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Study of a class of arbitrary order differential equations by a coincidence degree method

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Abstract

In this manuscript, we investigate some appropriate conditions which ensure the existence of at least one solution to a class of fractional order differential equations (FDEs) provided by

$$\begin{cases} -{}^{C}\mathbf{D}^{q}z(t) = \theta(t, z(t)); & t \in \mathfrak{J} = [0,1], q \in (1,2], \\ z(t)|_{t=0} = \phi(z), \qquad z(1) = \delta\, {}^{C}\mathbf{D}^{p}z(\eta), & p, \eta \in (0,1). \end{cases}$$

The nonlinear function $\theta : \mathfrak{J} \times \mathbf{R} \to \mathbf{R}$ is continuous. Further, $\delta \in (0,1)$ and $\phi \in C(\mathfrak{J}, \mathbf{R})$ is a non-local function. We establish some adequate conditions for the existence of at least one solution to the considered problem by using Grönwall's inequality and a priori estimate tools called the topological degree method. We provide two examples to verify the obtained results.

MSC: 34A08; 35R11

Keywords: fractional order differential equations; Caputo derivative; condensing operator; Grönwall's inequality; topological degree method

1 Introduction

The applications of non-integer order differential equations are increasing day by day in various areas of research. These applications can be traced out in many disciplines of science including technology and engineering. To see the recent applications of the mentioned area in various fields like physics, mechanics, chemical science, biological dynamics, material engineering, theory of control, signal and image propagation, communication and transform, economical problems and optimization theory, etc.; the reader is referred to [1–5]. Non-integer order differential equations provide a strong tool for the description of memory and hereditary properties of various materials and processes; see [6]. One of the key reasons of taking interest in fractional differential equations by the researchers is the presence of greater degree of freedom of fractional differential operator. In fact the fractional derivative is a global operator instead of classical derivative which is local in nature; see [7]. In the last few decades, various aspects of fractional differential equations have been investigated like existence theory, stability and numerical analysis, etc. One of the attractive areas of research is the existence theory of solutions for FDEs. In the last few decades, the aforesaid area has been extensively investigated by using various techniques of classical analysis, for further explanations, we refer the reader to [8–14]. It is well known that the standard Riemann-Liouville fractional derivative fails to provide the required physical interpretation for boundary value problems (BVPs) and initial value problems (IVPs) in most of the cases. However, these requirements of interpreting the initial and boundary conditions can better be fulfilled by the use of Caputo non-integer order derivatives. Existence theory of solutions or positive solutions to multi-points boundary value problems using different types of fixed point theorems like as Banach theorem of contraction type, Schaefer and Leray-Schauder theorem of fixed point is studied in detail [15–21]. Moreover, the existence of solutions to FDEs using coincidence degree theory for a contraction operator is studied in [22–28]. Recently the existence of a center stable manifold for planar fractional damped equations has been investigated. For the required solution, the authors in [29] constructed a suitable Lyapunov-Perron operator by giving the asymptotic behavior of the Mittag-Leffler function. Then they obtained an interesting center stable manifold result to prove center stable manifold theorem for planar fractional damped equations involving two Caputo fractional derivatives. In a similar manner, the authors of [30], studied finite time stability and existence theory of delay type differential equations of fractional order by using classical analysis. Wang *et al.* [26] investigated the existence theory and proved some conditions for uniqueness and derived some data dependency results of solutions using topological degree technique by considering some classes of non-local Cauchy problems including BVPs and impulsive Cauchy problems (ICPs) to FDEs. Chen *et al.* [27], obtained the existence results by coincidence degree theory to the following BVP involving a p-Laplacian operator:

$$\begin{cases} {}^{C}\mathbf{D}^{q}_{0+}\phi_p\big({}^{C}\mathbf{D}^{p}_{0+}z(t)\big) = \theta\big(t, z(t), {}^{C}\mathbf{D}^{p}_{0+}z(t)\big), \\ {}^{C}\mathbf{D}^{p}_{0+}z(t)|_{t=0} = {}^{C}\mathbf{D}^{p}_{0+}z(1) = 0, \end{cases}$$

where ${}^{C}\mathbf{D}^{q}_{0+}$ and ${}^{C}\mathbf{D}^{p}_{0+}$ represent non-integer order derivatives in the Caputo sense, $p, q \in (0,1]$, $p + q \in (1,2]$. Tang *et al.* [28] applied the aforesaid degree theory and established results for the following two point BVP of non-integer order p-Laplace DEs:

$$\begin{cases} {}^{C}\mathbf{D}^{q}_{0+}\phi_p\big({}^{C}\mathbf{D}^{p}_{0+}z(t)\big) = \theta\big(t, z(t), {}^{C}\mathbf{D}^{p}_{0+}z(t)\big), \\ z(t)\big|_{t=0} = 0, \qquad {}^{C}\mathbf{D}^{p}_{0+}z(t)\big|_{t=0} = {}^{C}\mathbf{D}^{p}_{0+}z(1), \end{cases}$$

where ${}^{C}\mathbf{D}^{q}_{0+}$ and ${}^{C}\mathbf{D}^{p}_{0+}$ are non-integer order derivatives of Caputo type, $p, q \in (0,1]$, $p+q \in (1,2]$. The mentioned theory of degree type has been studied recently in many papers; see [31–34].

The current manuscript is inspired from the aforesaid work. Our aim is to investigate the existence and uniqueness of at least one solution by applying the coincidence degree theory for a condensing mapping to the three-points BVP supplied as

$$\begin{cases} -{}^{C}\mathbf{D}^{q}z(t) = \theta\big(t, z(t)\big); \quad t \in \mathfrak{J}, q \in (1,2], \\ z(t)|_{t=0} = \phi(z), \qquad z(1) = \delta\, {}^{C}\mathbf{D}^{p}z(\eta). \end{cases} \tag{1}$$

The manuscript is organize as explained below.

Section 2 is concerned with some background material and lemmas required for the main results. In Section 3, the problem under consideration of FDEs is transformed to its equivalent Fredholm integral equation. Then the required theory devoted to the aims of this paper is developed via using coincidence degree of condensing maps and using the standard singular Grönwall inequality. At the end, an example is provided for justification of the established results.

2 Background material

This section contains basics materials and preliminaries related to of non-integer order calculus and degree theory of topological type. For further details, we refer to [2–5, 35–38].

The space consisting of all continuous functions $\mathfrak{J} \to \mathbf{R}$ is a Banach space endowed with a norm $\|z\|_{\mathbf{Z}} = \sup\{|z(t)| : t \in \mathfrak{J}\}$. For simplicity, we denote the defined space by $\mathbf{Z} = C(J, \mathbf{R})$.

Definition 2.1 Let $z \in C(\mathbf{R}^+, \mathbf{R})$ be a function. Then the non-integer order integral of order $q \in \mathbf{R}_+$ of the function $z(t)$ is defined as

$$\mathbf{I}_{a+}^{q} z(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{z(\tau)}{(t-\tau)^{q-1}}\, d\tau,$$

provided that integral on the right is pointwise defined on $(0, \infty)$.

Definition 2.2 The Caputo type non-integer order derivative of a function $z : \mathbf{R}^+ \to \mathbf{R}$ is defined by

$${}^{C}\mathbf{D}_{a+}^{q} z(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-\tau)^{m-q-1} z^{(m)}(\tau)\, d\tau,$$

where $m = [q] + 1$ and $[q]$ represents the integer part of q.

For further details on fractional derivatives and integrals; see [2–5].

Lemma 2.1 ([35]) *The unique solution of FDE of order* $q > 0$

$${}^{C}\mathbf{D}^{q} z(t) = 0, \quad q \in (m-1, m],$$

is given as

$$z(t) = e_0 + e_1 t + e_2 t^2 + \cdots + e_{m-1} t^{m-1}, \quad \textit{where } e_i \in \mathbf{R}, i = 0, 1, 2, \ldots, m-1.$$

Theorem 2.3 ([35]) *The given FDE*

$${}^{C}\mathbf{D}^{q} z(t) = \delta(t), \quad q \in (m-1, m],$$

has a solution given by

$$z(t) = \mathbf{I}^{q} \delta(t) + e_0 + e_1 t + e_2 t^2 + \cdots + e_{m-1} t^{m-1},$$

for arbitrary $e_i \in \mathbf{R}$, $i = 0, 1, 2, \ldots, m-1$.

Next, we present some important definitions, propositions and theorems from [36]. For the Banach space $\mathbf{Z}$, with $\mathbf{C} \in P(\mathbf{Z})$ represents the collection of all bounded sets.

Definition 2.4 The measure with non-compactness of Kuratowski type $\beta : \mathbf{C} \to \mathbf{R}_+$ as given by

$$\beta(\mathbf{B}) = \min\{d > 0\},$$

where $\mathbf{B} \in \mathbf{C}$ inserts a finite cover with a sets of diameter $\leq d$.

Proposition 2.1 *The measure of Kuratowski type denoted by β satisfies the following properties*:

(i) *the set $\mathbf{B}$ is relatively compact if and only if $\mathbf{B} \in \mathbf{C}$ has Kuratowski measure zero*;
(ii) *β is a seminorm, because it satisfies $\beta(\lambda\mathbf{B}) = |\lambda|\beta(\mathbf{B})$, $\lambda \in \mathbf{R}$ and* $\beta(\mathbf{B}_1 + \mathbf{B}_2) \leq \beta(\mathbf{B}_1) + \beta(\mathbf{B}_2)$;
(iii) $\beta(\mathbf{B}_1) \leq \beta(\mathbf{B}_2)$ *for* $\mathbf{B}_1 \subset \mathbf{B}_2$ *and* $\beta(\mathbf{B}_1 \cup \mathbf{B}_2) = \sup\{\beta(\mathbf{B}_1), \beta(\mathbf{B}_2)\}$;
(iv) $\beta(\operatorname{conv}\mathbf{B}) = \beta(\mathbf{B})$;
(v) $\beta(\bar{\mathbf{B}}) = \beta(\mathbf{B})$.

Definition 2.5 Assume that the function $\mathcal{F} : \Omega \to \mathbf{Z}$ is a continuous and bounded mapping for $\Omega \subset \mathbf{Z}$. Then $\mathcal{F}$ is β-Lipschitz if there exists $\mathcal{K} \geq 0$ such that

$$\beta(\mathcal{F}(\mathbf{B})) \leq \mathcal{K}\beta(\mathbf{B}), \quad \text{for all } \mathbf{B} \subset \Omega \text{ bounded}.$$

Also if $\mathcal{K} < 1$, then $\mathcal{F}$ is said to be a strict β-contraction.

Definition 2.6 The function $\mathcal{F}$ is β-condensing if

$$\beta(\mathcal{F}(\mathbf{B})) < \beta(\mathbf{B}), \quad \text{for every } \mathbf{B} \subset \Omega \text{ bounded with } \beta(\mathbf{B}) > 0.$$

In other words, $\beta(\mathcal{F}(\mathbf{B})) \geq \beta(\mathbf{B})$ implies $\beta(\mathbf{B}) = 0$.

Here, we represent the family of all strict β-contraction mappings $\mathcal{F} : \Omega \to \mathbf{Z}$ by $\Theta\mathbf{C}_\beta(\Omega)$. Further, denoting the family of all β-condensing mappings $\mathcal{F} : \Omega \to \mathbf{Z}$ by $\mathbf{C}_\beta(\Omega)$.

Remark 1 Each $\mathcal{F} \in \mathbf{C}_\beta(\Omega)$ is β-Lipschitz with constant $\mathcal{K} = 1$, where $\Theta\mathbf{C}_\beta(\Omega) \subset \mathbf{C}_\beta(\Omega)$.

Moreover, if there exists $\mathcal{K} > 0$, then $\mathcal{F} : \Omega \to \mathbf{Z}$ is said to be Lipschitz if and only if

$$\|\mathcal{F}(z) - \mathcal{F}(\bar{z})\| \leq \mathcal{K}|z - \bar{z}|, \quad \text{for every } z, \bar{z} \in \Omega.$$

Also $\mathcal{F}$ is strict contraction if and only if $\mathcal{K} < 1$.

The provided propositions are necessarily required for our analysis throughout this paper.

Proposition 2.2 *Consider $\mathcal{F}, \mathcal{G} : \Omega \to \mathbf{Z}$ to be β-Lipschitz operators and there exist two constants $\mathcal{K}$ and $\mathcal{K}'$, respectively, then their sum $\mathcal{F} + \mathcal{G} : \Omega \to \mathbf{Z}$ is also β-Lipschitz with constant $\hat{\mathcal{K}} = \mathcal{K} + \mathcal{K}'$.*

Proposition 2.3 *The operator $\mathcal{F}$ is β-Lipschitz with constant $\mathcal{K} = 0$. Then the same operator $\mathcal{F} : \Omega \to \mathbf{Z}$ is compact.*

Proposition 2.4 *If an operator $\mathcal{F} : \Omega \to \mathbf{Z}$ is Lipschitz with constant $\mathcal{K}$. Then the same operator $\mathcal{F}$ will also be β-Lipschitz with the same constant $\mathcal{K}$.*

Theorem 2.7 *We recall some basic properties of proposed degree theory from Isaia* [37].
Let for the family of admissible triplets given by

$$\digamma = \bigl\{(\mathcal{I} - \mathcal{F}, \Omega, z) : \Omega \subset \mathbf{Z} \text{ be an open and bounded set}, \mathcal{F} \in \mathbf{C}_\beta(\bar{\Omega}),\\ z \in \mathbf{Z} \setminus (\mathcal{I} - \mathcal{F})(\partial\Omega)\bigr\},$$

there exists a function $\deg : \digamma \to \mathbf{Z}$ *of one degree which has the following properties.*

(D1) *Normalization*: $\deg(I, \Omega, z) = 1$ *at each* $z \in \Omega$;

(D2) *additivity on domain*: *For each pair of disjoint open sets* $\Omega_1, \Omega_2 \subset \Omega$ *and each* $z \notin (I - \mathcal{F})((\bar{\Omega}) \setminus (\Omega_1 \cup \Omega_2))$, *we have*

$$\deg(\mathfrak{I} - \mathcal{F}, \Omega, z) = \deg(\mathcal{I} - \mathcal{F}\Omega, \Omega_1, z) + \deg(\mathcal{I} - \mathcal{F}, \Omega_2, z);$$

(D3) *invariance property under homoptopy*: $\deg(\mathcal{I} - H(t, z), \Omega, z)$ *is independent of* $t \in \mathfrak{J}$ *for each continuous and bounded mapping* $H : \mathfrak{J} \times \bar{\Omega} \to \mathbf{Z}$ *which satisfies*

$$\beta\bigl(H(\mathfrak{J} \times \mathbf{B})\bigr) < \beta(\mathbf{B}), \quad \text{for all } \mathbf{B} \subset \bar{\Omega} \text{ with } \beta(\mathbf{B}) > 0$$

and every continuous function $z : \mathfrak{J} \to \mathbf{Z}$ *which satisfies*

$$z \neq z - H(t, z), \quad \text{for all } t \in \mathfrak{J}, \text{for every } z \in \partial\Omega;$$

(D4) *existence*: $\deg(\mathcal{I} - \mathcal{F}, \Omega, z) \neq 0$ *yields*

$$z \in (\mathcal{I} - \mathcal{F})(\Omega);$$

(D5) *excision*: $\deg(\mathcal{I} - \mathcal{F}, \Omega, z) = \deg(\mathcal{I} - \mathcal{F}, \Omega_1, z)$ *for each open set* $\Omega_1 \subset \Omega$ *and for all* $z \notin (\mathcal{I} - \mathcal{F})(\bar{\Omega} \setminus \Omega_1)$.

Theorem 2.8 *Assume that* $\mathcal{F} : \mathbf{Z} \to \mathbf{Z}$ *is a* β*-condensing operator and*

$$\Theta = \{z \in \mathbf{Z} : \text{there exists } \lambda \in \mathfrak{J} \text{ with } z = \lambda\mathcal{F}z\} \subset \mathbf{Z}$$

is a bounded set and there exists a real number $r > 0$ *with* $\Theta \subset \mathbf{B}_r(0)$. *Then*

$$\deg\bigl(\mathcal{I} - \lambda\mathcal{F}, \mathbf{B}_r(0), 0\bigr) = 1, \quad \text{for all } \lambda \in \mathfrak{J}.$$

Therefore, the operator $\mathcal{F}$ *has at least one fixed point and the set of fixed points of* $\mathcal{F}$ *lies in* $\mathbf{B}_r(0)$.

Theorem 2.9 ([38]) *Let $z \in \mathbf{Z}$ satisfies the following inequality*:

$$\left|z(t)\right| \leq \hat{a} + \hat{b}\int_0^t (t-\tau)^{q-1}\left|z(s)\right|^{\lambda}\,d\tau + \hat{c}\int_0^t (T-\tau)^{q-1}\left|z(\tau)\right|^{\lambda}\,d\tau, \quad q \in (0,1]. \tag{2}$$

Here $0 < \lambda < 1 - \frac{1}{r}$ for some $r \in (1, \frac{1}{1-q})$ and $\hat{a}, \hat{b}, \hat{c} \in (0,\infty)$ are constants. Then we have

$$\left|z(t)\right| \leq (\hat{a}+1)e^{\mathcal{M}T}.$$

3 Existence of at least one solution to BVP (1)

The purpose of this section is concerning to establish the required theory for existence of at least one solutions to the BVP (1).

Lemma 3.1

For $\omega \in L^1(\mathfrak{J}, \mathbf{R})$, the solution of the linear BVP of FDEs

$$\begin{aligned} &{}^C\mathbf{D}^q z(t) + \omega(t) = 0; \quad t \in \mathfrak{J}, q \in (1,2], \\ &u(t)|_{t=0} = \phi(z), \qquad z(1) = \delta\,{}^C\mathbf{D}^p z(\eta), \end{aligned} \tag{3}$$

is given as

$$z(t) = \phi(z)(1-td) + \int_0^1 \mathcal{H}(t,\tau)\omega(\tau)\,d\tau,$$

where $\mathcal{H}(t,s)$ is the Green's function given by

$$\mathcal{H}(t,\tau) = \begin{cases} \frac{td(1-\tau)^{q-1}}{\Gamma(q)} - \frac{1(t-\tau)^{q-1}}{\Gamma(q)} - \frac{\delta td(\eta-\tau)^{q-p-1}}{\Gamma(q-p)}; & 0 \leq \tau \leq t \leq \eta \leq 1, \\ \frac{td(1-\tau)^{q-1}}{\Gamma(q)} - \frac{\delta td(\eta-\tau)^{q-p-1}}{\Gamma(q-p)}; & 0 \leq t \leq \tau \leq \eta \leq 1, \\ \frac{td(1-\tau)^{q-1}}{\Gamma(q)} - \frac{(t-\tau)^{q-1}}{\Gamma(q)}; & 0 \leq \eta \leq \tau \leq t \leq 1, \\ \frac{td(1-\tau)^{q-1}}{\Gamma(q)}; & 0 \leq \eta \leq t \leq \tau \leq 1. \end{cases} \tag{4}$$

Proof Consider equation (3) with the associated given boundary conditions; applying $\mathbf{I}^q$ on $-{}^C\mathbf{D}^q z(t) = \omega(t)$ and thanks to Theorem 2.3, we have

$$z(t) = -\mathbf{I}^q(t) + e_0 + e_1 t \tag{5}$$

for some $e_0, e_1 \in \mathbf{R}$. From the non-local condition $z(t)_{t=0} = \phi(z)$ implies that $e_0 = \phi(z)$ and $z(1) = \delta\,{}^C\mathbf{D}^p u(\eta)$ yields $e_1 = d[\mathbf{I}^q\omega(1) - \delta\mathbf{I}^{q-p}\omega(\eta) - \phi(z)]$, where $d = \frac{\Gamma(2-p)}{\Gamma(2-p)-\delta\eta^{1-p}} > 1$. It implies that

$$z(t) = -\mathbf{I}^q\omega(t) + td\mathbf{I}^q\omega(1) - \delta td\mathbf{I}^{q-p}\omega(\eta) + (1-td)\phi(z). \tag{6}$$

Thus we get a solution z in the form

$$z(t) = \phi(z)(1-td) + \int_0^1 \mathcal{H}(t,\tau)\omega(\tau)\,d\tau,$$

where the kernel $\mathcal{H}(t,\tau)$ is the Green's function and is given as (4). □

Lemma 3.2 *A function $z \in \mathbf{Z}$ will be the solution of the fractional integral equation* (6) *if and only if z is a solution of* (1).

Proof The proof is obvious. □

To derive formally the required results as regards the data dependence and existence of at least one of solutions to the proposed BVP (1), we state the following hypotheses:

(A_1) For arbitrary $u, v \in \mathbf{Z}$ and if there exists a constant $\mathcal{K}_\phi \in [0,1)$, then one has

$$\left|\phi(z) - \phi(\bar{z})\right| \le \mathcal{K}_\phi \|z - \bar{z}\|_{\mathbf{Z}};$$

(A_2) for constants $\mathcal{C}_\phi$, $q_1 \in [0,1)$ and $\mathcal{M}_\phi > 0$, we have the following growth condition:

$$\left|\phi(z)\right| \le \mathcal{C}_\phi \|z\|_{\mathbf{Z}}^{q_1} + \mathcal{M}_\phi, \quad \text{for each } z \in \mathbf{Z};$$

(A_3) in the same fashion, for constants $\mathcal{C}_\theta$, $q_2 \in [0,1)$ and $\mathcal{M}_\theta > 0$, we have the following growth condition:

$$\left|\theta(t, z(t))\right| \le \mathcal{C}_\theta \|z\|_{\mathbf{Z}}^{q_2} + \mathcal{M}_\theta.$$

To show that equation (6) has at least one solution $z \in \mathbf{Z}$ based on assumptions (A_1)-(A_3), we define the operators by

$$\mathcal{F} : \mathbf{Z} \to \mathbf{Z}$$

as follows:

$$(\mathcal{F}z)(t) = (1 - td)\phi(z), \quad d > 1$$

and $\mathcal{G} : \mathbf{Z} \to \mathbf{Z}$ is defined by

$$\begin{aligned}(\mathcal{G}z)(t) = {}& \frac{td}{\Gamma(q)} \int_0^1 (1-s)^{q-1}\theta\bigl(\tau, z(\tau)\bigr)\,d\tau - \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1}\theta\bigl(\tau, z(\tau)\bigr)\,d\tau \\ & - \frac{1}{\Gamma(q-p)} \int_0^\eta (\eta-\tau)^{q-p-1}\theta\bigl(\tau, z(\tau)\bigr)\,d\tau,\end{aligned}$$

$$\mathbf{T} : \mathbf{Z} \to \mathbf{Z}, \qquad \mathbf{T}z = \mathcal{F}z + \mathcal{G}z.$$

Obviously, the operator $\mathbf{T}$ is well defined because the function θ is continuous. So, we can write (6) as an operator equation given by

$$z = \mathbf{T}z = \mathcal{F}z + \mathcal{G}z. \tag{7}$$

Here by the existence of a solution to equation (1), we mean the existence of a fixed point for operator $\mathbf{T}$ as defined afore and satisfying equation (7).

Lemma 3.3 *The operator $\mathcal{F} : \mathbf{Z} \to \mathbf{Z}$ is Lipschitz and consequently β-Lipschitz with constant $\mathcal{K}_{\mathcal{F}} < 1$. Moreover, the operator $\mathcal{F}$ satisfies the following growth condition*:

$$\|\mathcal{F}z\| \leq \mathcal{Q}\|z\|_{\mathbf{Z}}^{q_1} + \mathcal{M}_{\phi}, \quad \textit{for every } z \in \mathbf{Z}. \tag{8}$$

Proof For $\mathcal{F}$ to be Lipschitz, we consider $|\mathcal{F}z(t) - \mathcal{F}\bar{z}(t)|$, and apply assumptions (A_1) and (A_2), we have

$$\begin{aligned} \left|\mathcal{F}z(t) - \mathcal{F}\bar{z}(t)\right| &= \left|(1-td)\big(\phi\big(z(t)\big) - \phi\big(\bar{z}(t)\big)\big)\right| \\ &= \left|(1-td)\right|\left|\big(\phi\big(z(t)\big) - \phi\big(\bar{z}(t)\big)\big)\right| \\ &\leq \left|(1-td)\right|\mathcal{K}_{\phi}\|z-\bar{z}\|_{\mathbf{Z}} \\ &\leq \mathcal{K}_{\mathcal{F}}\|z-\bar{z}\|_{\mathbf{Z}}, \quad \text{where } \mathcal{K}_{\mathcal{F}} = \left|(1-td)\right|\mathcal{K}_{\phi} < 1. \end{aligned}$$

Hence, we get

$$\|\mathcal{F}z - \mathcal{F}\bar{z}\|_{\mathbf{Z}} \leq \mathcal{K}_{\mathcal{F}}\|z-\bar{z}\|_{\mathbf{Z}}, \quad \text{for every } z \in \mathbf{Z}.$$

Thanks to proposition 2.4, $\mathcal{F}$ is also β-Lipschitz with the same coefficient $K_{\mathcal{F}}$.

To derive the growth condition, we consider $(\mathcal{F}z)(t) = (1-td)\phi(z)$, and applying assumption (A_2), we get

$$\|\mathcal{F}z\| \leq \mathcal{Q}\|z\|_{\mathbf{Z}}^{q_1} + \mathcal{M}_{\phi}, \quad \text{for every } z \in \mathbf{Z},$$

where $\mathcal{Q} = |d|C_{\phi}$. □

Lemma 3.4 *The operator $\mathcal{G} : \mathbf{Z} \to \mathbf{Z}$ is continuous. Moreover, it also satisfies the growth condition as*

$$\|\mathcal{G}z\|_{\mathbf{Z}} \leq \frac{2d+1}{\Gamma(q-p+1)}\big(\mathcal{C}_{\theta}\|z\|_{\mathbf{Z}}^{q_2} + \mathcal{M}_{\theta}\big), \tag{9}$$

for every $z \in \mathbf{Z}$.

Proof Let $\{z_m\}$ be a sequence in the bounded set $\bar{\mathbf{B}} = \{\|z\|_{\mathbf{Z}} \leq \kappa : z \in \mathbf{Z}\}$ such that $z_m \to z$ as $m \to \infty$ in $\bar{\mathbf{B}}$. We need to show that $\|\mathcal{G}z_m - \mathcal{G}z\|_{\mathbf{Z}} \to 0$ as $m \to \infty$. Since θ is continuous and $z_m \to z$, therefore, $\theta(\tau, z_m(\tau)) \to \theta(\tau, z(\tau))$ as $m \to 0$. Now consider

$$\begin{aligned} &\left|(\mathcal{G}z_m)(t) - (\mathcal{G}z)(t)\right| \\ &\quad\leq \frac{td}{\Gamma(q)}\int_0^1 (1-\tau)^{q-1}\left|\theta\big(\tau, z_m(\tau)\big) - \theta\big(\tau, z(\tau)\big)\right| d\tau \\ &\qquad+ \frac{t\delta d}{\Gamma(q-p)}\int_0^{\eta} (\eta-\tau)^{q-p-1}\left|\theta\big(\tau, z_m(\tau)\big) - \theta\big(\tau, z(\tau)\big)\right| d\tau \\ &\qquad+ \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\left|\theta\big(\tau, z_m(\tau)\big) - \theta\big(\tau, z(\tau)\big)\right| d\tau. \end{aligned}$$

In view of assumption (A_3) and thanks to the Lebesgue dominated convergence theorem, one has

$$\left\|(\mathcal{G}z_m)(t)-(\mathcal{G}z)(t)\right\|_{\mathbf{Z}}\to 0 \quad \text{as } m\to\infty,$$

which shows that $\mathcal{G}$ is continuous. To obtain the growth condition for the nonlinear operator $\mathcal{G}$, consider

$$\begin{aligned}\left|(\mathcal{G}z)(t)\right| &= \left|\frac{td}{\Gamma(q)}\int_0^1(1-\tau)^{q-1}\theta\bigl(\tau,z(\tau)\bigr)\,d\tau\right.\\ &\quad -\frac{t\delta d}{\Gamma(q-p)}\int_0^{\eta}(\eta-\tau)^{q-p-1}\theta\bigl(\tau,z(\tau)\bigr)\,d\tau\\ &\quad \left.-\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}\theta\bigl(\tau,z(\tau)\bigr)\,d\tau\right|\\ &\le \frac{2d+1}{\Gamma(q-p+1)}\bigl(\|z\|_{\mathbf{Z}}^{q_2}+\mathcal{M}_\theta\bigr).\end{aligned}$$

Applying assumption (A_3), we obtain the condition (9). □

Lemma 3.5 *The operator $\mathcal{G}:\mathbf{Z}\to\mathbf{Z}$ is compact. Consequently, $\mathcal{G}$ is β-Lipschitz with zero constant.*

Proof To prove the required result, we take a bounded set $\mathbf{D}\subset\bar{B}\subseteq\mathbf{Z}$. Let $\{z_m\}$ be a sequence on $\mathbf{D}\subset\bar{\mathbf{B}}$, then from (9) for every $z_m\in\mathbf{D}$, we have

$$\|\mathcal{G}z_m\|_{\mathrm{Z}}\le\frac{2d+1}{\Gamma(q-p+1)}\bigl(\mathcal{C}_\theta\|z_m\|_{\mathbf{Z}}^{q_2}+\mathcal{M}_\theta\bigr),$$

which implies that $\mathcal{G}(\mathbf{D})$ is bounded in $\mathbf{Z}$. Next, we will show that $\{\mathcal{G}z_m\}$ is equi-continuous. For this purpose, let $t_1<t_2\in(0,1)$, and using these relations $\delta\eta^{q-p}<1$, $\frac{1}{\Gamma(q+1)}<\frac{1}{\Gamma(q-p+1)}$, we have

$$\begin{aligned}\left|(\mathcal{G}z_m)(t_1)-(\mathcal{G}z)(t_2)\right| &\le \frac{(t_2-t_1)d}{\Gamma(q)}\int_0^1(1-\tau)^{q-1}\left|\theta\bigl(\tau,z(\tau)\bigr)\right|d\tau\\ &\quad+\frac{(t_2-t_1)\delta d}{\Gamma(q-p)}\int_0^{\eta}(\eta-\tau)^{q-p-1}\left|\theta\bigl(\tau,z(\tau)\bigr)\right|d\tau\\ &\quad+\frac{1}{\Gamma(q)}\int_0^{t_1}\bigl((t_1-\tau)^{q-1}-(t_2-\tau)^{q-1}\bigr)\left|\theta\bigl(\tau,z(\tau)\bigr)\right|d\tau\\ &\quad-\int_{t_1}^{t_2}(t_2-\tau)^{q-1}\left|\theta\bigl(\tau,z(\tau)\bigr)\right|d\tau,\end{aligned}$$

which on simplification takes the form

$$\left|(\mathcal{G}z_m)(t_1)-(\mathcal{G}z)(t_2)\right|\le\frac{(\mathcal{C}_\theta\kappa^{q_2}+\mathcal{M}_\theta)}{\Gamma(q-p+1)}\bigl[(t_2-t_1)2d+\bigl(t_2^q-t_1^q\bigr)-2(t_2-t_1)^q\bigr]. \tag{10}$$

The right hand side of the inequality (10) goes to zero as $t_2\to t_1$. Thus, $\{\mathcal{G}z_m\}$ is equi-continuous and also $\mathcal{G}(\mathbf{D})$ is relatively compact in $\mathbf{Z}$ by using the Arzelá-Ascoli theorem.

Furthermore, in view of Proposition 2.3, the nonlinear operator $\mathcal{G}$ is β-Lipschitz with constant zero. □

From now on, we will prove our main results.

Theorem 3.1 *Under the hypotheses* (A_1)-(A_3) *equation* (1) *has at least one solution* $u \in \mathbf{Z}$. *Also, the set of solutions for* (1) *is bounded in* $\mathbf{Z}$.

Proof Thank to Proposition 2.2, the operator $\mathbf{T}$ is a strict β-contraction with constant $\mathcal{K}_\phi$. Consider the set

$$\mathbf{S}_0 = \{z \in \mathbf{Z} : \text{there exists } \lambda \in [0,1] \text{ such that } z = \lambda \mathbf{T} z\}.$$

We need to show that $\mathbf{S}_0 \subset \mathbf{Z}$ is bounded. For this purpose, consider

$$|z| = |\lambda \mathbf{T} z| = \lambda |\mathbf{T} z| \leq \lambda \left(\|Fz\|_{\mathbf{Z}} + \|\mathcal{G}z\|_{\mathbf{Z}}\right),$$

using (8) and (9), we have

$$|z| \leq \lambda \left[\mathcal{Q}\|z\|_{\mathbf{Z}}^{q_1} + \mathcal{M}_\phi + \frac{2d+1}{\Gamma(q-p+1)}\left(\|z\|_{\mathbf{Z}}^{q_2} + \mathcal{M}_\theta\right)\right],$$

which implies using $\lambda < 1$ that

$$\|z\|_{\mathbf{Z}} \leq \left[\mathcal{Q}\|z\|_{\mathbf{Z}}^{q_1} + \mathcal{M}_\phi + \frac{2d+1}{\Gamma(q-p+1)}\left(\|z\|_{\mathbf{Z}}^{q_2} + \mathcal{M}_\theta\right)\right]. \tag{11}$$

Hence, (11) and $q_1, q_2 \in (0,1)$ shows that $\mathbf{S}_0$ is bounded in $\mathbf{Z}$. If it is not bounded then assume that $\|z\|_{\mathbf{Z}} = \rho$ and consider that $\rho \to \infty$. Then from (11), we have

$$1 \leq \frac{[\mathcal{Q}\|z\|_{\mathbf{Z}}^{q_1} + \mathcal{M}_\phi + \frac{2d+1}{\Gamma(q-p+1)}(\|z\|_{\mathbf{Z}}^{q_2} + \mathcal{M}_\theta)]}{\rho}, \tag{12}$$

where due to assumption

$$\rho \to \infty \quad \text{yields } 1 \leq 0.$$

This is impossible, so we take $\mathbf{S}_0$ to be bounded.

Therefore, we conclude that the operator $\mathbf{T}$ has at least one fixed point and the set of fixed points is bounded in $\mathbf{Z}$. □

We make the following assumption for discussion of data dependence of solutions:

(A_4) There exist constants $\mathcal{L}_\theta > 0$, $\lambda \in [0, 1 - \frac{1}{r}]$ for some $r \in (0, 1 - \frac{1}{1-q})$ such that

$$\left|\theta(t,z) - \theta(t,\bar{z})\right| \leq \mathcal{L}_\theta |z - \bar{z}|^\lambda, \quad \text{for each } t \in \mathfrak{J}, \text{and for all } z, \bar{z} \in \mathbf{R}.$$

Theorem 3.2 *Assuming that* (A_1)-(A_6) *hold, let* $z(t)\bar{z}(t)$ *be the solutions of* (*FDE*) (1) *with associated boundary conditions. Then there exists a constant* $\mathcal{M}^* > 0$ *such that*

$$\left|z(t) - \bar{z}(t)\right| \leq \mathcal{M}^* \left(\frac{1}{(p(1-q)+1)}\right)^{\frac{1}{p}}.$$

Proof Consider $|z(t) - \bar{z}(t)|$, and thanks to (A_1), (A_3), and (A_4), we get

$$\begin{aligned}\left|z(t) - \bar{z}(t)\right| \leq{}& \mathcal{K}_\phi \left|z(t) - \bar{z}(t)\right| + \frac{\mathcal{L}_\theta}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau \\ &+ \frac{d\mathcal{L}_\theta}{\Gamma(q)} \int_0^1 (1-\tau)^{q-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau \\ &+ \frac{\delta \mathcal{L}_\theta d}{\Gamma(q-p)} \int_0^\eta (\eta-\tau)^{q-p-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau.\end{aligned}$$

Upon further simplification and using (2), we get

$$\begin{aligned}\left|z(t) - \bar{z}(t)\right| \leq{}& \frac{1}{1-\mathcal{K}_\phi} \left[\frac{\mathcal{L}_\theta}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau\right] \\ &+ \frac{1}{1-\mathcal{K}_\phi} \left[\frac{d\mathcal{L}_\theta}{\Gamma(q)} \int_0^1 (1-\tau)^{q-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau \right.\\ &\left.+ \frac{\delta \mathcal{L}_\theta d}{\Gamma(q-p)} \int_0^\eta (\eta-\tau)^{q-p-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau\right].\end{aligned}$$

Hence using (2) and Theorem 2.9, we obtain

$$|z - \bar{z}| \leq \mathcal{M}^*,$$

where $\mathcal{M}^* = e^{\mathcal{M}}$ and

$$\begin{aligned}M ={}& \frac{1}{1-\mathcal{K}_\phi} \left[\frac{\mathcal{L}_\theta}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau\right] \\ &+ \frac{1}{1-\mathcal{K}_\phi} \left[\frac{d\mathcal{L}_\theta}{\Gamma(q)} \int_0^1 (1-\tau)^{q-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau \right.\\ &\left.+ \frac{\delta \mathcal{L}_\theta d}{\Gamma(q-p)} \int_0^\eta (\eta-\tau)^{q-p-1} \left|z(\tau) - \bar{z}(\tau)\right|^\lambda \, d\tau\right].\end{aligned}$$

□

Now, re-write assumption (A_4) as follows.

(A_5) For a $\mathcal{L}_\theta > 0$, the following relation holds:

$$\left|\theta(t,z) - \theta(t,\bar{z})\right| \leq \mathcal{L}_\theta |z - \bar{z}|, \quad \text{for each } t \in \mathfrak{J}, \text{and for each } z, \bar{z} \in \mathbf{R}.$$

Theorem 3.3 *Assume that the hypotheses* (A_1)-(A_5) *hold, then FDE* (1) *has a unique solution* $z \in \mathbf{Z}$ *if* $\frac{\mathcal{M}^*}{1-\mathcal{K}_\phi} < 1$.

Proof As we investigated in Theorem 3.1 $z(t) \in \mathbf{Z}$ is a solution of (1). Let $\bar{z}(t)$ be another solution of (1). Then, repeating the same procedure as in Theorem 3.2 and using assumptions (A_1), (A_3) and (A_5), we obtain

$$\begin{aligned}\big|\mathbf{T}z(t)-\mathbf{T}\bar{z}(t)\big| &\le \frac{1}{1-\mathcal{K}_\phi}\bigg[\mathcal{K}_\phi\|z-\bar{z}\|_{\mathbf{Z}}+\frac{\mathcal{L}_\theta}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}\big|z(\tau)-\bar{z}(\tau)\big|^\lambda\,d\tau\bigg]\\ &\quad+\frac{1}{1-\mathcal{K}_\phi}\bigg[\frac{\mathcal{L}_\theta td}{\Gamma q}\int_0^1(1-\tau)^{q-1}\big|z_1(\tau)-z_2(\tau)\big|^\lambda\,d\tau\\ &\quad+\frac{\delta d\mathcal{L}_\theta}{\Gamma(q-p)}\int_0^\eta(\eta-s)^{q-p-1}\big|z(\tau)-\bar{z}(\tau)\big|^\lambda\,d\tau\bigg],\end{aligned}$$

using the inequality in Theorem 2.9, we get

$$\|\mathbf{T}z-\mathbf{T}\bar{z}\|_{\mathbf{Z}}\le\frac{\mathcal{M}^*}{1-\mathcal{K}_\phi}\|z-\bar{z}\|_{\mathbf{Z}},\quad t\in\mathfrak{J},$$

which produces the uniqueness of z. □

4 Illustrative example

Example 1 Take the following FDE subject to the multi-points boundary conditions:

$$\begin{aligned}&{}^C\mathbf{D}^q z(t)=-\frac{\cos(t)}{10+t^2|z(t)|},\quad t\in\mathfrak{J},\\ &z(t)|_{t=0}=\phi(z)=\sum_{j=1}^4\frac{1}{20}\big|z(\eta_j)\big|,\qquad z(1)=\frac{1}{2}{}^C\mathbf{D}^{\frac{1}{2}}z\left(\frac{1}{2}\right).\end{aligned}\tag{13}$$

Here, we take $q=\frac{2}{3}$ and $\delta=\eta=\frac{1}{2}$, $\mathcal{K}_\phi=\frac{1}{20}$, $\eta_j=\frac{1}{2^j}$, $j=1,2,3,4$. $r=2\in(1,3)$, $\lambda=\frac{1}{2}\in[0,1]$, $\mathcal{L}_\theta=\mathcal{C}_\theta=\frac{1}{10}$, $\mathcal{C}_\phi=\frac{1}{20}$, $p=\frac{1}{2}$, $\mathcal{M}_\phi=\mathcal{M}_\theta=0$.

By simple computation, $d=1.5469$, $\mathcal{Q}=d\mathcal{C}_\phi=1.5469\times\frac{1}{20}=0.07734$. For the considered problem (13) all the data dependence results $(A1)$-$(A5)$ are satisfied. It is also obvious that the solution z

$$\|z\|_{\mathbf{Z}}\le 17.3984,$$

is bounded. Thus due to Theorem 3.1 there exists at least one solution for (13) which is bounded. Along the same line, one can derive the assumptions of Theorem 3.2 and 3.3.

Example 2 Consider the following boundary value problem of FDEs:

$$\begin{aligned}&{}^C\mathbf{D}^{\frac{5}{3}}z(t)=-\frac{\sin|z(t)|}{40+\exp(t^2)},\quad t\in\mathfrak{J},\\ &z(t)|_{t=0}=\phi(z)=\frac{1}{10}\cos|z|,\qquad z(1)=\frac{1}{3}{}^C\mathbf{D}^{\frac{1}{2}}z\left(\frac{1}{3}\right).\end{aligned}\tag{14}$$

From the given problem (14), we see that $q=\frac{5}{3}$ and we take $\delta=\eta=\frac{1}{3}$, $\mathcal{K}_\phi=\frac{1}{10}$. $r=2\in(1,3)$, $\lambda=\frac{1}{2}\in[0,1]$, $\mathcal{L}_\theta=\mathcal{C}_\theta=\frac{1}{41}$, $\mathcal{C}_\phi=\frac{1}{10}$, $p=\frac{1}{2}$, $\mathcal{M}_\phi=\mathcal{M}_\theta=0$.

Upon computation, we get $d=1.27739$, $\mathcal{Q}=d\mathcal{C}_\phi=0.127739$. Thus for the given boundary value problem (14) of FDEs, all the data dependence results (A_1)-(A_5) hold.

Further, it is easy to show by using Theorem 3.1 that there exists at least one solution for (14) which is bounded. Also, one can easily derive the assumptions of Theorems 3.2 and 3.3.

5 Concluding remarks

In this paper, we have successfully applied an *a priori* estimate method known as topological degree method rather than Schauder and Brouwer degree theory. Highly interesting results for the existence of at least one solution have been derived. In the future, we can extend the concerned theory to highly applicable nonlinear problems of applied analysis to investigate them for solutions.

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