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The Riemann problem for a one-dimensional nonlinear wave system with different gamma laws

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Abstract

The Riemann problem for a one-dimensional nonlinear wave system with different gamma laws is considered. By the properties of wave curves, we observe that this system does not contain the composite wave compared to the barotropic models of gas dynamics with different pressure laws. Under some initial value data, the Riemann solution is constructed. Using the interaction of the elementary waves, we consider the generalized Riemann problem and discover that the Riemann solution is stable for such perturbation of the initial data.

MSC: 35L65; 35J70; 35R35

Keywords: one-dimensional nonlinear wave system; different gamma laws; Riemann problem; perturbed initial data

1 Introduction

One-dimensional nonlinear wave system with the variable gamma laws which only depends on the spatial coordination is described as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + p(\rho, \gamma)_x = 0, \\ \gamma_t = 0, \end{cases} \tag{1.1}$$

where

$$p(\rho, \gamma) = A\rho^{\gamma(x)} \tag{1.2}$$

and $\gamma(x) > 1$ and $A > 0$. When $\gamma(x) = \gamma$ is constant, system (1.1) is changed into a one-dimensional nonlinear wave system, which can be obtained either by starting with the isentropic gas dynamics equations and neglecting the quadratic terms in the velocity or by writing the nonlinear wave equation as a first-order system [1, 2]. System (1.1) can also be deduced from the barotropic models of gas dynamics with different pressure laws in [3],

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and we refer the reader to this paper for details. Introducing $m = \rho u$, system (1.1) becomes

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + p(\rho, \gamma)_x = 0, \\ \gamma_t = 0, \end{cases} \tag{1.3}$$

which is similar to the model of a mixture of gases governed by different gamma laws in [4, 5]. In the two papers, they used the front tracking algorithm and the Glimm scheme to prove that the Cauchy problem has a global, weak solution under some conditions, respectively. We also see the results for the model of an inviscid fluid capable of undergoing phase transitions, which is a simplified version of the model proposed by Fan [6]. For the related results, we can see [7–13].

In this paper, we focus on the Riemann problem for system (1.3) with piecewise constant initial data

$$(\rho, m, \gamma) = \begin{cases} (\rho_l, m_l, \gamma_l), & x < 0, \\ (\rho_r, m_r, \gamma_r), & x > 0, \end{cases} \tag{1.4}$$

and the interaction between the elementary waves and the stationary contact wave. We observe that the corresponding Riemann solution is similar to that of the model of one-dimensional adiabatic flow in Lagrangian coordinates, we also see the related results [14–21] for details.

The organization of this paper is as follows. In Section 2, we describe the properties of wave curves and construct the Riemann solution. In Section 3, we consider the initial value problem with three constant states. By the interaction between the stationary contact wave and the shock wave or rarefaction wave, the global solutions are constructed. Moreover, we obtain that the solution of the perturbed initial value problem converges to the corresponding Riemann solution as ε approaches zero, which shows the stability of the Riemann solution for the small perturbation.

2 The solution to the Riemann problem

Setting the dependent variable $U = (\rho, m, \gamma)$, the Jacobian matrix of system (1.3) is in the form

$$A(U) = \begin{pmatrix} 0 & 1 & 0 \\ p_\rho & 0 & p_\gamma \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

which has three eigenvalues

$$\lambda_1 = -\sqrt{p_\rho}, \qquad \lambda_2 = 0, \qquad \lambda_3 = +\sqrt{p_\rho}, \tag{2.2}$$

together with the corresponding eigenvectors

$$r_1 = \begin{pmatrix} 1 \\ \lambda_1 \\ 0 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} p_\gamma \\ 0 \\ -p_\rho \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 \\ \lambda_3 \\ 0 \end{pmatrix}. \tag{2.3}$$

The first and the third families are genuinely nonlinear, and the second family is linearly degenerate.

In order to analyze the solutions to system (1.3), we need to look at the wave curves.

2.1 Wave curves

Since Eqs. (1.3) and the Riemann data are invariant under uniform stretching of coordinates

$$(x,t)\to(\kappa x,\kappa t),\quad \kappa \text{ is constant.}$$

By taking the self-similar transform $\xi = x/t$, the Riemann problem is reduced to the boundary value problem of the ordinary differential equation

$$\begin{cases} -\xi\rho_\xi + m_\xi = 0,\\ -\xi m_\xi + p(\rho,\gamma)_\xi = 0,\\ -\xi\gamma_\xi = 0 \end{cases} \tag{2.4}$$

with $(\rho,m,\gamma)(\infty) = (\rho_r,m_r,\gamma_r)$ and $(\rho,m,\gamma)(-\infty) = (\rho_l,m_l,\gamma_l)$.

For smooth solutions, Eqs. (2.4) can be rewritten as

$$\begin{pmatrix} -\xi & 1 & 0\\ p_\rho & -\xi & p_\gamma\\ 0 & 0 & -\xi \end{pmatrix}\begin{pmatrix}\rho\\ m\\ \gamma\end{pmatrix}_\xi = 0. \tag{2.5}$$

It follows from (2.5) that besides the constant solution ($\rho > 0$), it provides a rarefaction wave which is a continuous solution of (2.5) in the form $U(\xi)$. Given a left state $U_\ell = (\rho_\ell, m_\ell, \gamma_\ell)$, the rarefaction wave curves are the set of all right states $U = (\rho, m, \gamma_l)$ that can be connected to the left by a rarefaction wave in the first family, and they are as follows:

$$R_1(U_l,U): \begin{cases} \lambda_1 = \xi = -\sqrt{A\gamma_l}\rho^{\frac{\gamma_l-1}{2}},\\ m + \frac{2\sqrt{A\gamma_l}}{1+\gamma_l}\rho^{\frac{1+\gamma_l}{2}} = m_l + \frac{2\sqrt{A\gamma_l}}{1+\gamma_l}\rho_l^{\frac{1+\gamma_l}{2}}, \quad \rho<\rho_l. \end{cases} \tag{2.6}$$

Similarly, for a given right state $U_r = (\rho_r, m_r, \gamma_r)$, the rarefaction wave curve which is the sets of states $U = (\rho, m, \gamma_r)$ that can be connected on the right in the third family is described as follows:

$$R_3(U_r,U): \begin{cases} \lambda_3 = \xi = \sqrt{A\gamma_r}\rho^{\frac{\gamma_r-1}{2}},\\ m - \frac{2\sqrt{A\gamma_r}}{1+\gamma_r}\rho^{\frac{1+\gamma_r}{2}} = m_r - \frac{2\sqrt{A\gamma_r}}{1+\gamma_r}\rho_r^{\frac{1+\gamma_r}{2}}, \quad \rho<\rho_r. \end{cases} \tag{2.7}$$

For a bounded discontinuous solutions, the Rankine-Hugoniot condition holds:

$$\begin{cases} -\sigma[\rho] + [m] = 0,\\ -\sigma[m] + [p] = 0,\\ -\sigma[\gamma] = 0, \end{cases} \tag{2.8}$$

where, and in what follows, we use the notation $[h] = h_+ - h_-$ with $h_- = h(x(t) - 0, t)$ and $h_+ = h(x(t) + 0, t)$, and $\sigma = \frac{dx}{dt}$ is the velocity of the discontinuity. It follows from the third equation of (2.8) that

$$\sigma = 0 \quad \text{or} \quad [\gamma] = 0. \tag{2.9}$$

Under the condition $[\gamma] = 0$ and the Lax shock inequalities, the possible state $U = (\rho, m, \gamma_l)$ can be connected to the left state U_l on the right by a one-shock wave given by

$$S_1(U_l, U) : m - m_l = -\big(\big(p(\rho, \gamma_l) - p(\rho_l, \gamma_l)\big)(\rho - \rho_l)\big)^{1/2}, \quad \rho > \rho_l, \tag{2.10}$$

with the shock velocity

$$\sigma_1(U_l, U, \gamma_l) = -\sqrt{\frac{p(\rho, \gamma_l) - p(\rho_l, \gamma_l)}{\rho - \rho_l}}. \tag{2.11}$$

Similarly, for a given right state $U_r = (\rho_r, m_r, \gamma_r)$, we can obtain that the possible state $U = (\rho, m, \gamma_r)$ can be connected to the right state U_r on the left by a three-shock wave given as follows:

$$S_3(U_r, U) : m - m_r = \big(\big(p(\rho, \gamma_r) - p(\rho_r, \gamma_r)\big)(\rho - \rho_r)\big)^{1/2}, \quad \rho > \rho_r, \tag{2.12}$$

with the shock velocity

$$\sigma_3(U_r, U, \gamma_r) = \sqrt{\frac{p(\rho, \gamma_r) - p(\rho_r, \gamma_r)}{\rho - \rho_r}}. \tag{2.13}$$

The second family is linearly degenerate, that is, $\sigma = 0$, which implies that it is a stationary contact wave. From system (1.3), it satisfies

$$[m] = 0, \qquad [p] = 0. \tag{2.14}$$

The curve which is the sets of states $U_+ = (\rho_+, m_+, \gamma_r)$ that can be connected to $U_- = (\rho_-, m_-, \gamma_l)$ by the stationary contact wave is

$$J_2(U_-, U_+) : \begin{cases} m_+ = m_-, \\ p_+ = A\rho_+^{\gamma_r} = A\rho_-^{\gamma_l} = p_-. \end{cases} \tag{2.15}$$

Remark 1 As a consequence, the $\gamma(x)$ remains constant across a rarefaction wave or a shock wave and only changes along the contact wave. In addition, the shock speed σ does not vanish, i.e., there is not a stationary shock. So system (1.3) does not contain the curves of composite waves [22–25].

2.2 The properties of the elementary waves

To solve (1.3) and (1.4), we project all the wave curves on the (ρ, m)-plane. Now, let us investigate the properties of the wave curves.

Lemma 2.1 *The curve $R_1(U_l, U)$ is monotonic decreasing and concave, while $R_3(U_r, U)$ is monotonic increasing and convex.*

Proof By (2.6), the curve $R_1(U, U_l)$ can be rewritten as

$$m = m_l + \frac{2\sqrt{A\gamma_l}}{1+\gamma_l}\big(\rho_l^{\frac{1+\gamma_l}{2}} - \rho^{\frac{1+\gamma_l}{2}}\big).$$

Differentiating the above equation with respect to ρ gives

$$\begin{aligned} &\frac{dm}{d\rho} = -\sqrt{A\gamma_l}\rho^{\frac{\gamma_l-1}{2}} < 0, \\ &\frac{d^2m}{d\rho^2} = -\frac{(\gamma_l-1)\sqrt{A\gamma_l}}{2}\rho^{\frac{\gamma_l-3}{2}} < 0 \end{aligned} \tag{2.16}$$

for $\gamma_l > 1$. So, the curve $R_1(U_l, U)$ is monotonic decreasing and concave. Similarly, we can prove that $R_3(U_r, U)$ is monotonic increasing and convex. □

Lemma 2.2 *The curve $S_1(U_l, U)$ is monotonic decreasing and concave, while $S_3(U, U_r)$ is monotonic increasing and convex.*

Proof Here, we only prove the result for the case $S_1(U_l, U)$, and the other case can be studied in a similar method.

We obtain from (2.10) that

$$m = m_l - \big(\big(p(\rho, \gamma_l) - p(\rho_l, \gamma_l)\big)(\rho - \rho_l)\big)^{1/2}.$$

Differentiating the above equation with respect to ρ reduces to

$$\frac{dm}{d\rho} = -\frac{(\rho-\rho_l)p_\rho + p - p_l}{2\sqrt{(p-p_l)(\rho-\rho_l)}} < 0, \tag{2.17}$$

where we use the fact that $\rho > \rho_l$ for $S_1(U_l, U)$. Furthermore, we have

$$\frac{d^2m}{d\rho^2} = -\frac{2(\rho-\rho_l)^2(p-p_l)p_{\rho\rho} - [(\rho-\rho_l)p_\rho - (p-p_l)]^2}{4[(p-p_l)(\rho-\rho_l)]^{3/2}}. \tag{2.18}$$

Next, let us consider the term

$$f(\rho) = 2(\rho-\rho_l)^2(p-p_l)p_{\rho\rho} - \big[(\rho-\rho_l)p_\rho - (p-p_l)\big]^2, \tag{2.19}$$

which gives

$$f'(\rho) = 2(\rho-\rho_l)(p-p_l)\big[3p_{\rho\rho} + (\rho-\rho_l)p_{\rho\rho\rho}\big].$$

Using (1.2), we have

$$\begin{aligned} f'(\rho) &= 2A\gamma(\gamma-1)\rho^{\gamma-3}\big[3\rho+(\gamma-2)(\rho-\rho_l))\big] \\ &= 2A\gamma(\gamma-1)\rho^{\gamma-3}\big[(\gamma+1)(\rho-\rho_l)+3\rho_l\big] > 0 \end{aligned}$$

for $\gamma > 1$ and $\rho > \rho_l$. According to $f(\rho_l) = 0$, we have $f(\rho) > 0$ for $\rho > \rho_l$. So, it is clear from (2.18) that

$$\frac{d^2m}{d\rho^2} < 0. \tag{2.20}$$

Combining (2.17) and (2.20) gives that the curve $S_1(U_l, U)$ is monotonic decreasing and concave.

Similar arguments lead to the result that $S_3(U_r, U)$ is monotonic increasing and convex. Therefore, we complete the proof. □

Let the one-rarefaction wave $R_1(U, U_0)$ and the one-shock wave $S_1(U, U_0)$ (or the three-rarefaction wave $R_3(U, U_0)$ and the three-shock wave $S_3(U, U_0)$) pass through the point $P_0 = (\rho_0, m_0, \gamma_l)$ respectively, we have the following lemma.

Lemma 2.3 *The curves $R_1(U, U_0)$ ($R_3(U, U_0)$) contact with $S_1(U, U_0)$ ($S_3(U, U_0)$) at point P_0 up to the second order, respectively.*

Proof It follows from (2.17) that

$$\begin{aligned} \lim_{\rho\to\rho_0} \frac{dm}{d\rho}\Big|_{S_1(U,U_0)} &= -\frac{1}{2}\lim_{\rho\to\rho_0}\left\{\sqrt{\frac{\rho-\rho_0}{p-p_0}}p_\rho + \sqrt{\frac{p-p_0}{\rho-\rho_0}}\right\} \\ &= -\sqrt{p_\rho(\rho_0)} = \lim_{\rho\to\rho_0}\frac{dm}{d\rho}\Big|_{R_1(U,U_0)}. \end{aligned} \tag{2.21}$$

Similarly, we obtain from (2.18) that

$$\begin{aligned} \lim_{\rho\to\rho_0} \frac{d^2m}{d\rho^2}\Big|_{S_1(U,U_0)} &= -\frac{1}{2}\lim_{\rho\to\rho_0}\sqrt{\frac{\rho-\rho_0}{p-p_0}}p_{\rho\rho} + \lim_{\rho\to\rho_0}\frac{[(\rho-\rho_0)p_\rho-(p-p_0)]^2}{4[(p-p_0)(\rho-\rho_0)]^{3/2}} \\ &= -\frac{p_{\rho\rho}}{2\sqrt{p_\rho}}\Big|_{\rho=\rho_0} = \lim_{\rho\to\rho_0}\frac{d^2m}{d\rho^2}\Big|_{R_1(U,U_0)}. \end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we know that the curve $R_1(U, U_0)$ contacts with $S_1(U, U_0)$ at point P_0 up to the second order.

Similar calculations show that the other result of the lemma is true. So, the proof is completed. □

2.3 The Riemann solution

In this paper, we only consider the case $\gamma_l > \gamma_r$ and can obtain the corresponding results for the other case $\gamma_l < \gamma_r$ in a similar way. For simplicity, we take $A = 1$ in (1.2).

Given the left state $U_l = (\rho_l, m_l, \gamma_l)$ and the right state $U_r = (\rho_r, m_r, \gamma_r)$, we consider the projection of the wave curves in the (ρ, m)-plane. Let $U_- = (\rho_-, m_-, \gamma_l)$ and $U_+ =$

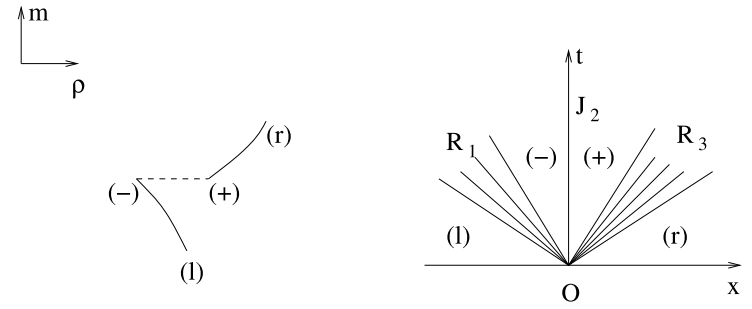


Figure 1 Case 1: $RP(U_l, U_r) = U_l + R_1 + U_- + J_2 + U_+ + R_3 + U_r$.

(ρ_+, m_+, γ_r) satisfy that $U_- \in R_1(U_l, U)$ or $S_1(U_l, U)$, $U_+ \in R_3(U_r, U)$ or $S_r(U_r, U)$ and Eq. (2.15). Denote $w_l = m_l + \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho_l^{\frac{1+\gamma_l}{2}}$ and $z_r = m_r - \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_r^{\frac{1+\gamma_r}{2}}$.

We assume $\rho_- > 1$ and describe the Riemann solution $RP(U_l, U_r)$ as the following five cases (for the other case $\rho_- < 1$, one easily obtains similar results).

Case 1. $w_l > z_r$ and $m_l < z_r$, the $RP(U_l, U_r)$ is $U_l + R_1 + U_- + J_2 + U_+ + R_3 + U_r$, where $U_\pm$ satisfies

$$\begin{cases} w_l - z_r = \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho_-^{\frac{1+\gamma_l}{2}} + \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_+^{\frac{1+\gamma_r}{2}}, \\ \rho_-^{\gamma_l} = \rho_+^{\gamma_r}, \\ m_- = m_+ = w_l - \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho_-^{\frac{1+\gamma_l}{2}} = z_r + \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_+^{\frac{1+\gamma_r}{2}}. \end{cases} \tag{2.23}$$

In addition, $R_1(U, U_l)$ is determined by (2.6), where $\rho_- \le \rho \le \rho_l$ and $R_3(U, U_r)$ is determined by (2.7), where $\rho_+ \le \rho \le \rho_r$. In order to show the Riemann solution, we only need to prove that (2.23) has a unique solution. Setting $\rho_- = \rho$, it follows from the first two equations of (2.23) that

$$f(\rho) := \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho^{\frac{1+\gamma_l}{2}} + \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho^{\frac{\gamma_l(1+\gamma_r)}{2\gamma_r}} - (w_l - z_r) = 0. \tag{2.24}$$

We have $f'(\rho) > 0$, which implies that the function $f(\rho)$ is increasing with respect to the variable ρ. It is obvious that $f(0) = -(w_l - z_r) < 0$ and

$$\begin{aligned} f(\rho_l) &= \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho_l^{\frac{1+\gamma_l}{2}} + \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_l^{\frac{\gamma_l(1+\gamma_r)}{2\gamma_r}} - (w_l - z_r) \\ &= \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_l^{\frac{\gamma_l(1+\gamma_r)}{2\gamma_r}} + (z_r - m_l) > 0. \end{aligned}$$

In view of the properties of the function $f(\rho)$, we conclude that Eq. (2.24) has a unique solution. $RP(U_l, U_r)$ is shown in Figure 1.

In the following cases, we can obtain the uniqueness of the corresponding Riemann solution and omit the details.

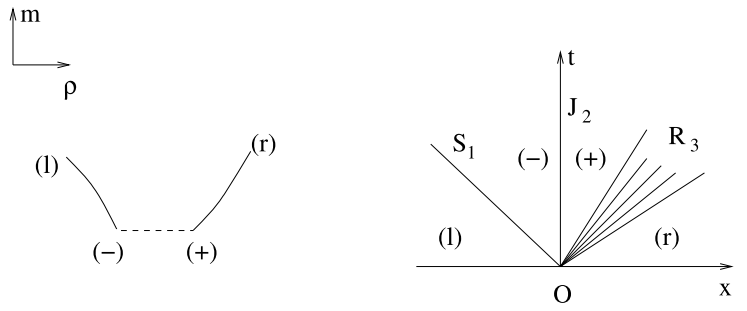


Figure 2 Case 2: $RP(U_l, U_r) = U_l + S_1 + U_- + J_2 + U_+ + R_3$.

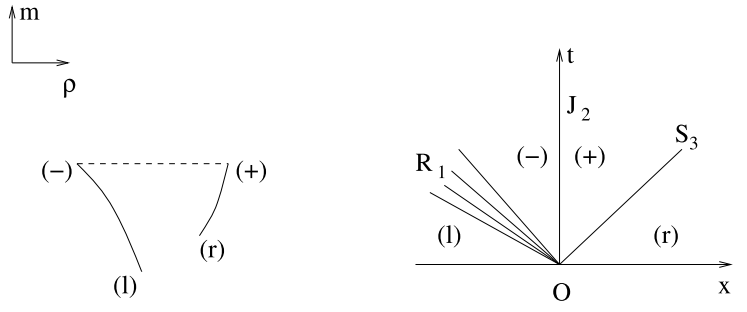


Figure 3 Case 3: $RP(U_l, U_r) = U_l + R_1 + U_- + J_2 + U_+ + S_3 + U_r$.

Case 2. $z_r < m_l < m_r$ and $w_l < m_r$. By virtue of (2.7) and (2.10), we have

$$\begin{cases} m_l - z_r = ((p(\rho_-, \gamma_l) - p(\rho_l, \gamma_l))(\rho_- - \rho_l))^{1/2} + \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_+^{\frac{1+\gamma_r}{2}}, \\ \rho_-^{\gamma_l} = \rho_+^{\gamma_r}, \\ m_- = m_+ = m_l - ((p(\rho_-, \gamma_l) - p(\rho_l, \gamma_l))(\rho_- - \rho_l))^{1/2} = z_r + \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\rho_+^{\frac{1+\gamma_r}{2}}. \end{cases} \tag{2.25}$$

$RP(U_l, U_r)$ is $U_l + S_1 + U_- + J_2 + U_+ + R_3 + U_r$ and see Figure 2. It is clear that $S_1 = S_1(U_-, U_l)$ is demonstrated by (2.10) and $R_3(U, U_r)$ is determined by (2.7) $\rho_+ \le \rho \le \rho_r$.

Case 3. $w_l > m_r$ and $m_l < m_r$, $RP(U_l, U_r)$ is $U_l + R_1 + U_- + J_2 + U_+ + S_3 + U_r$, where $U_\pm$ are determined by

$$\begin{cases} w_l - m_r = \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho_-^{\frac{1+\gamma_l}{2}} + ((p(\rho_+, \gamma_r) - p(\rho_r, \gamma_r))(\rho_+ - \rho_r))^{1/2}, \\ \rho_-^{\gamma_l} = \rho_+^{\gamma_r}, \\ m_- = m_+ = w_l - \frac{2\sqrt{\gamma_l}}{1+\gamma_l}\rho_-^{\frac{1+\gamma_l}{2}} = m_r + ((p(\rho_+, \gamma_r) - p(\rho_r, \gamma_r))(\rho_+ - \rho_r))^{1/2}, \end{cases} \tag{2.26}$$

which is shown in Figure 3. For $R_1(U, U_l)$, we can see Case 1, and $S_3 = S_3(U_+, U_r)$ is shown by (2.12).

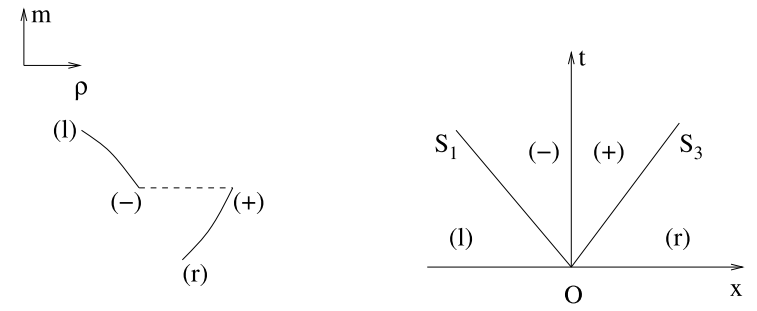


Figure 4 Case 4: $RP(U_l, U_r) = U_l + S_1 + U_- + J_2 + U_+ + S_3$.

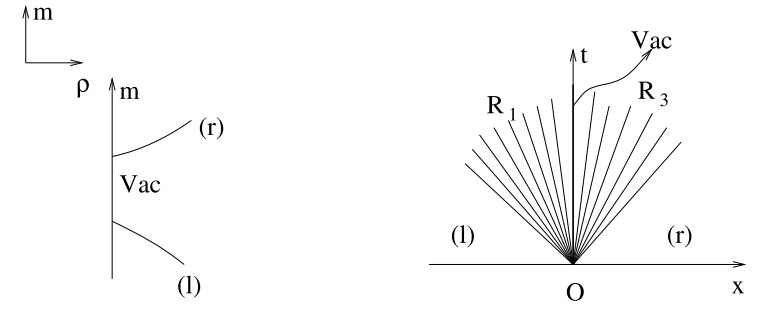


Figure 5 Case 5: $RP(U_l, U_r) = U_l + R_1 + Vac + R_3 + U_r$.

Case 4. $m_l > m_r$. Combining (2.10) and (2.12) with (2.15), we obtain

$$\begin{cases} w_l - m_r = ((p(\rho_-, \gamma_l) - p(\rho_l, \gamma_l))(\rho_- - \rho_l))^{1/2} + ((p(\rho_+, \gamma_r) - p(\rho_r, \gamma_r))(\rho_+ - \rho_r))^{1/2}, \\ \rho_-^{\gamma_l} = \rho_+^{\gamma_r}, \\ m_- = m_+ = m_l - ((p(\rho_-, \gamma_l) - p(\rho_l, \gamma_l))(\rho_- - \rho_l))^{1/2} \\ \qquad = m_r + ((p(\rho_+, \gamma_r) - p(\rho_r, \gamma_r))(\rho_+ - \rho_r))^{1/2}. \end{cases}$$

The corresponding Riemann solution $RP(U_l, U_r)$ is $U_l + S_1 + U_- + J_2 + U_+ + S_3 + U_r$, which is shown in Figure 4. Similarly, we also obtain $S_1 = S_1(U_l, U_-)$ and $S_3 = S_3(U_+, U_r)$.

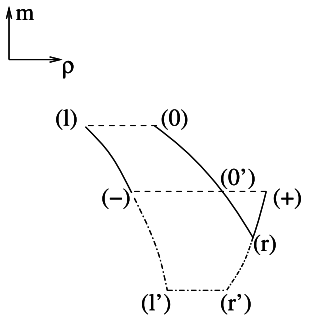
Case 5. $w_l < z_r$, we get that $R_1(U, U_l)$ does not intersect $R_3(U, U_r)$ in the region $\rho > 0$. So $RP(U_l, U_r)$ is $R_1 + U_0 + R_3$, where the state U_0 represents the vacuum, that is, $\rho_0 = 0$ at $x = 0$, see Figure 5. $R_1 = R_1(U_l, U_0)$ and $R_3 = R_3(U_0, U_r)$ can be constructed by Case 1, where $\rho_- = \rho_+ = 0$.

Remark 2 We have demonstrated the corresponding Riemann solutions for the case $\rho_- > 1$. For the case $0 < \rho_- < 1$, we note $\rho_- > \rho_+$ for $\gamma_l > \gamma_r$ and easily obtain the Riemann solutions by similar methods.

We have constructed the Riemann solutions for all the cases and have the following theorem.

Theorem 2.1 *There exists a unique solution for Riemann problem* (1.3) *and* (1.4).

Figure 6 Case 1:$J_2 + S_1$ ($\rho_- > 1$).



3 Interaction between the stationary contact wave and the elementary waves

In this section, we only consider the interaction of the stationary contact wave with the rarefaction wave or the shock wave. As for the interaction between the rarefaction wave and the shock wave, we may see [14, 21] or other related results. Since the speed of one-wave (R_1 or S_1) is less than zero and that of three-wave (R_3 or S_3) is greater than zero, the interaction of the stationary contact wave with the shock or the rarefaction wave may be divided into four cases:

$J_2 + S_1, J_2 + R_1, S_3 + J_2, R_3 + J_2$.

Case 1. The collision of J_2 and S_1.

We give the corresponding initial value with three-piece constant states as follows:

$$U = (\rho, m, \gamma) = \begin{cases} U_l, & x < 0, \\ U_0, & 0 < x < \varepsilon, \\ U_r, & x > \varepsilon, \end{cases} \tag{3.1}$$

where ε is a small positive number.

Assume that there are a stationary contact wave $J_2(U_l, U_0)$ and a one-shock wave $S_1(U_0, U_r)$, as shown in Figure 6, where $(l) = U_l = (\rho_l, m_l, \gamma_l)$, $(0) = U_0 = (\rho_0, m_0, \gamma_r)$ and $(r) = U_r = (\rho_r, m_r, \gamma_r)$.

First, we consider the subcase $\rho_- > 1$, see Figure 6. It is clear that when J_2 collides with S_1 at some point, the new Riemann problem is formed. We claim that the corresponding Riemann solution $RP(U_l, U_r)$ is $U_l + S_1(U_l, U_-) + U_- + J_2(U_-, U_+) + U_+ + S_3(U_+, U_r) + U_r$ and is not $U_l + S_1(U_l, U_{l'}) + J_2(U_{l'}, U_{r'}) + R_3(U_{r'}, U_r) + U_r$.

In order to obtain the above statement, let us claim that $\rho_+ > \rho_{0'}$, where $U_{0'} \in S_1(U_0, U)$ and $m_- = m_{0'} = m_+$. The reason is as follows.

We note that

$$\begin{cases} \rho_l^{\gamma_l} = p_l = p_0 = \rho_0^{\gamma_r}, \qquad \rho_-^{\gamma_l} = p_- = p_+ = \rho_+^{\gamma_r}, \\ m_- - m_l = -((p_- - p_l)(\rho_- - \rho_l))^{1/2}, \\ m_{0'} - m_0 = -((p_{0'} - p_0)(\rho_{0'} - \rho_0))^{1/2}, \\ m_l = m_0, \qquad m_- = m_{0'}. \end{cases} \tag{3.2}$$

Setting $\Delta\rho = \rho_- - \rho_l > 0$, $\Delta\hat{\rho} = \rho_+ - \rho_0 > 0$ and $\Delta\widetilde{\rho} = \rho_{0'} - \rho_0 > 0$, by the first equality of (3.2), we have

$$(\rho_l + \Delta\rho)^{\gamma_l} = (\rho_0 + \Delta\hat{\rho})^{\gamma_r} = \left(\rho_l^{\frac{\gamma_l}{\gamma_r}} + \Delta\hat{\rho}\right)^{\gamma_r},$$

which gives

$$\left(1 + \frac{\Delta\rho}{\rho_l}\right)^{\gamma_l} = \left(1 + \frac{\Delta\hat{\rho}}{\rho_l^{\gamma_l/\gamma_r}}\right)^{\gamma_r}.$$

Note $\gamma_l > \gamma_r$, if $\rho_l > 1$, and we have

$$\frac{\Delta\rho}{\rho_l} < \frac{\Delta\hat{\rho}}{\rho_l^{\gamma_l/\gamma_r}} < \frac{\Delta\hat{\rho}}{\rho_l}.$$

So, we get

$$\Delta\hat{\rho} > \Delta\rho. \tag{3.3}$$

We obtain from (3.2) that

$$(p_+ - p_0)\Delta\rho = (p_{0'} - p_0)\Delta\widetilde{\rho},$$

which gives

$$\Delta\widetilde{\rho} = \frac{p_+ - p_0}{p_{0'} - p_0}\Delta\rho.$$

If $p_{0'} \geq p_+$, that is, $\rho_{0'} \geq \rho_+$, it implies

$$\Delta\widetilde{\rho} \leq \Delta\rho. \tag{3.4}$$

Combining (3.2) and (3.4), we get $\Delta\hat{\rho} > \Delta\widetilde{\rho}$, which is contradiction to $p_{0'} \geq p_+$. So, we have $p_{0'} < p_+$, which implies the claim.

If $\rho_l < 1$, we have $\rho_0 < 1$ and also obtain similar results.

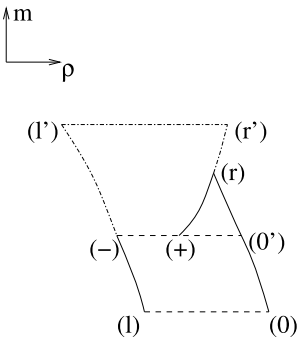
For the subcase $\rho_- < 1$, we can obtain similar results and omit the details here. In the following cases, we only consider the subcase $\rho_- > 1$.

Furthermore, we observe that as $\varepsilon \to 0$, the limit of the solution of (1.3) and (3.1) is the corresponding Riemann solution of (1.3) and (1.4).

Case 2. The collision of J_2 and R_1.

Suppose that there are a stationary contact wave $J_2(U_l, U_0)$ and a one-rarefaction wave $R_1(U_0, U_r)$, as shown in Figure 7. For the initial value problem, we may see (3.1) in the case. Similarly, when J_2 collides with R_1 at some point, the new Riemann problem is formed. We draw a one-rarefaction wave $R_1(U_l, U_-)$ from U_l to U_- and a three-rarefaction wave $R_3(U_+, U_r)$ from U_r to U_+. Here $U_\pm$ satisfies that $m_- = m_+$ and $m_l < m_\pm < m_r$. Then the new Riemann solution is $U_l + R_1(U_l, U_-) + U_- + J_2(U_-, U_+) + U_+ + R_3(U_+, U_r) + U_r$.

Figure 7 Case 2: $J_2 + R_1$ ($\rho_- > 1$).



Next, we prove that the Riemann solution is not $U_l + R_1(U_{l'}, U_l) + J_2(U_{l'}, U_{r'}) + S_3(U_{r'}, U_r) + U_r$, see Figure 7. To prove the result, we only need to show that $\rho_+ < \rho_{0'}$, where $U_{0'} \in R_1(U_0, U_r)$ and $m_- = m_+ = m_{0'}$. Then we have

$$\begin{cases} \rho_l^{\gamma_l} = p_l = p_0 = \rho_0^{\gamma_r}, \qquad \rho_-^{\gamma_l} = p_- = p_+ = \rho_+^{\gamma_r}, \\ m_- - m_l = \frac{2\sqrt{\gamma_l}}{1+\gamma_l}(\rho_l^{\frac{1+\gamma_l}{2}} - \rho_-^{\frac{1+\gamma_l}{2}}), \\ m_{0'} - m_0 = \frac{2\sqrt{\gamma_r}}{1+\gamma_r}(\rho_0^{\frac{1+\gamma_r}{2}} - \rho_{0'}^{\frac{1+\gamma_r}{2}}), \\ m_l = m_0, \qquad m_- = m_{0'}. \end{cases} \tag{3.5}$$

We can obtain from the first equation of (3.5)

$$\rho_- = \rho_+^{\frac{\gamma_r}{\gamma_l}}, \qquad \rho_l = \rho_0^{\frac{\gamma_r}{\gamma_l}}.$$

Together with the last three equalities of (3.5), we have

$$\frac{2\sqrt{\gamma_l}}{1+\gamma_l}\big(\rho_0^{\frac{(1+\gamma_l)\gamma_r}{2\gamma_l}} - \rho_+^{\frac{(1+\gamma_l)\gamma_r}{2\gamma_l}}\big) = \frac{2\sqrt{\gamma_r}}{1+\gamma_r}\big(\rho_0^{\frac{1+\gamma_r}{2}} - \rho_{0'}^{\frac{1+\gamma_r}{2}}\big). \tag{3.6}$$

By the fact that

$$\frac{2\sqrt{\gamma_l}}{1+\gamma_l} < \frac{2\sqrt{\gamma_r}}{1+\gamma_r}, \qquad \frac{1+\gamma_r}{2} > \frac{(1+\gamma_l)\gamma_r}{2\gamma_l}$$

for $\gamma_l > \gamma_r > 1$, we obtain from (3.6) that

$$\rho_+^{\frac{1+\gamma_r}{2}} < \rho_{0'}^{\frac{1+\gamma_r}{2}},$$

which implies

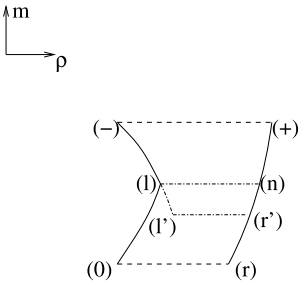
$$\rho_+ < \rho_{0'}.$$

Thus, the result is verified.

Moreover, it is clear that as $\varepsilon \to 0$, the solution of the initial value problem transforms into $RP(U_l, U_r)$.

Case 3. The collision of S_3 and J_2.

Figure 8 Case 3: $S_3 + J_2$ ($\rho_- > 1$).



To discuss the interaction between S_3 and J_2, we consider the initial value problem with three-piece constant states

$$U = (\rho, m, \gamma) = \begin{cases} U_l, & x < -\varepsilon, \\ U_0, & -\varepsilon < x < 0, \\ U_r, & x > 0, \end{cases} \tag{3.7}$$

where $(l) = U_l = (\rho_l, m_l, \gamma_l)$, $(0) = U_0 = (\rho_0, m_0, \gamma_l)$, $U_r = (\rho_r, m_r, \gamma_r)$ and $\varepsilon > 0$.

Assume that there are a three-shock wave $S_3(U_l, U_0)$ and a stationary contact wave $J_2(U_0, U_r)$, and they collide with each other at a finite time, as shown in Figure 8. Meanwhile, a new Riemann problem is formed, the corresponding Riemann solution is denoted by $RP(U_l, U_r)$. We draw a one-rarefaction wave $R_1(U_l, U_-)$ from U_l to U_- and a three-shock wave $S_3(U_+, U_r)$ from U_r to U_+. Here $U_\pm$ satisfies that $m_- = m_+$, $m_\pm > m_l$ and $m_\pm > m_r$. Then the new Riemann solution $RP(U_l, U_r)$ is $U_l + R_1(U_-, U_l) + U_- + J_2(U_-, U_+) + U_+ + S_3(U_+, U_r) + U_r$.

If $\rho_- > 1$, which indicates $\rho_+ > 1$, we have $\rho_r > 1$, which implies $\rho_0 > 1$. If $\rho_- < 1$, we have $\rho_+ < 1$, which implies $\rho_r < 1$.

For this subcase $\rho_- > 1$, we claim that the Riemann solution is not $U_l + S_1(U_{l'}, U_l) + J_2(U_{l'}, U_{r'}) + S_3(U_{r'}, U_r) + U_r$, see Figure 8. To prove the result, it is sufficient to show that $\rho_{\bar{l}} > \rho_n$, where $U_n \in S_3(U_+, U_r)$, $U_{\bar{l}} \in J_2(U_l, U)$ and $m_{\bar{l}} = m_l = m_n$. So, we have

$$\begin{cases} \rho_0^{\gamma_l} = p_0 = p_r = \rho_r^{\gamma_r}, \qquad \rho_l^{\gamma_l} = p_l = p_{\bar{l}} = \rho_{\bar{l}}^{\gamma_r}, \\ m_l - m_0 = ((p_l - p_0)(\rho_l - \rho_0))^{1/2}, \\ m_n - m_r = ((p_n - p_r)(\rho_n - \rho_r))^{1/2}, \\ m_l = m_n, \qquad m_0 = m_r. \end{cases} \tag{3.8}$$

It is clear from (3.8) that

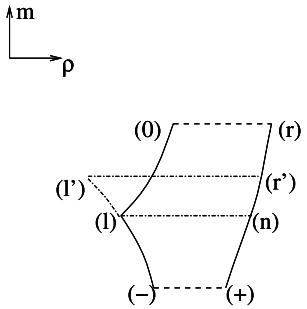
$$(p_l - p_0)(\rho_l - \rho_0) = (p_n - p_0)(\rho_n - \rho_r). \tag{3.9}$$

If $\rho_{\bar{l}} \le \rho_n$, then $p_l = p_{\bar{l}} \le p_n$, which implies $\rho_l - \rho_0 > \rho_n - \rho_r$. Setting $\Delta\rho = \rho_l - \rho_0$ and $\Delta\widetilde{\rho} = \rho_n - \rho_r$, we have $\Delta\rho > \Delta\widetilde{\rho}$. From the first equality of (3.11), we get

$$\rho_{\bar{l}} = (\rho_0 + \Delta\rho)^{\frac{\gamma_l}{\gamma_r}}, \qquad \rho_n = \rho_0^{\frac{\gamma_l}{\gamma_r}} + \Delta\widetilde{\rho},$$

which implies $\rho_{\bar{l}} > \rho_n$ and $\rho_0 > 1$. This is not true. So, we prove the above statement.

Figure 9 $R_3 + J_2$ ($\rho_- > 1$).



As $\varepsilon \to 0$, the solution of the initial value problem reduces to $RP(U_l, U_r)$.

Case 4. The collision of R_3 and J_2.

Suppose the initial value described as (3.7), there are a three-shock wave $R_3(U_l, U_0)$ and a stationary contact wave $J_2(U_0, U_r)$, and they collide with each other at a finite time, as shown in Figure 9. Then a new Riemann problem is formed. In order to clarify the construction of the corresponding Riemann solution $RP(U_l, U_r)$, we show $\rho_{\bar{l}} < \rho_n$, where $U_n \in R_3(U_+, U_r)$, $U_{\bar{l}} \in J_2(U, U_l)$ and $m_{\bar{l}} = m_l = m_n$. In fact, we have

$$\begin{cases} \rho_0^{\gamma_l} = p_0 = p_r = \rho_r^{\gamma_r}, \qquad \rho_l^{\gamma_l} = p_l = p_{\bar{l}} = \rho_{\bar{l}}^{\gamma_r}, \\ m_0 - m_l = ((p_0 - p_l)(\rho_0 - \rho_l))^{1/2}, \\ m_r - m_n = ((p_r - p_n)(\rho_r - \rho_n))^{1/2}, \\ m_l = m_n, \qquad m_0 = m_r. \end{cases} \tag{3.10}$$

We obtain from (3.10) that

$$(p_0 - p_l)(\rho_0 - \rho_l) = (p_0 - p_n)(\rho_r - \rho_n). \tag{3.11}$$

If $\rho_{\bar{l}} \geq \rho_n$, then $p_l = p_{\bar{l}} \geq p_n$. Setting $\Delta\rho = \rho_0 - \rho_l$ and $\Delta\widetilde{\rho} = \rho_r - \rho_n$, we have $\Delta\rho \geq \Delta\widetilde{\rho}$. From the first equality of (3.10), we get

$$\rho_{\bar{l}} = (\rho_0 - \Delta\rho)^{\frac{\gamma_l}{\gamma_r}}, \qquad \rho_n = \rho_0^{\frac{\gamma_1}{\gamma_r}} - \Delta\widetilde{\rho},$$

which implies $\rho_{\bar{l}} < \rho_n$ for $\gamma_l > \gamma_r$. This is contradiction to the assumption. Then we prove the above statement.

Now, we draw a one-shock wave $S_1(U_-, U_l)$ from the state U_l and a three-rarefaction wave $R_3(U_+, U_r)$ from the state U_r. Here $U_\pm$ satisfies that $m_- = m_+$, $m_\pm < m_l$ and $m_\pm < m_r$. Then the Riemann solution $RP(U_l, U_r)$ is $U_l + S_1(U_-, U_l) + U_- + J_2(U_-, U_+) + U_+ + R_3(U_+, U_r) + U_r$, see Figure 9.

In addition, it is not difficult to find that $RP(U_l, U_r)$ is the limit of the solution of the initial value problem.

So far, we have discussed the interactions of the contact wave with the rarefaction wave or the shock wave and have constructed the solutions for the initial value problem (1.3) and (3.1) or (3.7). Therefore, we obtain the following theorem.

Theorem 3.1 *There exists a unique solution to the perturbed initial value problem* (1.3) *and* (3.1) *or* (3.7). *The limit of the perturbed Riemann solution of* (1.3) *and* (3.1) *or* (3.7)

is exactly the corresponding Riemann solution of (1.3) *and* (1.4). *The Riemann solution of* (1.3) *and* (1.4) *is stable with respect to such small perturbations of the initial data*.

4 Concluding remarks

In this paper, we present the Riemann problem and the interactions of the stationary contact discontinuity with the elementary waves. We discover the stability of the generalized Riemann problem, but do not observe the composite wave, which motivates us to consider the related problems including the coupling of two different hyperbolic systems.

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Competing interests
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Authors' contributions
All authors carried out the proofs and the authors conceived of the study. All authors read and approved the final manuscript.

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