = \frac{u_n^*}{\|u_n\|_{1,\Phi_1}}$ implies

$$\bigl|u_n^*(x)\bigr| = \bigl|\bar{u}_n(x)\bigr| \|u_n\|_{1,\Phi_1} \to \infty, \quad \text{a.e. } x \in [\bar{u} \neq 0]. \tag{3.22}$$

Then, it follows from (F_5), (3.21), (3.22) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \overline{F}\bigl(x, u_n^*, v_n^*\bigr)\,dx \ge \int_{[\bar{u}\neq 0]} \overline{F}\bigl(x, u_n^*, v_n^*\bigr)\,dx \to +\infty,$$

which contradicts (3.20). Therefore, $\lambda_1 = 0$ and

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} \Phi_1\bigl(|\tilde{u}_n|\bigr)\,dx = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} \Phi_2\bigl(|\tilde{v}_n|\bigr)\,dx = 0. \tag{3.23}$$

By Lemma 2.5, (ϕ_3) and the fact that

$$\limsup_{t\to+\infty} \frac{t^{l_i}}{\Phi_{i*}(t)} \le \limsup_{t\to+\infty} \frac{t^{l_i}}{\Phi_{i*}(1)\min\{t^{l_i^*}, t^{m_i^*}\}} = 0, \quad i = 1, 2,$$

imply that the embeddings $W^{1,\Phi_i}(\mathbb{R}^N) \hookrightarrow L^{l_i}(\mathbb{R}^N)(i = 1,2)$ are continuous. Hence, there exists a constant $M_1 > 0$ such that

$$\|\tilde{u}_n\|_{l_1}^{l_1} + \|\tilde{v}_n\|_{l_2}^{l_2} \le M_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.24}$$

For $p_i \in (l_i, l_i^*)(i = 1,2)$, by (ϕ_2) and (ϕ_3), we have

$$\lim_{t\to 0^+} \frac{t^{p_i}}{\Phi_i(t)} = 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{t^{p_i}}{\Phi_{i*}(t)} \le \lim_{t\to +\infty} \frac{t^{p_i}}{\Phi_{i*}(1)\min\{t^{l_i^*}, t^{m_i^*}\}} = 0, \quad i = 1,2. \tag{3.25}$$

Then, by the Lions type result for Orlicz-Sobolev spaces (see Theorem 1.3 in [8]), (3.23) and (3.25) imply that

$$\begin{aligned}
&\tilde{u}_n \to \mathbf{0} \quad \text{in } L^{p_1}(\mathbb{R}^N) \quad \text{and} \\
&\tilde{v}_n \to \mathbf{0} \quad \text{in } L^{p_2}(\mathbb{R}^N) \text{ for all } p_1 \in (l_1, l_1^*), p_2 \in (l_2, l_2^*).
\end{aligned} \tag{3.26}$$

Now, by (3.6), (V_2) and Lemma 2.2, we have

$$\begin{aligned}
&\frac{I_1(u_n, v_n)}{\|u_n\|_{1,\Phi_1}^{l_1} + \|v_n\|_{1,\Phi_2}^{l_2}} \\
&\quad \ge \frac{\int_{\mathbb{R}^N} \Phi_1(|\nabla u_n|)\,dx + \alpha_1 \int_{\mathbb{R}^N} \Phi_1(|u_n|)\,dx + \int_{\mathbb{R}^N} \Phi_2(|\nabla v_n|)\,dx + \alpha_1 \int_{\mathbb{R}^N} \Phi_2(|v_n|)\,dx}{\|u_n\|_{1,\Phi_1}^{l_1} + \|v_n\|_{1,\Phi_2}^{l_2}} \\
&\quad \ge \frac{\min\{\|\nabla u_n\|_{\Phi_1}^{l_1}, \|\nabla u_n\|_{\Phi_1}^{m_1}\} + \alpha_1 \min\{\|u_n\|_{\Phi_1}^{l_1}, \|u_n\|_{\Phi_1}^{m_1}\}}{\|u_n\|_{1,\Phi_1}^{l_1} + \|v_n\|_{1,\Phi_2}^{l_2}} \\
&\qquad + \frac{\min\{\|\nabla v_n\|_{\Phi_2}^{l_2}, \|\nabla v_n\|_{\Phi_2}^{m_2}\} + \alpha_1 \min\{\|v_n\|_{\Phi_2}^{l_2}, \|v_n\|_{\Phi_2}^{m_2}\}}{\|u_n\|_{1,\Phi_1}^{l_1} + \|v_n\|_{1,\Phi_2}^{l_2}} \\
&\quad \ge \frac{\|\nabla u_n\|_{\Phi_1}^{l_1} + \alpha_1 \|u_n\|_{\Phi_1}^{l_1} + \|\nabla v_n\|_{\Phi_2}^{l_2} + \alpha_1 \|v_n\|_{\Phi_2}^{l_2} - 2 - 2\alpha_1}{\|u_n\|_{1,\Phi_1}^{l_1} + \|v_n\|_{1,\Phi_2}^{l_2}} \\
&\quad \ge \frac{\min\{1, \alpha_1\} C_{l_1} \|u_n\|_{1,\Phi_1}^{l_1} + \min\{1, \alpha_1\} C_{l_2} \|v_n\|_{1,\Phi_2}^{l_2} - 2 - 2\alpha_1}{\|u_n\|_{1,\Phi_1}^{l_1} + \|v_n\|_{1,\Phi_2}^{l_2}} \\
&\quad \ge \min\{1, \alpha_1\} \min\{C_{l_1}, C_{l_2}\} + o_n(1).
\end{aligned} \tag{3.27}$$

Moreover, (3.11) and (2) in Lemma 2.1 imply that

$$\lim_{|(u,v)|\to 0} \frac{F(x,u,v)}{|u|^{l_1} + |v|^{l_2}} = 0$$

uniformly in $x \in \mathbb{R}^N$. Then, for any given constant $\varepsilon > 0$, there exists a constant $R_\varepsilon > 0$ such that

$$\frac{|F(x,u,v)|}{|u|^{l_1} + |v|^{l_2}} \le \varepsilon \quad \text{for all } x \in \mathbb{R}^N, \big|(u,v)\big| \le R_\varepsilon, \tag{3.28}$$

and by (F_1) and (F_5), for above $R_\varepsilon > 0$, there exists a constant $C_R > 0$ such that

$$\left(\frac{|F(x,u,v)|}{|u|^{l_1}+|v|^{l_2}}\right)^k \le C_R\overline{F}(x,u,v) \quad \text{for all } x\in\mathbb{R}^N, \bigl|(u,v)\bigr| > R_\varepsilon. \tag{3.29}$$

Let

$$X_n = \bigl\{x\in\mathbb{R}^N : \bigl|\bigl(u_n(x),v_n(x)\bigr)\bigr| \le R_\varepsilon\bigr\} \quad \text{and} \quad Y_n = \bigl\{x\in\mathbb{R}^N : \bigl|\bigl(u_n(x),v_n(x)\bigr)\bigr| > R_\varepsilon\bigr\}.$$

Then

$$\frac{|I_2(u_n,v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}} \le \int_{X_n}\frac{|F(x,u_n,v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}}\,dx + \int_{Y_n}\frac{|F(x,u_n,v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}}\,dx. \tag{3.30}$$

By (3.28) and (3.24), we have

$$\begin{aligned}\int_{X_n}\frac{|F(x,u_n,v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}}\,dx &= \int_{X_n}\frac{|F(x,u_n,v_n)|}{\frac{|u_n|^{l_1}}{|\tilde{u}_n|^{l_1}}+\frac{|v_n|^{l_2}}{|\tilde{v}_n|^{l_2}}}\,dx \\ &\le \int_{X_n}\frac{|F(x,u_n,v_n)|}{|u_n|^{l_1}+|v_n|^{l_2}}\bigl(|\tilde{u}_n|^{l_1}+|\tilde{v}_n|^{l_2}\bigr)\,dx \le \varepsilon M_1.\end{aligned} \tag{3.31}$$

Since $k > \max\{\frac{N}{l_1},\frac{N}{l_2}\}$, then $\frac{l_i k}{k-1}\in(l_i,l_i^*)(i=1,2)$. Hence, by (3.29), (3.18), (3.26) and the fact $\overline{F}(x,u,v)\ge 0$, for n large, we have

$$\begin{aligned}&\int_{Y_n}\frac{|F(x,u_n,v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}}\,dx \\ &\quad\le \int_{Y_n}\frac{|F(x,u_n,v_n)|}{|u_n|^{l_1}+|v_n|^{l_2}}\bigl(|\tilde{u}_n|^{l_1}+|\tilde{v}_n|^{l_2}\bigr)\,dx \\ &\quad\le \left(\int_{Y_n}\left(\frac{|F(x,u_n,v_n)|}{|u_n|^{l_1}+|v_n|^{l_2}}\right)^k dx\right)^{\frac{1}{k}}\left(\int_{Y_n}\bigl(|\tilde{u}_n|^{l_1}+|\tilde{v}_n|^{l_2}\bigr)^{\frac{k}{k-1}}\,dx\right)^{\frac{k-1}{k}} \\ &\quad\le \left(\int_{Y_n}C_R\overline{F}(x,u_n,v_n)\,dx\right)^{\frac{1}{k}}\left(\int_{\mathbb{R}^N}C_{\frac{k}{k-1}}\bigl(|\tilde{u}_n|^{\frac{l_1k}{k-1}}+|\tilde{v}_n|^{\frac{l_2k}{k-1}}\bigr)\,dx\right)^{\frac{k-1}{k}} \\ &\quad\le \bigl[C_R(c+1)\bigr]^{\frac{1}{k}}\Bigl[C_{\frac{k}{k-1}}\Bigl(\|\tilde{u}_n\|_{\frac{l_1k}{k-1}}^{\frac{l_1k}{k-1}}+\|\tilde{v}_n\|_{\frac{l_2k}{k-1}}^{\frac{l_2k}{k-1}}\Bigr)\Bigr]^{\frac{k-1}{k}} = o_n(1).\end{aligned} \tag{3.32}$$

Since ε is arbitrary, it follows from (3.30)-(3.32) that

$$\frac{I_2(u_n,v_n)}{\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}} \to 0 \quad \text{as } n\to\infty. \tag{3.33}$$

By dividing (3.17) by $\|u_n\|_{1,\Phi_1}^{l_1}+\|v_n\|_{1,\Phi_2}^{l_2}$ and letting $n\to\infty$, we get a contradiction via (3.27) and (3.33).

Case 2. Suppose that $\|u_n\|_{1,\Phi_1}\to\infty$ and $\|v_n\|_{1,\Phi_2}\le M_2$ for some constant $M_2>0$. Let $\tilde{u}_n = \frac{u_n}{\|u_n\|_{1,\Phi_1}}$ and $\tilde{v}_n = \frac{v_n}{\|u_n\|_{1,\Phi_1}}$. Then $\{\tilde{u}_n\}$ is bounded in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $\tilde{v}_n\to\mathbf{0}$ in $W^{1,\Phi_2}(\mathbb{R}^N)$. We claim that

$$\lambda_2 := \lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_2(y)}\bigl(\Phi_1\bigl(|\tilde{u}_n|\bigr)+\Phi_2\bigl(|\tilde{v}_n|\bigr)\bigr)\,dx = 0.$$

Indeed, if $\lambda_2 \neq 0$, there exist a constant $\delta > 0$, a subsequence of $\{(\tilde{u}_n, \tilde{v}_n)\}$, still denoted by $\{(\tilde{u}_n, \tilde{v}_n)\}$, and a sequence $\{z_n\} \in \mathbb{Z}^N$ such that

$$\int_{B_2(z_n)} \big(\Phi_1(|\tilde{u}_n|) + \Phi_2(|\tilde{v}_n|)\big)\,dx > \delta \quad \text{for all } n \in \mathbb{N}. \tag{3.34}$$

Let $\bar{u}_n = \tilde{u}_n(\cdot + z_n)$ and $\bar{v}_n = \tilde{v}_n(\cdot + z_n)$. Then $\|\bar{u}_n\|_{1,\Phi_1} = \|\tilde{u}_n\|_{1,\Phi_1}$ and $\|\bar{v}_n\|_{1,\Phi_2} = \|\tilde{v}_n\|_{1,\Phi_2}$, that is, $\{\bar{u}_n\}$ is bounded in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $\bar{v}_n \to \mathbf{0}$ in $W^{1,\Phi_2}(\mathbb{R}^N)$. Passing to a subsequence of $\{(\bar{u}_n, \bar{v}_n)\}$, still denoted by $\{(\bar{u}_n, \bar{v}_n)\}$, by Remark 2.7, there exists $(\bar{u}, \mathbf{0}) \in W$ such that

- $\star$ $\bar{u}_n \rightharpoonup \bar{u}$ in $W^{1,\Phi_1}(\mathbb{R}^N)$, $\bar{u}_n \to \bar{u}$ in $L^{\Phi_1}(B_2)$ and $\bar{u}_n(x) \to \bar{u}(x)$ a.e. in B_2;
- $\star$ $\bar{v}_n \to \mathbf{0}$ in $W^{1,\Phi_2}(\mathbb{R}^N)$, $\bar{v}_n \to \mathbf{0}$ in $L^{\Phi_2}(B_2)$ and $\bar{v}_n(x) \to \mathbf{0}$ a.e. in B_2.

Since

$$\int_{B_2} \big(\Phi_1(|\bar{u}_n|) + \Phi_2(|\bar{v}_n|)\big)\,dx = \int_{B_2(z_n)} \big(\Phi_1(|\tilde{u}_n|) + \Phi_2(|\tilde{v}_n|)\big)\,dx,$$

then, by (3.34), $\star$ and (2.3), we obtain that $\bar{u} \neq \mathbf{0}$ in $L^{\Phi_1}(B_2)$, that is, $[\bar{u} \neq 0] := \{x \in B_2 : \bar{u}(x) \neq 0\}$ has nonzero Lebesgue measure. Let $u_n^* = u_n(\cdot + z_n)$ and $v_n^* = v_n(\cdot + z_n)$. Then $\|(u_n^*, v_n^*)\| = \|(u_n, v_n)\|$ and

$$\big|u_n^*(x)\big| = \big|\bar{u}_n(x)\big|\|u_n\|_{1,\Phi_1} \to \infty, \quad \text{a.e. } x \in [\bar{u} \neq 0]. \tag{3.35}$$

Since $V_i (i = 1, 2)$ and F are 1-periodic in x, $\{(u_n^*, v_n^*)\}$ is also a $(C)_c$-sequence of I. Then, by (3.18), for n large, we have

$$\int_{\mathbb{R}^N} \overline{F}(x, u_n^*, v_n^*)\,dx \leq c + 1. \tag{3.36}$$

However, it follows from (F_5), (3.35), (3.21) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \overline{F}(x, u_n^*, v_n^*)\,dx \geq \int_{[\bar{u} \neq 0]} \overline{F}(x, u_n^*, v_n^*)\,dx = +\infty,$$

which contradicts (3.36). Therefore, $\lambda_2 = 0$ and

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_2(y)} \Phi_1(|\tilde{u}_n|)\,dx = 0. \tag{3.37}$$

Then, by the Lions type result for Orlicz-Sobolev spaces (see Theorem 1.3 in [8]) again, (3.37), (3.25) and the fact $\frac{l_1 k}{k-1} \in (l_1, l_1^*)$ imply that

$$\tilde{u}_n \to \mathbf{0} \quad \text{in } L^{\frac{l_1 k}{k-1}}(\mathbb{R}^N). \tag{3.38}$$

Since the embeddings $W^{1,\Phi_i}(\mathbb{R}^N) \hookrightarrow L^{l_i}(\mathbb{R}^N) (i = 1, 2)$ are continuous, there exists a constant $M_3 > 0$ such that

$$\|\tilde{u}_n\|_{l_1}^{l_1} + \|v_n\|_{l_2}^{l_2} \leq M_3 \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

Moreover, $\frac{l_2 k}{k-1} \in (l_2, l_2^*)$, (3.25) and Lemma 2.5 imply that the embedding $W^{1,\Phi_2}(\mathbb{R}^N) \hookrightarrow L^{\frac{l_2 k}{k-1}}(\mathbb{R}^N)$ is continuous. Hence, there exists a constant $M_4 > 0$ such that

$$\|v_n\|_{\frac{l_2 k}{k-1}}^{l_2} \le M_4 \quad \text{for all } n \in \mathbb{N}. \tag{3.40}$$

So, for any given constant $M > 1$, by (3.6), (V_2) and Lemma 2.2, we have

$$\begin{aligned}
\frac{I_1(u_n, v_n)}{\|u_n\|_{1,\Phi_1}^{l_1} + M} &\ge \frac{\min\{\|\nabla u_n\|_{\Phi_1}^{l_1}, \|\nabla u_n\|_{\Phi_1}^{m_1}\} + \alpha_1 \min\{\|u_n\|_{\Phi_1}^{l_1}, \|u_n\|_{\Phi_1}^{m_1}\}}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \\
&\ge \frac{\|\nabla u_n\|_{\Phi_1}^{l_1} + \alpha_1 \|u_n\|_{\Phi_1}^{l_1} - 1 - \alpha_1}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \\
&\ge \frac{\min\{1, \alpha_1\} C_{l_1} \|u_n\|_{1,\Phi_1}^{l_1} - 1 - \alpha_1}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \\
&= \min\{1, \alpha_1\} C_{l_1} + o_n(1).
\end{aligned} \tag{3.41}$$

It is obvious that (3.28) and (3.29) still hold for this case. Based on this fact, let

$$\begin{aligned}
X_n &= \{x \in \mathbb{R}^N : |(u_n(x), v_n(x))| \le R_\varepsilon\} \quad \text{and} \\
Y_n &= \{x \in \mathbb{R}^N : |(u_n(x), v_n(x))| > R_\varepsilon\}.
\end{aligned}$$

Then

$$\frac{|I_2(u_n, v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \le \int_{X_n} \frac{|F(x, u_n, v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \, dx + \int_{Y_n} \frac{|F(x, u_n, v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \, dx. \tag{3.42}$$

By (3.28) and (3.39), we have

$$\begin{aligned}
\int_{X_n} \frac{|F(x, u_n, v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \, dx &\le \int_{X_n} \frac{|F(x, u_n, v_n)|}{|u_n|^{l_1} + |v_n|^{l_2}} \left(|\tilde{u}_n|^{l_1} + \frac{1}{M} |v_n|^{l_2} \right) dx \\
&\le \varepsilon \left(\|\tilde{u}_n\|_{l_1}^{l_1} + \frac{1}{M} \|v_n\|_{l_2}^{l_2} \right) \le \varepsilon M_3.
\end{aligned} \tag{3.43}$$

Note that $\frac{l_i k}{k-1} \in (l_i, l_i^*) (i = 1, 2)$. By (3.29), (3.18), (3.38), (3.40) and the fact $\overline{F}(x, u, v) \ge 0$, for n large, we have

$$\begin{aligned}
&\int_{Y_n} \frac{|F(x, u_n, v_n)|}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \, dx \\
&\quad \le \int_{Y_n} \frac{|F(x, u_n, v_n)|}{|u_n|^{l_1} + |v_n|^{l_2}} \left(|\tilde{u}_n|^{l_1} + \frac{1}{M} |v_n|^{l_2} \right) dx \\
&\quad \le \left(\int_{Y_n} \left(\frac{|F(x, u_n, v_n)|}{|u_n|^{l_1} + |v_n|^{l_2}} \right)^k dx \right)^{\frac{1}{k}} \left(\int_{Y_n} \left(|\tilde{u}_n|^{l_1} + \frac{1}{M} |v_n|^{l_2} \right)^{\frac{k}{k-1}} dx \right)^{\frac{k-1}{k}} \\
&\quad \le \left(\int_{Y_n} C_R \overline{F}(x, u_n, v_n) \, dx \right)^{\frac{1}{k}} \left[C_{\frac{k}{k-1}} \left(\|\tilde{u}_n\|_{\frac{l_1 k}{k-1}}^{\frac{l_1 k}{k-1}} + \left(\frac{1}{M} \right)^{\frac{k}{k-1}} \|v_n\|_{\frac{l_2 k}{k-1}}^{\frac{l_2 k}{k-1}} \right) \right]^{\frac{k-1}{k}}
\end{aligned}$$

$$\leq \big[C_R(c+1)\big]^{\frac{1}{k}} C_{\frac{k}{k-1}}^{\frac{k-1}{k}} C_{\frac{k-1}{k}} \left(\|\tilde{u}_n\|_{\frac{l_1 k}{k-1}}^{l_1} + \frac{1}{M}\|v_n\|_{\frac{l_2 k}{k-1}}^{l_2} \right)$$

$$\leq \big[C_R(c+1)\big]^{\frac{1}{k}} C_{\frac{k}{k-1}}^{\frac{k-1}{k}} C_{\frac{k-1}{k}} \left(o_n(1) + \frac{M_4}{M} \right). \tag{3.44}$$

Since $\varepsilon > 0$ and $M > 1$ are arbitrary, it follows from (3.42)-(3.44) that

$$\frac{I_2(u_n, v_n)}{\|u_n\|_{1,\Phi_1}^{l_1} + M} \to 0 \quad \text{as } n \to \infty. \tag{3.45}$$

By dividing (3.17) by $\|u_n\|_{1,\Phi_1}^{l_1} + M$ and letting $n \to \infty$, we get a contradiction via (3.41) and (3.45).

Case 3. Suppose that $\|\nabla u_n\|_{\Phi_1} \leq M_5$ for some constant $M_5 > 0$ and $\|\nabla v_n\|_{\Phi_2} \to \infty$. For this case, with the same discussion as Case 2, we can also get a contradiction. □

Lemma 3.18 *System* (1.1) *has a nontrivial solution under the assumptions of Theorems* 3.1 *and* 3.2, *respectively.*

Proof For the level $c > 0$ given in (3.13), there exists a $(C)_c$-sequence $\{(u_n, v_n)\}$ for I in W. Moreover, Lemmas 3.16 and 3.17 show that the sequence $\{(u_n, v_n)\}$ is bounded in W. We claim that

$$\lambda_3 := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} \big(\Phi_1\big(|u_n|\big) + \Phi_2\big(|v_n|\big)\big)\,dx > 0.$$

Indeed, if $\lambda_3 = 0$, then

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} \Phi_1\big(|u_n|\big)\,dx = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} \Phi_2\big(|v_n|\big)\,dx = 0.$$

By using the Lions type result for Orlicz-Sobolev spaces (see Theorem 1.3 in [8]) again, we have

$$\begin{aligned} &u_n \to \mathbf{0} \quad \text{in } L^{q_1}\big(\mathbb{R}^N\big) \quad \text{and} \\ &v_n \to \mathbf{0} \quad \text{in } L^{q_2}\big(\mathbb{R}^N\big), \text{ for all } q_1 \in \big(m_1, l_1^*\big), q_2 \in \big(m_2, l_2^*\big). \end{aligned} \tag{3.46}$$

Given $q_i \in (m_i, l_i^*)(i = 1,2)$, by (F_1), (F_2), (ϕ_1), (ϕ_2) and (2.1), for any given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} \big|F(x,u_n,v_n)\big| &\leq \varepsilon\big(\Phi_1\big(|u_n|\big) + \Phi_2\big(|v_n|\big) + \Phi_{1*}\big(|u_n|\big) + \Phi_{2*}\big(|v_n|\big)\big) + C_\varepsilon\big(|u_n|^{q_1} + |v_n|^{q_2}\big), \\ \big|u_n F_u(x,u_n,v_n)\big| &\leq \varepsilon\big(\Phi_1\big(|u_n|\big) + \Phi_2\big(|v_n|\big) + \Phi_{1*}\big(|u_n|\big) + \Phi_{2*}\big(|v_n|\big)\big) \\ &\quad + C_\varepsilon\big(|u_n|^{q_1} + |v_n|^{q_2}\big), \\ \big|v_n F_v(x,u_n,v_n)\big| &\leq \varepsilon\big(\Phi_1\big(|u_n|\big) + \Phi_2\big(|v_n|\big) + \Phi_{1*}\big(|u_n|\big) + \Phi_{2*}\big(|v_n|\big)\big) \\ &\quad + C_\varepsilon\big(|u_n|^{q_1} + |v_n|^{q_2}\big), \end{aligned} \tag{3.47}$$

for all $x \in \mathbb{R}^N$. Then it follows from Lemma 2.2, (2) in Lemma 2.4, (2.8), (3.46) and the arbitrariness of $\varepsilon > 0$ that

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_n,v_n)\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N} u_nF_u(x,u_n,v_n)\,dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N} v_nF_v(x,u_n,v_n)\,dx = 0.\end{aligned} \tag{3.48}$$

Hence, by (3.4), (3.5), (3.14), (ϕ_2), (V_2) and (3.48), we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left\{I(u_n,v_n) - \left\langle I'(u_n,v_n),\left(\frac{1}{l_1}u_n,\frac{1}{l_2}v_n\right)\right\rangle\right\} \\ &= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^N}\Phi_1\bigl(|\nabla u_n|\bigr)\,dx - \int_{\mathbb{R}^N}\frac{1}{l_1}a_1\bigl(|\nabla u_n|\bigr)|\nabla u_n|^2\,dx\right. \\ &\quad + \int_{\mathbb{R}^N}V_1(x)\Phi_1\bigl(|u_n|\bigr)\,dx - \int_{\mathbb{R}^N}\frac{1}{l_1}V_1(x)a_1\bigl(|u_n|\bigr)|u_n|^2\,dx \\ &\quad + \int_{\mathbb{R}^N}\Phi_2\bigl(|\nabla v_n|\bigr)\,dx - \int_{\mathbb{R}^N}\frac{1}{l_2}a_2\bigl(|\nabla v_n|\bigr)|\nabla v_n|^2\,dx \\ &\quad + \int_{\mathbb{R}^N}V_2(x)\Phi_2\bigl(|v_n|\bigr)\,dx - \int_{\mathbb{R}^N}\frac{1}{l_2}V_2(x)a_2\bigl(|v_n|\bigr)|v_n|^2\,dx \\ &\quad \left.+ \int_{\mathbb{R}^N}\left(\frac{1}{l_1}u_nF_u(x,u_n,v_n) + \frac{1}{l_2}v_nF_v(x,u_n,v_n) - F(x,u_n,v_n)\right)dx\right\} \\ &\le \lim_{n\to\infty}\left\{\int_{\mathbb{R}^N}\left(\frac{1}{l_1}u_nF_u(x,u_n,v_n) + \frac{1}{l_2}v_nF_v(x,u_n,v_n) - F(x,u_n,v_n)\right)dx\right\} = 0,\end{aligned}$$

which contradicts $c > 0$. Therefore, $\lambda_3 > 0$, which implies that there exist a constant $\delta > 0$, a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, and a sequence $\{z_n\} \in \mathbb{Z}^N$ such that

$$\begin{aligned}&\int_{B_2(z_n)}\bigl(\Phi_1\bigl(|u_n|\bigr) + \Phi_2\bigl(|v_n|\bigr)\bigr)\,dx = \int_{B_2}\bigl(\Phi_1\bigl(|u_n^*|\bigr) + \Phi_2\bigl(|v_n^*|\bigr)\bigr)\,dx > \delta \\ &\quad\text{for all } n \in \mathbb{N},\end{aligned} \tag{3.49}$$

where $u_n^* := u_n(\cdot + z_n)$ and $v_n^* := v_n(\cdot + z_n)$. Since $\|u_n^*\|_{1,\Phi_1} = \|u_n\|_{1,\Phi_1}$ and $\|v_n^*\|_{1,\Phi_2} = \|v_n\|_{1,\Phi_2}$, then $\{u_n^*\}$ and $\{v_n^*\}$ are bounded in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. Passing to a subsequence of $\{(u_n^*,v_n^*)\}$, still denoted by $\{(u_n^*,v_n^*)\}$, there exists $(u^*,v^*) \in W$ such that $u_n^* \rightharpoonup u^*$ in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $v_n^* \rightharpoonup v^*$ in $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. Moreover, for any given constant $r > 0$, by Remark 2.7 and the similar arguments as those in Lemma 4.3 in [8], we can assume that

- ⋆ $u_n^* \to u^*$ in $L^{\Phi_1}(B_r)$ and $u_n^*(x) \to u^*(x)$, $\nabla u_n^*(x) \to \nabla u^*(x)$ a.e. in B_r;
- ⋆ $v_n^* \to v^*$ in $L^{\Phi_2}(B_r)$ and $v_n^*(x) \to v^*(x)$, $\nabla v_n^*(x) \to \nabla v^*(x)$ a.e. in B_r.

Then, by (3.49), ⋆ and (2.3), we obtain that $(u^*,v^*) \neq (\mathbf{0},\mathbf{0})$. Since $V_i (i = 1,2)$ and F are 1-periodic in x, $\{(u_n^*,v_n^*)\}$ is also a $(C)_c$-sequence of I. Then, for any given point $(w_1,w_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ with $\mathrm{supp}\{w_1\} \cup \mathrm{supp}\{w_2\} \subset B_r$ for some $r > 0$, we have

$$\lim_{n\to\infty}\bigl\langle I'\bigl(u_n^*,v_n^*\bigr),(w_1,w_2)\bigr\rangle = 0.$$

We claim that

$$\lim_{n\to\infty}\langle I'(u_n^*,v_n^*),(w_1,w_2)\rangle = \langle I'(u^*,v^*),(w_1,w_2)\rangle. \tag{3.50}$$

First, we claim

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)a_1(|u_n^*|)u_n^* w_1\,dx = \int_{\mathbb{R}^N} V_1(x)a_1(|u^*|)u^* w_1\,dx. \tag{3.51}$$

Indeed, it follows from (ϕ_1), (ϕ_2), (V_2), $\star$ and the boundedness of sequence $\{u_n^*\}$ in $W^{1,\Phi_1}(\mathbb{R}^N)$ that the sequence $\{V_1(x)a_1(|u_n^*|)u_n^*\}$ is bounded in $L^{\widetilde{\Phi}_1}(B_r)$ and $V_1(x)\times a_1(|u_n^*(x)|)u_n^*(x)\to V_1(x)a_1(|u^*(x)|)u^*(x)$ a.e. $x\in B_r$. Then, by applying Lemma 2.1 in [8], we get (3.51) because $w_1\in L^{\Phi_1}(B_r)$. Similarly, we can get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_2(x)a_2(|v_n^*|)v_n^* w_2\,dx = \int_{\mathbb{R}^N} V_2(x)a_2(|v^*|)v^* w_2\,dx. \tag{3.52}$$

Next, we claim

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F_u(x,u_n^*,v_n^*)w_1\,dx = \int_{\mathbb{R}^N} F_u(x,u^*,v^*)w_1\,dx. \tag{3.53}$$

Indeed, it follows from (ϕ_1), (ϕ_2), (F_1), (F_2), $\star$, the boundedness of sequence $\{(u_n^*,v_n^*)\}$ in W and Remark 2.7 that the sequence $\{F_u(x,u_n^*,v_n^*)\}$ is bounded in $L^{\widetilde{\Phi}_{1*}}(B_r)$ and $F_u(x,u_n^*(x),v_n^*(x))\to F_u(x,u^*(x),v^*(x))$ a.e. $x\in B_r$. Then, by applying Lemma 2.1 in [8] again, we get (3.53) because $w_1\in L^{\Phi_{1*}}(B_r)$. Similarly, we can get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F_v(x,u_n^*,v_n^*)w_2\,dx = \int_{\mathbb{R}^N} F_v(x,u^*,v^*)w_2\,dx. \tag{3.54}$$

Finally, we claim

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a_1(|\nabla u_n^*|)\nabla u_n^*\nabla w_1\,dx = \int_{\mathbb{R}^N} a_1(|\nabla u^*|)\nabla u^*\nabla w_1\,dx \tag{3.55}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a_2(|\nabla v_n^*|)\nabla v_n^*\nabla w_2\,dx = \int_{\mathbb{R}^N} a_2(|\nabla v^*|)\nabla v^*\nabla w_2\,dx. \tag{3.56}$$

In fact, the boundedness of sequence $\{(u_n^*,v_n^*)\}$ implies that sequences $\{a_1(|\nabla u_n^*|)\frac{\partial u_n^*}{\partial x_j}\}$ $(j=1,2,\dots,N)$ are bounded in $L^{\widetilde{\Phi}_1}(B_r)$. Moreover, (ϕ_1) and $\star$ imply that $a_1(|\nabla u_n^*(x)|)\frac{\partial u_n^*(x)}{\partial x_j}\to a_1(|\nabla u^*(x)|)\frac{\partial u^*(x)}{\partial x_j}$ $(j=1,2,\dots,N)$ a.e. $x\in B_r$. Then, by applying Lemma 2.1 in [8] again, we get (3.55) because $\frac{\partial w_1}{\partial x_j}\in L^{\Phi_1}(B_r)(j=1,2,\dots,N)$. Similarly, we can get (3.56). Hence, it follows from (3.51)-(3.56) that (3.50) holds, that is, $\langle I'(u^*,v^*),(w_1,w_2)\rangle=0$ for all $(w_1,w_2)\in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$. Now, we can conclude that $I'(u^*,v^*)=0$ because $C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$ is dense in W. □

Proof of Theorem 3.1 Lemma 3.18 shows that system (1.1) has at least a nontrivial solution. Next, we prove that system (1.1) has a ground state. Let

$$d=\inf\{I(u,v):(u,v)\neq(\mathbf{0},\mathbf{0})\text{ and } I'(u,v)=0\}.$$

First, we claim that $d \geq 0$. Indeed, for any given nontrivial critical point (u,v) of I, by (3.4), (3.5), (ϕ_2), (V_2) and (F_3), we have

$$\begin{aligned}
I(u,v) &= I(u,v) - \left\langle I'(u,v), \left(\frac{1}{\mu_1}u, \frac{1}{\mu_2}v\right)\right\rangle \\
&= \int_{\mathbb{R}^N} \left(\Phi_1(|\nabla u|) - \frac{1}{\mu_1}a_1(|\nabla u|)|\nabla u|^2\right) dx \\
&\quad + \int_{\mathbb{R}^N} V_1(x)\left(\Phi_1(|u|) - \frac{1}{\mu_1}a_1(|u|)|u|^2\right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\Phi_2(|\nabla v|) - \frac{1}{\mu_2}a_2(|\nabla v|)|\nabla v|^2\right) dx \\
&\quad + \int_{\mathbb{R}^N} V_2(x)\left(\Phi_2(|v|) - \frac{1}{\mu_2}a_2(|v|)|v|^2\right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\mu_1}uF_u(x,u,v) + \frac{1}{\mu_2}vF_v(x,u,v) - F(x,u,v)\right) dx \\
&\geq \left(1 - \frac{m_1}{\mu_1}\right)\int_{\mathbb{R}^N} \left(\Phi_1(|\nabla u|) + \alpha_1\Phi_1(|u|)\right) dx \\
&\quad + \left(1 - \frac{m_2}{\mu_2}\right)\int_{\mathbb{R}^N} \left(\Phi_2(|\nabla v|) + \alpha_1\Phi_2(|v|)\right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\mu_1}uF_u(x,u,v) + \frac{1}{\mu_2}vF_v(x,u,v) - F(x,u,v)\right) dx \geq 0.
\end{aligned}$$

Since the nontrivial critical point (u,v) of I is arbitrary, we conclude $d \geq 0$. Choose a sequence $\{(u_n,v_n)\} \subset \{(u,v) \in W : (u,v) \neq (\mathbf{0},\mathbf{0}) \text{ and } I'(u,v) = 0\}$ such that $I(u_n,v_n) \to d$ as $n \to \infty$. Then it is obvious that $\{(u_n,v_n)\}$ is a $(C)_d$-sequence of I for the level $d \geq 0$. Lemma 3.16 shows that $\{(u_n,v_n)\}$ is bounded in W. Moreover, Lemma A.3 in the Appendix implies that there exists a constant $M_6 > 0$ such that

$$\|(u_n,v_n)\| \geq M_6 \quad \text{for all } n \in \mathbb{N}. \tag{3.57}$$

We claim that

$$\lambda_4 := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} \left(\Phi_1(|u_n|) + \Phi_2(|v_n|)\right) dx > 0.$$

Indeed, if $\lambda_4 = 0$, similar to (3.48), we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} u_nF_u(x,u_n,v_n)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} v_nF_v(x,u_n,v_n)\, dx = 0. \tag{3.58}$$

Then, by (3.5), (ϕ_2), (V_2) and (3.58), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \left\{\langle I'(u_n,v_n),(u_n,v_n)\rangle + \int_{\mathbb{R}^N} u_nF_u(x,u_n,v_n)\, dx + \int_{\mathbb{R}^N} v_nF_v(x,u_n,v_n)\, dx\right\} \\
&= \lim_{n\to\infty} \left\{\int_{\mathbb{R}^N} a_1(|\nabla u_n|)|\nabla u_n|^2\, dx + \int_{\mathbb{R}^N} V_1(x)a_1(|u_n|)|u_n|^2\, dx \right. \\
&\quad \left. + \int_{\mathbb{R}^N} a_2(|\nabla v_n|)|\nabla v_n|^2\, dx + \int_{\mathbb{R}^N} V_2(x)a_2(|v_n|)|v_n|^2\, dx\right\}
\end{aligned}$$

$$\begin{aligned}&\geq \lim_{n\to\infty}\biggl\{l_1\int_{\mathbb{R}^N}\Phi_1\bigl(|\nabla u_n|\bigr)\,dx + l_1\alpha_1\int_{\mathbb{R}^N}\Phi_1\bigl(|u_n|\bigr)\,dx \\ &\quad + l_2\int_{\mathbb{R}^N}\Phi_2\bigl(|\nabla v_n|\bigr)\,dx + l_2\alpha_1\int_{\mathbb{R}^N}\Phi_2\bigl(|v_n|\bigr)\,dx\biggr\} \\ &\geq 0,\end{aligned}$$

which, together with (2.3), implies that $\|(u_n,v_n)\| = \|\nabla u_n\|_{\Phi_1} + \|u_n\|_{\Phi_1} + \|\nabla v_n\|_{\Phi_2} + \|v_n\|_{\Phi_2} \to 0$, which contradicts (3.57). Therefore, $\lambda_4 > 0$, which implies that there exist a constant $\delta > 0$, a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, and a sequence $\{z_n\} \in \mathbb{Z}^N$ such that

$$\begin{aligned}&\int_{B_2(z_n)}\bigl(\Phi_1\bigl(|u_n|\bigr)+\Phi_2\bigl(|v_n|\bigr)\bigr)\,dx = \int_{B_2}\bigl(\Phi_1\bigl(\bigl|u_n^*\bigr|\bigr)+\Phi_2\bigl(\bigl|v_n^*\bigr|\bigr)\bigr)\,dx > \delta \\ &\text{for all } n\in\mathbb{N},\end{aligned} \tag{3.59}$$

where $u_n^* := u_n(\cdot + z_n)$ and $v_n^* := v_n(\cdot + z_n)$. Since $\|u_n^*\|_{1,\Phi_1} = \|u_n\|_{1,\Phi_1}$ and $\|v_n^*\|_{1,\Phi_2} = \|v_n\|_{1,\Phi_2}$, then $\{u_n^*\}$ and $\{v_n^*\}$ are bounded in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. Passing to a subsequence of $\{(u_n^*,v_n^*)\}$, still denoted by $\{(u_n^*,v_n^*)\}$, there exists $(u_0,v_0) \in W$ such that $u_n^* \rightharpoonup u_0$ in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $v_n^* \rightharpoonup v_0$ in $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. Moreover, for any given constant $r > 0$, by Remark 2.7, we can assume that

- ⋆ $u_n^* \to u_0$ in $L^{\Phi_1}(B_r)$ and $u_n^*(x) \to u_0(x)$ a.e. in B_r;
- ⋆ $v_n^* \to v_0$ in $L^{\Phi_2}(B_r)$ and $v_n^*(x) \to v_0(x)$ a.e. in B_r.

Then, by (3.59), ⋆ and (2.3), we obtain that $(u_0,v_0) \neq (\mathbf{0},\mathbf{0})$. Since $V_i (i = 1,2)$ and F are 1-periodic in x, $\{(u_n^*,v_n^*)\}$ is also a $(C)_d$-sequence of I with $\{(u_n^*,v_n^*)\} \subset \{(u,v) \in W : (u,v) \neq (\mathbf{0},\mathbf{0}) \text{ and } I'(u,v) = 0\}$. Then similar arguments as those in Lemma 3.18 show that $I'(u_0,v_0) = 0$, and thus $I(u_0,v_0) \geq d$. However, for any given constant $r > 0$, it follows from (3.4), (3.5), (ϕ_2), (V_2), (F_3), ⋆ and Fatou's lemma that

$$\begin{aligned}&\int_{B_r}\biggl(\Phi_1\bigl(|\nabla u_0|\bigr) - \frac{1}{\mu_1}a_1\bigl(|\nabla u_0|\bigr)|\nabla u_0|^2\biggr)\,dx \\ &\quad + \int_{B_r}V_1(x)\biggl(\Phi_1\bigl(|u_0|\bigr) - \frac{1}{\mu_1}a_1\bigl(|u_0|\bigr)|u_0|^2\biggr)\,dx \\ &\quad + \int_{B_r}\biggl(\Phi_2\bigl(|\nabla v_0|\bigr) - \frac{1}{\mu_2}a_2\bigl(|\nabla v_0|\bigr)|\nabla v_0|^2\biggr)\,dx \\ &\quad + \int_{B_r}V_2(x)\biggl(\Phi_2\bigl(|v_0|\bigr) - \frac{1}{\mu_2}a_2\bigl(|v_0|\bigr)|v_0|^2\biggr)\,dx \\ &\quad + \int_{B_r}\biggl(\frac{1}{\mu_1}u_0F_u(x,u_0,v_0) + \frac{1}{\mu_2}v_0F_v(x,u_0,v_0) - F(x,u_0,v_0)\biggr)\,dx \\ &\leq \liminf_{n\to\infty}\biggl\{\int_{B_r}\biggl(\Phi_1\bigl(\bigl|\nabla u_n^*\bigr|\bigr) - \frac{1}{\mu_1}a_1\bigl(\bigl|\nabla u_n^*\bigr|\bigr)\bigl|\nabla u_n^*\bigr|^2\biggr)\,dx \\ &\quad + \int_{B_r}V_1(x)\biggl(\Phi_1\bigl(\bigl|u_n^*\bigr|\bigr) - \frac{1}{\mu_1}a_1\bigl(\bigl|u_n^*\bigr|\bigr)\bigl|u_n^*\bigr|^2\biggr)\,dx \\ &\quad + \int_{B_r}\biggl(\Phi_2\bigl(\bigl|\nabla v_n^*\bigr|\bigr) - \frac{1}{\mu_2}a_2\bigl(\bigl|\nabla v_n^*\bigr|\bigr)\bigl|\nabla v_n^*\bigr|^2\biggr)\,dx\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_r} V_2(x)\left(\Phi_2\left(\left|v_n^*\right|\right)-\frac{1}{\mu_2}a_2\left(\left|v_n^*\right|\right)\left|v_n^*\right|^2\right)dx \\
&+\int_{B_r}\left(\frac{1}{\mu_1}u_n^*F_u\left(x,u_n^*,v_n^*\right)+\frac{1}{\mu_2}v_n^*F_v\left(x,u_n^*,v_n^*\right)-F\left(x,u_n^*,v_n^*\right)\right)dx\Bigg\} \\
\le{}& \liminf_{n\to\infty}\left\{I\left(u_n^*,v_n^*\right)-\left\langle I'\left(u_n^*,v_n^*\right),\left(\frac{1}{\mu_1}u_n^*,\frac{1}{\mu_2}v_n^*\right)\right\rangle\right\} \\
={}& d.
\end{aligned}
$$

Since $r > 0$ is arbitrary, then $I(u_0,v_0) = I(u_0,v_0) - \langle I'(u_0,v_0),(\frac{1}{\mu_1}u_0,\frac{1}{\mu_2}v_0)\rangle \le d$. Therefore, $I(u_0,v_0) = d$, that is, (u_0,v_0) is a ground state of system (1.1). □

Proof of Theorem 3.2 Lemma 3.18 shows that system (1.1) has at least a nontrivial solution under the assumptions of Theorem 3.2. Moreover, following the same steps as in the above proof of Theorem 3.1 but replacing μ_i with $m_i (i = 1,2)$, we can find a ground state of system (1.1). □

4 Examples

For system (1.1), $\phi_i (i = 1,2)$ defined by (1.2) can be chosen from the following cases which satisfy all (ϕ_1)-(ϕ_3) type conditions:

Case 1. Let $\phi(t) = |t|^{p-1}t$ for $t \neq 0$, $\phi(0) = 0$ with $1 < p+1 < N$. In this case, simple computations show that $l = m = p+1$;

Case 2. Let $\phi(t) = |t|^{p-1}t + |t|^{q-1}t$ for $t \neq 0$, $\phi(0) = 0$ with $1 < p+1 < q+1 < N < \frac{(p+1)(q+1)}{q-p}$. In this case, simple computations show that $l = p+1, m = q+1$;

Case 3. Let $\phi(t) = \frac{|t|^{q-1}t}{\log(1+|t|^p)}$ for $t \neq 0$, $\phi(0) = 0$ with $1 < p+1 < q+1 < N < \frac{(q-p+1)(q+1)}{p}$. In this case, simple computations show that $l = q-p+1, m = q+1$.

Moreover, we also give a case that satisfies (ϕ_1) and (ϕ_2) but does not satisfy (ϕ_3) type conditions:

Case 4. Let $\phi(t) = |t|^{q-1}t\log(1+|t|^p)$ for $t \neq 0$, $\phi(0) = 0$ with $p,q > 0$ and $p+q+1 < N < \frac{(q+1)(p+q+1)}{p}$. In this case, simple computations show that $l = q+1, m = p+q+1$.

Example 4.1 Assume that $V_i (i = 1,2)$ and $\phi_i (i = 1,2)$ defined by (1.2) satisfy (V_1), (V_2), (ϕ_1) and (ϕ_2) with $m_i \ge 4 (i = 1,2)$. Let $F(x,u,v) = |u|^{\frac{m_1+l_1^*}{2}} + |v|^{\frac{m_2+l_2^*}{2}} + |u|^{\frac{m_1+l_1^*}{4}}|v|^{\frac{m_2+l_2^*}{4}}$. Choose $\mu_i = \frac{m_i+l_i^*}{2} (i = 1,2)$. Then it is easy to check that F satisfies (F_1), $(F_2)'$ and (F_3).

Example 4.2 Assume that $V_i (i = 1,2)$ and $\phi_i (i = 1,2)$ defined by (1.2) satisfy (V_1), (V_2), (ϕ_1)-(ϕ_3) with $m_i \ge 4 (i = 1,2)$, $\max\{\frac{N}{l_1},\frac{N}{l_2}\} < \min\{\frac{m_1}{m_1-l_1},\frac{m_2}{m_2-l_2}\}$. Then $F(x,u,v) = |u|^{m_1}\log(1+|u|) + |v|^{m_2}\log(1+|v|) + |u|^{\frac{m_1+\epsilon}{2}}|v|^{\frac{m_2+\epsilon}{2}}$, where constant $\epsilon > 0$ satisfying $\epsilon < \frac{2l_1^*l_2^*-m_1l_2^*-m_2l_1^*}{l_1^*+l_2^*}$ and $\max\{\frac{N}{l_1},\frac{N}{l_2}\} < \min\{\frac{m_1}{m_1-l_1+\epsilon},\frac{m_2}{m_2-l_2+\epsilon}\}$ satisfies (F_1), $(F_2)'$, $(F_4)'$ and (F_5). In fact,

$$
\begin{aligned}
&F_u(x,u,v) = m_1|u|^{m_1-2}u\log\left(1+|u|\right) + \frac{|u|^{m_1-1}u}{1+|u|} + \frac{m_1+\epsilon}{2}|u|^{\frac{m_1+\epsilon-4}{2}}|v|^{\frac{m_2+\epsilon}{2}}u, \\
&F_v(x,u,v) = m_2|v|^{m_2-2}v\log\left(1+|v|\right) + \frac{|v|^{m_2-1}v}{1+|v|} + \frac{m_2+\epsilon}{2}|u|^{\frac{m_2+\epsilon}{2}}|v|^{\frac{m_2+\epsilon-4}{2}}v,
\end{aligned}
$$

then

$$\begin{aligned}\overline{F}(x,u,v) &= \frac{|u|^{m_1+1}}{m_1(1+|u|)} + \frac{|v|^{m_2+1}}{m_2(1+|v|)} + \frac{(m_1+m_2)\epsilon}{2m_1m_2}|u|^{\frac{m_1+\epsilon}{2}}|v|^{\frac{m_2+\epsilon}{2}} \\ &\geq \frac{|u|^{m_1+1}}{m_1(1+|u|)} + \frac{|v|^{m_2+1}}{m_2(1+|v|)}.\end{aligned}$$

It is obvious that F satisfies (F_1) and $(F_4)'$. Since $0 < \epsilon < \frac{2l_1^* l_2^* - m_1 l_2^* - m_2 l_1^*}{l_1^* + l_2^*}$, then it is easy to check $(F_2)'$ by Young's inequality. Next, we check (F_5). It is obvious that $\overline{F}(x,u,v) > 0$ for all $(u,v) \neq (0,0)$. Moreover, choose $\max\{\frac{N}{l_1}, \frac{N}{l_2}\} < k \leq \min\{\frac{m_1}{m_1-l_1+\epsilon}, \frac{m_2}{m_2-l_2+\epsilon}\}$, then

$$\begin{aligned}&\limsup_{|(u,v)|\to\infty}\left(\frac{|F(x,u,v)|}{|u|^{l_1}+|v|^{l_2}}\right)^k \frac{1}{\overline{F}(x,u,v)} \\ &\quad\leq \limsup_{|(u,v)|\to\infty}\frac{(|u|^{m_1}\log(1+|u|)+|v|^{m_2}\log(1+|v|)+|u|^{\frac{m_1+\epsilon}{2}}|v|^{\frac{m_2+\epsilon}{2}})^k}{(|u|^{l_1}+|v|^{l_2})^k(\frac{|u|^{m_1+1}}{m_1(1+|u|)}+\frac{|v|^{m_2+1}}{m_2(1+|v|)})} \\ &\quad\leq C_k\limsup_{|(u,v)|\to\infty}\frac{|u|^{km_1}(\log(1+|u|))^k+|v|^{km_2}(\log(1+|v|))^k+|u|^{k(m_1+\epsilon)}+|v|^{k(m_2+\epsilon)}}{\frac{|u|^{kl_1+m_1+1}}{m_1(1+|u|)}+\frac{|v|^{kl_2+m_2+1}}{m_2(1+|v|)}} \\ &\quad< \infty.\end{aligned}$$

Appendix

Lemma A.1 *If Φ is an N-function and* (2.5) *holds, then for any sequence $\{u_n\}$ converging to u in $L^\Phi(\mathbb{R}^N)$, there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $h \in L^1(\mathbb{R}^N)$ such that*

(a) $u_n(x) \to u(x)$ *a.e.* $x \in \mathbb{R}^N$;

(b) $\widetilde{\Phi}(\phi(|u_n(x)|)) \leq h(x)$ *for all* $n \in \mathbb{N}$, *a.e.* $x \in \mathbb{R}^N$;

(c) $\Phi(|u_n(x)|) \leq h(x)$ *for all* $n \in \mathbb{N}$, *a.e.* $x \in \mathbb{R}^N$.

Proof Since $u_n \to u$ in $L^\Phi(\mathbb{R}^N)$, by Lemma 2.2, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\Phi\big(4|u_n-u|\big)\,dx \\ &\quad\leq \max\big\{4^l\|u_n-u\|_\Phi^l, 4^m\|u_n-u\|_\Phi^m\big\} \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

which implies $\Phi(4|u_n-u|) \to 0$ in $L^1(\mathbb{R}^N)$. Hence, by ([24], Theorem 4.9), there exist a subsequence of $\{\Phi(4|u_n-u|)\}$, still denoted by $\{\Phi(4|u_n-u|)\}$, and functions $h_1 \in L^1(\mathbb{R}^N)$ such that

$$\Phi\big(4\big|u_n(x)-u(x)\big|\big) \to 0, \quad \text{a.e. } x \in \mathbb{R}^N \tag{A.1}$$

and

$$\Phi\big(4\big|u_n(x)-u(x)\big|\big) \leq h_1(x) \quad \text{for all } n\in\mathbb{N},\ \text{a.e. } x\in\mathbb{R}^N. \tag{A.2}$$

Then, by (2.2), the monotonicity and convexity of Φ, (A.2) and the fact $4u \in L^{\Phi}(\mathbb{R}^N)$, for all $n \in \mathbb{N}$, a.e. $x \in \mathbb{R}^N$, we have

$$\begin{aligned}\widetilde{\Phi}\bigl(\phi\bigl(\bigl|u_n(x)\bigr|\bigr)\bigr) &\le \Phi\bigl(2\bigl|u_n(x)\bigr|\bigr) \le \frac{1}{2}\Phi\bigl(4\bigl|u_n(x)-u(x)\bigr|\bigr) + \frac{1}{2}\Phi\bigl(4\bigl|u(x)\bigr|\bigr) \\ &\le \frac{1}{2}h_1(x) + \frac{1}{2}\Phi\bigl(4\bigl|u(x)\bigr|\bigr) \in L^1\bigl(\mathbb{R}^N\bigr)\end{aligned}$$

and

$$\Phi\bigl(\bigl|u_n(x)\bigr|\bigr) \le \Phi\bigl(2\bigl|u_n(x)\bigr|\bigr) \le \frac{1}{2}h_1(x) + \frac{1}{2}\Phi\bigl(4\bigl|u(x)\bigr|\bigr) \in L^1\bigl(\mathbb{R}^N\bigr).$$

Moreover, (A.1) implies that $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. □

Lemma A.2 *Suppose that* (ϕ_1), (ϕ_2), (V_2), $(F1)$ *and* (F_2) *hold. Then* $I : W \to \mathbb{R}$ *is well defined and of class* $C^1(W,\mathbb{R})$ *and*

$$\begin{aligned}\bigl\langle I'(u,v),(\tilde{u},\tilde{v})\bigr\rangle = {}& \int_{\mathbb{R}^N} a_1\bigl(|\nabla u|\bigr)\nabla u\nabla\tilde{u}\,dx + \int_{\mathbb{R}^N} V_1(x)a_1\bigl(|u|\bigr)u\tilde{u}\,dx \\ &+ \int_{\mathbb{R}^N} a_2\bigl(|\nabla v|\bigr)\nabla v\nabla\tilde{v}\,dx + \int_{\mathbb{R}^N} V_2(x)a_2\bigl(|v|\bigr)v\tilde{v}\,dx \\ &- \int_{\mathbb{R}^N} F_u(x,u,v)\tilde{u}\,dx - \int_{\mathbb{R}^N} F_v(x,u,v)\tilde{v}\,dx\end{aligned}$$

for all $(\tilde{u},\tilde{v}) \in W$.

Proof Under assumptions (ϕ_1), (ϕ_2) and (V_2), by similar arguments as those in [25], we can prove that $I_1 : W \to \mathbb{R}$ is well defined and of class $C^1(W,\mathbb{R})$ and

$$\begin{aligned}\bigl\langle I_1'(u,v),(\tilde{u},\tilde{v})\bigr\rangle = {}& \int_{\mathbb{R}^N} a_1\bigl(|\nabla u|\bigr)\nabla u\nabla\tilde{u}\,dx + \int_{\mathbb{R}^N} V_1(x)a_1\bigl(|u|\bigr)u\tilde{u}\,dx \\ &+ \int_{\mathbb{R}^N} a_2\bigl(|\nabla v|\bigr)\nabla v\nabla\tilde{v}\,dx + \int_{\mathbb{R}^N} V_2(x)a_2\bigl(|v|\bigr)v\tilde{v}\,dx\end{aligned}\tag{A.3}$$

for all $(\tilde{u},\tilde{v}) \in W$. So, it is sufficient to prove that $I_2 : W \to \mathbb{R}$ is well defined and of class $C^1(W,\mathbb{R})$ and

$$\bigl\langle I_2'(u,v),(\tilde{u},\tilde{v})\bigr\rangle = \int_{\mathbb{R}^N} F_u(x,u,v)\tilde{u}\,dx + \int_{\mathbb{R}^N} F_v(x,u,v)\tilde{v}\,dx\tag{A.4}$$

for all $(\tilde{u},\tilde{v}) \in W$.

By (3.7) and (3.10), we have

$$\begin{aligned}I_2(u,v) &\le \int_{\mathbb{R}^N} \bigl|F(x,u,v)\bigr|\,dx \\ &\le C_3\int_{\mathbb{R}^N} \bigl(\Phi_1\bigl(|u|\bigr) + \Phi_2\bigl(|v|\bigr) + \Phi_{1*}\bigl(|u|\bigr) + \Phi_{2*}\bigl(|v|\bigr)\bigr)\,dx,\end{aligned}$$

which, together with (2.8), implies that I_2 is well defined in W.

We now prove that (A.4) holds. For any given $(u,v),(\tilde{u},\tilde{v}) \in W$, we have

$$\begin{aligned}
&\langle I_2'(u,v),(\tilde{u},\tilde{v})\rangle \\
&\quad = \lim_{h\to 0}\frac{1}{h}\big(I_2(u+h\tilde{u},v+h\tilde{v})-I_2(u,v)\big) \\
&\quad = \lim_{h\to 0}\int_{\mathbb{R}^N}\frac{F(x,u+h\tilde{u},v+h\tilde{v})-F(x,u,v+h\tilde{v})}{h}\,dx \\
&\qquad + \lim_{h\to 0}\int_{\mathbb{R}^N}\frac{F(x,u,v+h\tilde{v})-F(x,u,v)}{h}\,dx \\
&\quad = \lim_{h\to 0}\int_{\mathbb{R}^N}F_u\big(x,u+\theta_1(x)h\tilde{u},v+h\tilde{v}\big)\tilde{u}\,dx + \lim_{h\to 0}\int_{\mathbb{R}^N}F_v\big(x,u,v+\theta_2(x)h\tilde{v}\big)\tilde{v}\,dx,
\end{aligned}\tag{A.5}$$

where $\theta_1,\theta_2:\mathbb{R}^N\to(0,1)$. By the continuity of F_u and F_v, we have that

$$F_u\big(x,u+\theta_1(x)h\tilde{u},v+h\tilde{v}\big)\tilde{u}\to F_u(x,u,v)\tilde{u}\tag{A.6}$$

and

$$F_v\big(x,u,v+\theta_2(x)h\tilde{v}\big)\tilde{v}\to F_v(x,u,v)\tilde{v}$$

as $h\to 0$ for a.e. $x\in\mathbb{R}^N$. Moreover, for all $h\in(-1,1)$, by (3.8), the monotonicity of functions, (2.1), (ϕ_2), (1) in Lemma 2.4 and (2.8), we have

$$\begin{aligned}
&\big|F_u\big(x,u+\theta_1(x)h\tilde{u},v+h\tilde{v}\big)\tilde{u}\big| \\
&\quad \le C_1\big(\phi_1\big(|u+\theta_1(x)h\tilde{u}|\big)+\widetilde{\Phi}_1^{-1}\big(\Phi_2\big(|v+h\tilde{v}|\big)\big)+\Phi_{1*}'\big(|u+\theta_1(x)h\tilde{u}|\big) \\
&\qquad +\widetilde{\Phi}_{1*}^{-1}\big(\Phi_{2*}\big(|v+h\tilde{v}|\big)\big)\big)|\tilde{u}| \\
&\quad \le C_1\big(\big(|u|+|\tilde{u}|\big)\phi_1\big(|u|+|\tilde{u}|\big)+|\tilde{u}|\widetilde{\Phi}_1^{-1}\big(\Phi_2\big(|v|+|\tilde{v}|\big)\big) \\
&\qquad +\big(|u|+|\tilde{u}|\big)\Phi_{1*}'\big(|u|+|\tilde{u}|\big)+|\tilde{u}|\widetilde{\Phi}_{1*}^{-1}\big(\Phi_{2*}\big(|v|+|\tilde{v}|\big)\big)\big) \\
&\quad \le C_1\big(m_1\Phi_1\big(|u|+|\tilde{u}|\big)+\Phi_1\big(|\tilde{u}|\big)+\Phi_2\big(|v|+|\tilde{v}|\big)+m_1^*\Phi_{1*}\big(|u|+|\tilde{u}|\big) \\
&\qquad +\Phi_{1*}\big(|\tilde{u}|\big)+\Phi_{2*}\big(|v|+|\tilde{v}|\big)\big) \\
&\quad =: g_1(x)\in L^1\big(\mathbb{R}^N\big).
\end{aligned}\tag{A.7}$$

Then it follows from (A.6), (A.7) and Lebesgue's dominated convergence theorem that

$$\lim_{h\to 0}\int_{\mathbb{R}^N}F_u\big(x,u+\theta_1(x)h\tilde{u},v+h\tilde{v}\big)\tilde{u}\,dx=\int_{\mathbb{R}^N}F_u(x,u,v)\tilde{u}\,dx.\tag{A.8}$$

Similarly, we can obtain that

$$\lim_{h\to 0}\int_{\mathbb{R}^N}F_v\big(x,u,v+\theta_2(x)h\tilde{v}\big)\tilde{v}\,dx=\int_{\mathbb{R}^N}F_v(x,u,v)\tilde{v}\,dx.\tag{A.9}$$

Combining (A.8) and (A.9) with (A.5), we can conclude that (A.4) holds.

Next, we prove the continuity of I_2'. Let $(u_n, v_n) \to (u, v)$ in W. We claim that $I_2'(u_n, v_n) \to I_2'(u, v)$ in W^* (the dual space of W). Otherwise, there exist a constant $\varepsilon_0 > 0$ and a subsequence of $\{(u_n, v_n)\}$, denoted by $\{(u_{ni}, v_{ni})\}$, such that

$$\left\| I_2'(u_{ni}, v_{ni}) - I_2'(u, v) \right\|_{W^*} \geq \varepsilon_0 > 0 \quad \text{for all } i \in \mathbb{N}. \tag{A.10}$$

Since $(u_{ni}, v_{ni}) \to (u, v)$ in W, then $u_{ni} \to u$ in $W^{1,\Phi_1}(\mathbb{R}^N)$ and $v_{ni} \to v$ in $W^{1,\Phi_2}(\mathbb{R}^N)$, respectively. It follows from (2.8) that $u_{ni} \to u$ in $L^{\Phi_{1*}}(\mathbb{R}^N)$ and $v_{ni} \to v$ in $L^{\Phi_{2*}}(\mathbb{R}^N)$, respectively. By Lemma A.1, there exist a subsequence of $\{(u_{ni}, v_{ni})\}$, still denoted by $\{(u_{ni}, v_{ni})\}$, and a function $h \in L^1(\mathbb{R}^N)$ such that

$$u_{ni}(x) \to u(x), \qquad v_{ni}(x) \to v(x), \quad \text{a.e. } x \in \mathbb{R}^N \tag{A.11}$$

and

$$\begin{aligned}
&\widetilde{\Phi}_1\bigl(\phi_1\bigl(\left|u_{ni}(x)\right|\bigr)\bigr) \leq h(x), \qquad \widetilde{\Phi}_{1*}\bigl(\Phi_{1*}'\bigl(\left|u_{ni}(x)\right|\bigr)\bigr) \leq h(x), \\
&\Phi_1\bigl(\left|u_{ni}(x)\right|\bigr) \leq h(x), \qquad \Phi_{1*}\bigl(\left|u_{ni}(x)\right|\bigr) \leq h(x), \\
&\widetilde{\Phi}_2\bigl(\phi_2\bigl(\left|v_{ni}(x)\right|\bigr)\bigr) \leq h(x), \qquad \widetilde{\Phi}_{2*}\bigl(\Phi_{2*}'\bigl(\left|v_{ni}(x)\right|\bigr)\bigr) \leq h(x), \\
&\Phi_2\bigl(\left|v_{ni}(x)\right|\bigr) \leq h(x), \qquad \Phi_{2*}\bigl(\left|v_{ni}(x)\right|\bigr) \leq h(x)
\end{aligned} \tag{A.12}$$

for all $i \in \mathbb{N}$, a.e. $x \in \mathbb{R}^N$. For this subsequence $\{(u_n, v_n)\}$ and all $(\tilde{u}, \tilde{v}) \in W$, by (A.4) we have

$$\begin{aligned}
&\left|\bigl\langle I_2'(u_{ni}, v_{ni}) - I_2'(u, v), (\tilde{u}, \tilde{v})\bigr\rangle\right| \\
&\quad = \left| \int_{\mathbb{R}^N} F_u(x, u_{ni}, v_{ni})\tilde{u}\, dx + \int_{\mathbb{R}^N} F_v(x, u_{ni}, v_{ni})\tilde{v}\, dx \right. \\
&\qquad \left. - \int_{\mathbb{R}^N} F_u(x, u, v)\tilde{u}\, dx - \int_{\mathbb{R}^N} F_v(x, u, v)\tilde{v}\, dx \right| \\
&\quad \leq \int_{\mathbb{R}^N} \left|F_u(x, u_{ni}, v_{ni}) - F_u(x, u, v)\right| |\tilde{u}|\, dx \\
&\qquad + \int_{\mathbb{R}^N} \left|F_v(x, u_{ni}, v_{ni}) - F_v(x, u, v)\right| |\tilde{v}|\, dx.
\end{aligned} \tag{A.13}$$

Firstly, we claim that

$$\int_{\mathbb{R}^N} \left|F_u(x, u_{ni}, v_{ni}) - F_u(x, u, v)\right| |\tilde{u}|\, dx = o_i(1) \left\|(\tilde{u}, \tilde{v})\right\|. \tag{A.14}$$

In fact,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|F_u(x, u_{ni}, v_{ni}) - F_u(x, u, v)\right| |\tilde{u}|\, dx \\
&\quad = \int_{\Omega_1} \left|F_u(x, u_{ni}, v_{ni}) - F_u(x, u, v)\right| |\tilde{u}|\, dx \\
&\qquad + \int_{\Omega_2} \left|F_u(x, u_{ni}, v_{ni}) - F_u(x, u, v)\right| |\tilde{u}|\, dx,
\end{aligned} \tag{A.15}$$

where $\Omega_1 = \{x \in \mathbb{R}^N : |u(x)| \leq 1, |v(x)| \leq 1 \text{ and } h(x) \leq 1\}$, $\Omega_2 = \mathbb{R}^N \setminus \Omega_1$. It is obvious that $|\Omega_1| = \infty$ and $|\Omega_2| < \infty$. Then, by (2.4) and (2.8), we have

$$\begin{aligned}
&\int_{\Omega_1} \big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big||\tilde{u}|\,dx + \int_{\Omega_2} \big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big||\tilde{u}|\,dx \\
&\quad = \int_{\mathbb{R}^N} \big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big||\tilde{u}|\chi\{\Omega_1\}\,dx \\
&\qquad + \int_{\mathbb{R}^N} \big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big||\tilde{u}|\chi\{\Omega_2\}\,dx \\
&\quad \leq 2\big\|\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\chi\{\Omega_1\}\big\|_{\widetilde{\Phi}_1}\|\tilde{u}\|_{\Phi_1} \\
&\qquad + 2\big\|\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\chi\{\Omega_2\}\big\|_{\widetilde{\Phi}_{1*}}\|\tilde{u}\|_{\Phi_{1*}} \\
&\quad \leq 2(1 + C_{\Phi_{1*}})\big(\big\|\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\chi\{\Omega_1\}\big\|_{\widetilde{\Phi}_1} \\
&\qquad + \big\|\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\chi\{\Omega_2\}\big\|_{\widetilde{\Phi}_{1*}}\big)\big\|(\tilde{u},\tilde{v})\big\|,
\end{aligned}$$

where χ denotes the characteristic function. Then, to get (A.14), by (2.3) it is sufficient to prove

$$\begin{aligned}
&\int_{\mathbb{R}^N} \widetilde{\Phi}_1\big(\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\chi\{\Omega_1\}\big)\,dx \\
&\qquad + \int_{\mathbb{R}^N} \widetilde{\Phi}_{1*}\big(\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\chi\{\Omega_2\}\big)\,dx \\
&\quad = \int_{\Omega_1} \widetilde{\Phi}_1\big(\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\big)\,dx \\
&\qquad + \int_{\Omega_2} \widetilde{\Phi}_{1*}\big(\big|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)\big|\big)\,dx = o_i(1).
\end{aligned} \tag{A.16}$$

By (A.11), the continuity of F_u, $\widetilde{\Phi}_1$, $\widetilde{\Phi}_{1*}$ and the fact $\widetilde{\Phi}_1(0) = \widetilde{\Phi}_{1*}(0) = 0$, we have

$$\widetilde{\Phi}_1\big(\big|F_u\big(x,u_{ni}(x),v_{ni}(x)\big) - F_u\big(x,u(x),v(x)\big)\big|\big) \to 0, \quad \text{a.e. } x \in \Omega_1 \tag{A.17}$$

and

$$\widetilde{\Phi}_{1*}\big(\big|F_u\big(x,u_{ni}(x),v_{ni}(x)\big) - F_u\big(x,u(x),v(x)\big)\big|\big) \to 0, \quad \text{a.e. } x \in \Omega_2. \tag{A.18}$$

By (A.12) we have

$$\Phi_1\big(\big|u_{ni}(x)\big|\big) \leq h(x) \leq 1, \qquad \Phi_2\big(\big|v_{ni}(x)\big|\big) \leq h(x) \leq 1 \quad \text{for all } i \in \mathbb{N}, \text{ a.e. } x \in \Omega_1,$$

which, together with the monotonicity of Φ_1 and Φ_2, implies that

$$\big|u_{ni}(x)\big| \leq \Phi_1^{-1}(1), \qquad \big|v_{ni}(x)\big| \leq \Phi_2^{-1}(1) \quad \text{for all } i \in \mathbb{N}, \text{ a.e. } x \in \Omega_1.$$

Then, by (F_2), there exists a constant $M_7 > 0$ such that

$$\big|F_u\big(x,u_{ni}(x),v_{ni}(x)\big)\big| \leq M_7\big(\phi_1\big(\big|u_{ni}(x)\big|\big) + \widetilde{\Phi}_1^{-1}\big(\Phi_2\big(\big|v_{ni}(x)\big|\big)\big)\big) \quad \text{for all } i \in \mathbb{N}, \text{ a.e. } x \in \Omega_1$$

and

$$\left|F_u\left(x,u(x),v(x)\right)\right| \le M_7\left(\phi_1\left(\left|u(x)\right|\right) + \widetilde{\Phi}_1^{-1}\left(\Phi_2\left(\left|v(x)\right|\right)\right)\right) \quad \text{for all } x \in \Omega_1.$$

Then, by the monotonicity and convexity of $\widetilde{\Phi}_1$, the fact that $\widetilde{\Phi}_1$ satisfies the Δ_2-condition globally, (A.12) and (2.2), for all $i \in \mathbb{N}$, a.e. $x \in \Omega_1$, we have

$$\begin{aligned}
&\widetilde{\Phi}_1\left(\left|F_u\left(x,u_{ni}(x),v_{ni}(x)\right) - F_u\left(x,u(x),v(x)\right)\right|\right)\\
&\quad \le \widetilde{\Phi}_1\left(\left|F_u\left(x,u_{ni}(x),v_{ni}(x)\right)\right| + \left|F_u\left(x,u(x),v(x)\right)\right|\right)\\
&\quad \le \widetilde{\Phi}_1\left[M_7\left(\phi_1\left(\left|u_{ni}(x)\right|\right) + \widetilde{\Phi}_1^{-1}\left(\Phi_2\left(\left|v_{ni}(x)\right|\right)\right) + \phi_1\left(\left|u(x)\right|\right) + \widetilde{\Phi}_1^{-1}\left(\Phi_2\left(\left|v(x)\right|\right)\right)\right)\right]\\
&\quad \le C\left(\widetilde{\Phi}_1\left(\phi_1\left(\left|u_{ni}(x)\right|\right)\right) + \Phi_2\left(\left|v_{ni}(x)\right|\right) + \widetilde{\Phi}_1\left(\phi_1\left(\left|u(x)\right|\right)\right) + \Phi_2\left(\left|v(x)\right|\right)\right)\\
&\quad \le C\left(h(x) + \Phi_1\left(2\left|u(x)\right|\right) + \Phi_2\left(\left|v(x)\right|\right)\right) =: g_2(x) \in L^1(\Omega_1),
\end{aligned} \tag{A.19}$$

where C is a positive constant. Moreover, by (F_2), there exists a constant $M_8 > 0$ such that

$$\begin{aligned}
&\left|F_u\left(x,u_{ni}(x),v_{ni}(x)\right)\right| \le M_8 + \Phi_{1*}'\left(\left|u_{ni}(x)\right|\right) + \widetilde{\Phi}_{1*}^{-1}\left(\Phi_{2*}\left(\left|v_{ni}(x)\right|\right)\right)\\
&\quad \text{for all } i \in \mathbb{N},\ \text{a.e. } x \in \Omega_2
\end{aligned}$$

and

$$\left|F_u\left(x,u(x),v(x)\right)\right| \le M_8 + \Phi_{1*}'\left(\left|u(x)\right|\right) + \widetilde{\Phi}_{1*}^{-1}\left(\Phi_{2*}\left(\left|v(x)\right|\right)\right) \quad \text{for all } x \in \Omega_2.$$

Then, by the monotonicity and convexity of $\widetilde{\Phi}_{1*}$, the fact that $\widetilde{\Phi}_{1*}$ satisfies the Δ_2-condition globally, (A.12) and (2.2), for all $i \in \mathbb{N}$, a.e. $x \in \Omega_2$, we have

$$\begin{aligned}
&\widetilde{\Phi}_{1*}\left(\left|F_u\left(x,u_{ni}(x),v_{ni}(x)\right) - F_u\left(x,u(x),v(x)\right)\right|\right)\\
&\quad \le \widetilde{\Phi}_{1*}\left(\left|F_u\left(x,u_{ni}(x),v_{ni}(x)\right)\right| + \left|F_u\left(x,u(x),v(x)\right)\right|\right)\\
&\quad \le \widetilde{\Phi}_{1*}\left(2M_8 + \Phi_{1*}'\left(\left|u_{ni}(x)\right|\right) + \widetilde{\Phi}_{1*}^{-1}\left(\Phi_{2*}\left(\left|v_{ni}(x)\right|\right)\right) + \Phi_{1*}'\left(\left|u(x)\right|\right) + \widetilde{\Phi}_{1*}^{-1}\left(\Phi_{2*}\left(\left|v(x)\right|\right)\right)\right)\\
&\quad \le C\left(1 + \widetilde{\Phi}_{1*}\left(\Phi_{1*}'\left(\left|u_{ni}(x)\right|\right)\right) + \Phi_{2*}\left(\left|v_{ni}(x)\right|\right) + \widetilde{\Phi}_{1*}\left(\Phi_{1*}'\left(\left|u(x)\right|\right)\right) + \Phi_{2*}\left(\left|v(x)\right|\right)\right)\\
&\quad \le C\left(1 + h(x) + \Phi_{1*}\left(2\left|u(x)\right|\right) + \Phi_{2*}\left(\left|v(x)\right|\right)\right) =: g_3(x) \in L^1(\Omega_2),
\end{aligned} \tag{A.20}$$

where C is a positive constant. Combining (A.17)-(A.20) with Lebesgue's dominated convergence theorem, we can conclude that (A.16) holds. Then (A.14) holds. Similarly, we can obtain that

$$\int_{\mathbb{R}^N} \left|F_v(x,u_{ni},v_{ni}) - F_v(x,u,v)\right| |\tilde{v}|\,dx = o_i(1)\left\|(\tilde{u},\tilde{v})\right\|. \tag{A.21}$$

Therefore, combining (A.14) and (A.21) with (A.13), we can conclude that $I_2'(u_{ui},v_{ni}) \to I_2'(u,v)$ in W^*, which contradicts (A.10). □

Lemma A.3 *Assume that* (ϕ_1), (ϕ_2), (V_2), (F_1) *and* (F_2) *hold. Then*

$$\left\langle I'(u,v),(u,v)\right\rangle = \left\langle I_1'(u,v),(u,v)\right\rangle - o\left(\left\langle I_1'(u,v),(u,v)\right\rangle\right) \quad \text{as } \left\|(u,v)\right\| \to 0.$$

Proof Since $\langle I'(u,v),(u,v)\rangle = \langle I_1'(u,v),(u,v)\rangle - \langle I_2'(u,v),(u,v)\rangle$ and $\langle I_i'(u,v),(u,v)\rangle = o(1)(i = 1,2)$ as $\|(u,v)\| \to 0$, we need to prove $\langle I_2'(u,v),(u,v)\rangle = o(\langle I_1'(u,v),(u,v)\rangle)$ as $\|(u,v)\| \to 0$. By (F_1), (F_2), (ϕ_2), 1) in Lemma 2.4 and (2.1), for any given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\bigl|uF_u(x,u,v)\bigr| + \bigl|vF_v(x,u,v)\bigr| \le \varepsilon\bigl(\Phi_1\bigl(|u|\bigr) + \Phi_2\bigl(|v|\bigr)\bigr) + C_\varepsilon\bigl(\Phi_{1*}\bigl(|u|\bigr) + \Phi_{2*}\bigl(|v|\bigr)\bigr)$$

for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$. Then, by (A.4), 3) in Lemma 2.4, (2.8) and (2.7), we have

$$\begin{aligned}
&\bigl|\bigl\langle I_2'(u,v),(u,v)\bigr\rangle\bigr| \\
&\quad\le \int_{\mathbb{R}^N}\bigl(\bigl|F_u(x,u,v)\bigr||u| + \bigl|F_v(x,u,v)\bigr||v|\bigr)\,dx \\
&\quad\le \varepsilon\int_{\mathbb{R}^N}\bigl(\Phi_1\bigl(|u|\bigr) + \Phi_2\bigl(|v|\bigr)\bigr)\,dx + C_\varepsilon\int_{\mathbb{R}^N}\bigl(\Phi_{1*}\bigl(|u|\bigr) + \Phi_{2*}\bigl(|v|\bigr)\bigr)\,dx \\
&\quad\le \varepsilon\int_{\mathbb{R}^N}\bigl(\Phi_1\bigl(|u|\bigr) + \Phi_2\bigl(|v|\bigr)\bigr)\,dx + C_\varepsilon\bigl(\|u\|_{\Phi_{1*}}^{l_1^*} + \|u\|_{\Phi_{1*}}^{m_1^*} + \|v\|_{\Phi_{2*}}^{l_2^*} + \|v\|_{\Phi_{2*}}^{m_2^*}\bigr) \\
&\quad\le \varepsilon\int_{\mathbb{R}^N}\bigl(\Phi_1\bigl(|u|\bigr) + \Phi_2\bigl(|v|\bigr)\bigr)\,dx + C\bigl(\|u\|_{1,\Phi_1}^{l_1^*} + \|u\|_{1,\Phi_1}^{m_1^*} + \|v\|_{1,\Phi_2}^{l_2^*} + \|v\|_{1,\Phi_2}^{m_2^*}\bigr),
\end{aligned} \tag{A.22}$$

where $C = C_\varepsilon \max\{C_{\Phi_{1*}}^{l_1^*}, C_{\Phi_{1*}}^{m_1^*}, C_{\Phi_{2*}}^{l_2^*}, C_{\Phi_{2*}}^{m_2^*}\}$. Moreover, by (A.3), (ϕ_2), (V_2) and Lemma 2.2, when $\|(u,v)\| = \|\nabla u\|_{\Phi_1} + \|u\|_{\Phi_1} + \|\nabla v\|_{\Phi_2} + \|v\|_{\Phi_2} \le 1$, we have

$$\begin{aligned}
&\bigl\langle I_1'(u,v),(u,v)\bigr\rangle \\
&\quad= \int_{\mathbb{R}^N} a_1\bigl(|\nabla u|\bigr)|\nabla u|^2\,dx + \int_{\mathbb{R}^N} V_1(x)a_1\bigl(|u|\bigr)|u|^2\,dx \\
&\qquad+ \int_{\mathbb{R}^N} a_2\bigl(|\nabla v|\bigr)|\nabla v|^2\,dx + \int_{\mathbb{R}^N} V_2(x)a_2\bigl(|v|\bigr)|v|^2\,dx \\
&\quad\ge l_1\int_{\mathbb{R}^N}\Phi_1\bigl(|\nabla u|\bigr)\,dx + \alpha_1 l_1\int_{\mathbb{R}^N}\Phi_1\bigl(|u|\bigr)\,dx \\
&\qquad+ l_2\int_{\mathbb{R}^N}\Phi_2\bigl(|\nabla v|\bigr)\,dx + \alpha_1 l_2\int_{\mathbb{R}^N}\Phi_2\bigl(|v|\bigr)\,dx \\
&\quad\ge \min\{l_1,l_2\}\min\{1,\alpha_1\}\bigl(\|\nabla u\|_{\Phi_1}^{m_1} + \|u\|_{\Phi_1}^{m_1} + \|\nabla v\|_{\Phi_2}^{m_2} + \|v\|_{\Phi_2}^{m_2}\bigr) \\
&\quad\ge \min\{l_1,l_2\}\min\{1,\alpha_1\}\bigl(C_{m_1}\|u\|_{1,\Phi_1}^{m_1} + C_{m_2}\|v\|_{1,\Phi_2}^{m_2}\bigr).
\end{aligned} \tag{A.23}$$

Then (A.22), (A.23) and the fact that $1 < m_i < l_i^* \le m_i^* (i = 1,2)$ imply that

$$\begin{aligned}
&\lim_{\|(u,v)\|\to 0}\frac{|\langle I_2'(u,v),(u,v)\rangle|}{\langle I_1'(u,v),(u,v)\rangle} \\
&\quad\le \lim_{\|(u,v)\|\to 0}\frac{\varepsilon\int_{\mathbb{R}^N}(\Phi_1(|u|) + \Phi_2(|v|))\,dx}{\alpha_1\min\{l_1,l_2\}\int_{\mathbb{R}^N}(\Phi_1(|u|) + \Phi_2(|v|))\,dx} \\
&\qquad+ \lim_{\|(u,v)\|\to 0}\frac{C(\|u\|_{1,\Phi_1}^{l_1^*} + \|u\|_{1,\Phi_1}^{m_1^*} + \|v\|_{1,\Phi_2}^{l_2^*} + \|v\|_{1,\Phi_2}^{m_2^*})}{\min\{l_1,l_2\}\min\{1,\alpha_1\}(C_{m_1}\|u\|_{1,\Phi_1}^{m_1} + C_{m_2}\|v\|_{1,\Phi_2}^{m_2})} \\
&\quad= \frac{\varepsilon}{\alpha_1\min\{l_1,l_2\}}.
\end{aligned}$$

Since ε is arbitrary, we conclude that $|\langle I_2'(u,v),(u,v)\rangle| = o(\langle I_1'(u,v),(u,v)\rangle)$ as $\|(u,v)\| \to 0$. Hence, $\langle I_2'(u,v),(u,v)\rangle = o(\langle I_1'(u,v),(u,v)\rangle)$ as $\|(u,v)\| \to 0$. □

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