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Dynamic bifurcation for the granulation convection in the solar photosphere

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Abstract

The main objective of this paper is to investigate the dynamic bifurcation for the granulation convection in the solar photosphere. Based on the dynamic bifurcation theory, which is established by Ma and Wang (Phase transition dynamics, pp. 380-459, 2014), the critical parameters (R, F) condition for the granulation convection is derived. Furthermore, the corresponding R-F-phase diagram is generated. In addition, the bifurcation solution is also obtained with certain assumptions.

Keywords: granulation; eigenvalue; solar photosphere; dynamic bifurcation

1 Introduction

The atmosphere of the sun is divided into three parts: the bottom layer of the atmosphere is photosphere, the middle layer is chromosphere and the outermost layer is the corona. The grainy appearance of solar photosphere is produced by the tops of the convective cells and is called granulation. It worth noting that Kutner analyzed the structure of granulation in his book [2]. He pointed out that the temperature decreases with the increasing height in the photosphere. And he concluded that the temperature difference between the top and bottom of photosphere causes the granulation convection in photosphere which can be explained by circulation cells of material. In circulation cells, the hotter gas is brought up from below, producing the bright regions. The cooler gas, which produces the darker regions, is carried down to replace the gas that was brought up (see Figure 1).

In addition, Ustyugov introduced the solar activity and gave the magnetohydrodynamics (MHD) equations in [3, 4], which can explain the convection structure in the sun. There have been many studies on this kind of problem. For instance, Chae [5], Agélas [6], and Wu [7] considered the regularity of MHD equations, and Capone [8] and Fuchs [9] studied the stability bifurcation of MHD equations. For more research about the MHD equations refer to [10–17]. It is noticed that the granulation convection structure (see Figure 1) is similar to the structure of Rayleigh-Bénard convection and the Taylor problem. Inspired by the dynamic theory, which is established by Ma and Wang to study the Rayleigh-Bénard convection and the Taylor problem (see [1, 18–20]), we will investigate the dynamic bifurcation for the granulation convection. This paper is organized as follows.

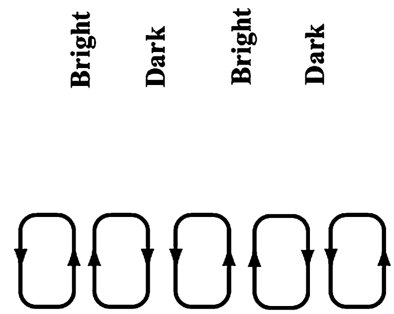

Figure 1 The structure of granulation. This is a lateral view showing the situation under the surface. The hotter gas is brought up from below, producing the bright regions. The cooler gas, which produces the darker regions, is carried down to replace the gas that was brought up.

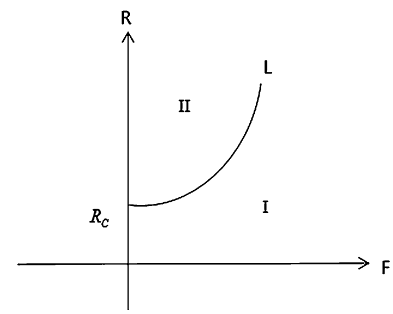

Figure 2 The RF-phase diagram. In region I, there is no granulation convection structure; and in region II, the granulation appears. So the transition occurs as the parameter (R,F) crosses the critical curve *L* from region I to region II.

1. In Section 2, we introduce the equations governing the atmospheric circulation with magnetic field and also give the bifurcation theory.
2. Section 3 is devoted to getting the RF-phase diagram of granulation under the parametric condition $K_1 = 0$ and $F \neq 0$, where R is the Rayleigh number, which is related to the difference of temperature $T_1 - T_0$, F is a dimensionless parameter, which is related to the magnetic field H, and K_1 is as in (2.5), which is related to the boundary condition H_0 and H_1. The transition occurs as the parameter (R, F) crosses the critical curve L from region I to region II; see Figure 2.
3. In Section 4, the bifurcation solution and the critical Rayleigh number R_C are obtained under the condition $F = 0$.

2 Governing equations

2.1 The model in spherical coordinates

Let (φ, θ, r) be the spherical coordinates, where φ, θ, r represent the longitude, the latitude, and the radial coordinate, respectively. The unknown functions include the velocity field $u = (u_\varphi, u_\theta, u_r)$, the temperature function T, the pressure function p, the electromagnetic press function Φ and the magnetic field $H = (H_\varphi, H_\theta, H_r)$. Then the equations governing the atmospheric circulation with magnetic field [9, 21–23] in the spherical coordinates are given by

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \bar{\nabla})u = \nu \bar{\Delta} u - \frac{1}{\rho_0} \nabla p - g\vec{k}[1 - \alpha(T - T_0)] + \frac{1}{\rho_0}(\nabla \times H) \times H, \\ \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \Delta T, \\ \frac{\partial H}{\partial t} = \nabla \times (u \times H) + \eta \bar{\Delta} H + \nabla \Phi, \\ \operatorname{div} u = 0, \\ \operatorname{div} H = 0, \end{cases} \tag{2.1}$$

where $0 \le \varphi \le 2\pi$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $r_0 < r < r_1$, $r_1 = r_0 + d$, r_0 is the radius of the sun, and d is the height of the troposphere, $\vec{k} = (0,0,1)$, g is the gravitative constant, ν is the kinetic viscosity, κ is the thermal diffusivity, and η is the resistivity. The differential operators in the spherical coordinates are given as follows:

$$
\begin{aligned}
(u\cdot\bar{\nabla})u = \Bigg(& (u\cdot\nabla)u_\varphi + \frac{u_r u_\varphi}{r} + \frac{u_\varphi u_\theta}{r}\cot\theta, \\
& (u\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2}{r}\cot\theta, \\
& (u\cdot\nabla)u_r - \frac{u_\theta^2}{r} - \frac{u_\varphi^2}{r}\Bigg)^T, \\
\nabla p = {} & \left(\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi}, \frac{1}{r}\frac{\partial P}{\partial\theta}, \frac{\partial p}{\partial r}\right)^T, \\
\operatorname{div} u = {} & \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta u_\theta)}{\partial\theta} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r), \\
\bar{\triangle} u = \Bigg(& \triangle u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}, \\
& \triangle u_\theta - \frac{2\cos\theta}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}, \\
& \triangle u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(\sin\theta u_\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\Bigg)^T, \\
(\nabla\times H)\times H = \Bigg(& \frac{H_r}{r}\left[\frac{\partial(rH_\varphi)}{\partial r} - \frac{1}{\sin\theta}\frac{\partial H_r}{\partial\theta}\right] - \frac{H_\theta}{r\sin\theta}\left[\frac{\partial H_\theta}{\partial\varphi} - \frac{\partial(H_\varphi\sin\theta)}{\partial\theta}\right], \\
& \frac{H_\varphi}{r\sin\theta}\left[\frac{\partial H_\theta}{\partial\varphi} - \frac{\partial(H_\varphi\sin\theta)}{\partial\theta}\right] - \frac{H_r}{r}\left[\frac{\partial H_r}{\partial\theta} - \frac{\partial(rH_\theta)}{\partial r}\right], \\
& \frac{H_\theta}{r}\left[\frac{\partial H_r}{\partial\theta} - \frac{\partial(rH_\theta)}{\partial r}\right] - \frac{H_\varphi}{r}\left[\frac{\partial(rH_\varphi)}{\partial r} - \frac{1}{\sin\theta}\frac{\partial H_r}{\partial\theta}\right]\Bigg)^T, \\
\nabla\times(u\times H) = \Bigg(& \frac{1}{r}\frac{\partial}{\partial r}\big[r(u_\varphi H_r - u_r H_\varphi)\big] - \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta H_\varphi - u_\varphi H_\theta), \\
& \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(u_\theta H_\varphi - u_\varphi H_\theta) - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\big[r\sin\theta(u_r H_\theta - u_\theta H_r)\big], \\
& \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\big[r\sin\theta(u_r H_\theta - u_\theta H_r)\big] - \frac{1}{r^2\sin\theta}\big[r(u_\varphi H_r - u_r H_\varphi)\big]\Bigg)^T,
\end{aligned}
$$

$\bar{\triangle} H$, $\nabla\Phi$, $\operatorname{div} H$ are similar to $\bar{\triangle} u$, ∇p, $\operatorname{div} u$, and the operators Δ, $(u\cdot\nabla)$ are given by

$$
\begin{aligned}
(u\cdot\nabla) &= \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} + u_r\frac{\partial}{\partial r}, \\
\triangle &= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).
\end{aligned}
$$

In this paper, we mainly focus on the dynamic bifurcation for the granulation. For simplicity, we assume $\theta = \frac{\pi}{2}$. Then the velocity component u_θ and the magnetic field H_θ are zero, and equations (2.1) become

$$\begin{cases} \frac{\partial u_\varphi}{\partial t} + (u\cdot\nabla)u_\varphi + \frac{u_r u_\varphi}{r} = \nu(\triangle u_\varphi + \frac{2}{r^2}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r^2}) - \frac{1}{\rho_0 r}\frac{\partial p}{\partial\varphi} + \frac{H_r}{\rho_0 r}[\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial\varphi}], \\ \frac{\partial u_r}{\partial t} + (u\cdot\nabla)u_r - \frac{u_\varphi^2}{r} = \nu(\triangle u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}) - \frac{1}{\rho_0}\frac{\partial p}{\partial r} - g[1-\alpha(T-T_0)] \\ \qquad + \frac{H_r}{\rho_0 r}[\frac{\partial H_r}{\partial\varphi} - \frac{\partial(rH_\varphi)}{\partial r}], \\ \frac{\partial T}{\partial t} + (u\cdot\nabla)T = \kappa\triangle T, \\ \frac{\partial H_\varphi}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}[r(u_\varphi H_r - u_r H_\varphi)] + \eta(\triangle H_\varphi + \frac{2}{r^2}\frac{\partial H_r}{\partial\varphi} - \frac{H_\varphi}{r^2}) + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi}, \\ \frac{\partial H_r}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial\varphi}[r(u_r H_\varphi - u_\varphi H_r)] + \eta(\triangle H_r - \frac{2H_r}{r^2} - \frac{2}{r^2}\frac{\partial H_\varphi}{\partial\varphi}) + \frac{\partial\Phi}{\partial r}, \\ \operatorname{div} u = 0, \\ \operatorname{div} H = 0, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} (u\cdot\nabla) &= \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi} + u_r\frac{\partial}{\partial r}, \\ \triangle &= \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right), \\ \operatorname{div} u &= \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r). \end{aligned}$$

Furthermore, equations (2.2) are supplemented with the following boundary condition:

$$\begin{cases} (u,T,H)(\varphi,r) = (u,T,H)(\varphi+2k\pi,r), \\ u_r = 0, \quad H_r = H_0, \quad T = T_0, \quad \frac{\partial u_\varphi}{\partial r} = \frac{\partial H_\varphi}{\partial r} = 0, \quad r = r_0, \\ u_r = 0, \quad H_r = H_1, \quad T = T_1, \quad \frac{\partial u_\varphi}{\partial r} = \frac{\partial H_\varphi}{\partial r} = 0, \quad r = r_0 + d. \end{cases} \tag{2.3}$$

2.2 Perturbed dimensionless equations

We determine the basic flow by following assumptions.

1. $U = (u_\varphi, u_r, T, H_\varphi, H_r) = (0, 0, \tilde{T}(r), 0, \tilde{H}_r)$, $p = \tilde{p}(r)$, $\Phi = 0$; that is, the pressure function, the temperature function and the magnetic field function in r-direction are not zero, and which are only depending on r.
2. The functions $\tilde{T}(r)$, $\tilde{H}(r)$ and $\tilde{p}(r)$ satisfy

$$\begin{cases} -\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial r} - g[1-\alpha(\tilde{T}-T_0)] = 0, \\ \triangle\tilde{T} = 0, \\ \triangle\tilde{H}_r - \frac{2}{r^2}\tilde{H}_r = 0. \end{cases}$$

3. Based on the boundary condition (2.3), the value of basic flow on the boundary is given by

$$\begin{aligned} &\tilde{H}_r = H_0, \qquad \tilde{T} = T_0, \qquad r = r_0, \\ &\tilde{H}_r = H_1, \qquad \tilde{T} = T_1, \qquad r = r_0 + d. \end{aligned}$$

From the above assumptions, we derive the basic flow as follows:

$$\begin{cases} \tilde{T} = \frac{C_1}{r} + C_0, \\ \tilde{H_r} = K_1 r + \frac{K_0}{r^2}, \\ \tilde{p} = \int_{r_0}^{r_0+d} -\rho_0 g[1-\alpha(\tilde{T}-T_0)]\,dr, \end{cases} \tag{2.4}$$

where

$$\begin{aligned} &C_0 = \frac{T_1 r_1 - T_0 r_0}{r_1 - r_0}, \qquad C_1 = \frac{(T_0 - T_1) r_0 r_1}{r_1 - r_0}, \\ &H_0 = K_1 r_0 + \frac{K_0}{r_0^2}, \qquad H_1 = K_1 r_1 + \frac{K_0}{r_1^2}. \end{aligned} \tag{2.5}$$

It is noticed that K_0 and K_1 are related to the boundary value H_0 and H_1. Furthermore, in order to get the perturbation equations related to the variables r and φ, we make the following translations:

$$p = p' + \tilde{p}, \qquad T = T' + \tilde{T}, \qquad H_r = H_r' + \tilde{H}_r.$$

Omitting the primes, equations (2.2) can be rewritten as

$$\begin{cases} \frac{\partial u_\varphi}{\partial t} + (u\cdot\nabla)u_\varphi + \frac{u_r u_\varphi}{r} = \nu(\triangle u_\varphi + \frac{2}{r^2}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r^2}) - \frac{1}{\rho_0 r}\frac{\partial p}{\partial\varphi} + \frac{H_r}{\rho_0 r}[\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial\varphi}] \\ \qquad + \frac{\tilde{H}_r}{\rho_0 r}[\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial\varphi}], \\ \frac{\partial u_r}{\partial t} + (u\cdot\nabla)u_r - \frac{u_\varphi^2}{r} = \nu(\triangle u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}) - \frac{1}{\rho_0}\frac{\partial p}{\partial r} - g\alpha T, \\ \frac{\partial T}{\partial t} + (u\cdot\nabla)T = \kappa\triangle T + \frac{C_1}{r^2}u_r, \\ \frac{\partial H_\varphi}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}[r(u_\varphi H_r - u_r H_\varphi)] + \eta(\triangle H_\varphi + \frac{2}{r^2}\frac{\partial H_r}{\partial\varphi} - \frac{H_\varphi}{r^2}) + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial r}(ru_\varphi\tilde{H}_r), \\ \frac{\partial H_r}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial\varphi}[r(u_r H_\varphi - u_\varphi H_r)] + \eta(\triangle H_r - \frac{2H_r}{r^2} - \frac{2}{r^2}\frac{\partial H_\varphi}{\partial\varphi}) + \frac{\partial\Phi}{\partial r} - \frac{1}{r}\tilde{H}_r\frac{\partial u_\varphi}{\partial\varphi}, \\ \operatorname{div} u = 0, \\ \operatorname{div} H = 0. \end{cases} \tag{2.6}$$

As we know, the radius of the sun is about 7×10^5 km and the thickness of the photosphere is about 500 km. Then the ratio of the thickness of the photosphere to the radius of the sun is small. Hence, we adopt the approximations that $1/r \simeq 1/r_0$, $(r_0+d)/r_0 \simeq 1$. For simplicity, we assume $\nu/\kappa = \eta/\kappa = 1$. Also, we introduce the following dimensionless variables:

$$\begin{aligned} &u = \frac{\kappa}{d}u', \qquad r = dr', \qquad T = \frac{T_0 - T_1}{\sqrt{R}}T', \qquad p = \frac{\rho_0\kappa^2}{d^2}p', \\ &H = H_0 H', \qquad \Phi = H_0\frac{\kappa}{d}\Phi', \qquad t = \frac{d^2}{\kappa}t', \end{aligned}$$

where R, called the Rayleigh number, is a dimensionless parameter and

$$R = \frac{g\alpha(T_0 - T_1)}{\kappa\nu}d^3.$$

Let $(\varphi'', r'') = (r_0\varphi', r')$. With the above assumptions, we omit the primes and get the approximate equations, which are given as follows:

$$\begin{cases} \frac{\partial u_\varphi}{\partial t} + (u\cdot\nabla)u_\varphi + \frac{u_r u_\varphi}{r_0} = \triangle u_\varphi + \frac{2}{r_0}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r_0^2} - \frac{\partial p}{\partial\varphi} + \frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0^2 H_r(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial\varphi}) \\ \qquad + \frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0(K_1 d r_0 + \frac{K_0}{d^2 r_0^2})(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial\varphi}), \\ \frac{\partial u_r}{\partial t} + (u\cdot\nabla)u_r - \frac{u_\varphi^2}{r} = \triangle u_r - \frac{2u_r}{r_0^2} - \frac{2}{r_0}\frac{\partial u_\varphi}{\partial\varphi} - \frac{\partial p}{\partial r} + \sqrt{R}T, \\ \frac{\partial T}{\partial t} + (u\cdot\nabla)T = \triangle T + \sqrt{R}u_r, \\ \frac{\partial H_\varphi}{\partial t} = \frac{\partial}{\partial r}(u_\varphi H_r - u_r H_\varphi) + \triangle H_\varphi + \frac{2}{r_0}\frac{\partial H_r}{\partial\varphi} - \frac{H_\varphi}{r_0^2} + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} + \frac{K_1 d}{H_0}u_\varphi \\ \qquad + \frac{1}{H_0}(K_1 d r_0 + \frac{K_0}{d^2 r_0^2})\frac{\partial u_\varphi}{\partial r}, \\ \frac{\partial H_r}{\partial t} = \frac{\partial}{\partial\varphi}(u_r H_\varphi - u_\varphi H_r) + \triangle H_r - \frac{2H_r}{r_0^2} - \frac{2}{r_0}\frac{\partial H_\varphi}{\partial\varphi} + \frac{\partial\Phi}{\partial r} - \frac{1}{H_0}(K_1 d r_0 + \frac{K_0}{d^2 r_0^2})\frac{\partial u_\varphi}{\partial\varphi}, \\ \operatorname{div} u = 0, \\ \operatorname{div} H = 0, \end{cases} \tag{2.7}$$

where $(\varphi, r) \in M = (0, L) \times (r_0, r_0 + 1)$, r_0 is the radius of the sun with the unit of d, $L = 2\pi r_0$, and K_0, K_1 are given as (2.5). $(u\cdot\nabla)$, div and $\triangle$ are general differential operators.

$$(u\cdot\nabla) = u_\varphi\frac{\partial}{\partial\varphi} + u_r\frac{\partial}{\partial r}, \qquad \triangle = \frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial r^2}, \qquad \operatorname{div} u = \frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial u_r}{\partial r}.$$

The boundary conditions (2.3) are rewritten as

$$\begin{cases} (u, T, H)(\varphi, r) = (u, T, H)(\varphi + L, r), \\ u_r = 0, \quad H_r = 0, \quad T = 0, \quad \frac{\partial u_\varphi}{\partial r} = \frac{\partial H_\varphi}{\partial r} = 0, \quad r = r_0, \\ u_r = 0, \quad H_r = 0, \quad T = 0, \quad \frac{\partial u_\varphi}{\partial r} = \frac{\partial H_\varphi}{\partial r} = 0, \quad r = r_0 + 1. \end{cases} \tag{2.8}$$

2.3 Abstract operator equation

Now we will show that equations (2.7) can be written in the operator form. Since the velocity field u and the magnetic field H on M are divergence-free, there exist the following stream functions f_1 and f_2 satisfying the given boundary condition:

$$u_\varphi = \frac{\partial f_1}{\partial r}, \qquad u_r = -\frac{\partial f_1}{\partial\varphi},$$
$$H_\varphi = \frac{\partial f_2}{\partial r}, \qquad H_r = -\frac{\partial f_2}{\partial\varphi}.$$

Moreover, the following two vector fields:

$$\left(\frac{2}{r_0}\frac{\partial u_r}{\partial\varphi}, -\frac{2}{r_0}\frac{\partial u_\varphi}{\partial\varphi}\right) = -\frac{2}{r_0}\nabla\frac{\partial f_1}{\partial\varphi},$$
$$\left(\frac{2}{r_0}\frac{\partial H_r}{\partial\varphi}, -\frac{2}{r_0}\frac{\partial H_\varphi}{\partial\varphi}\right) = -\frac{2}{r_0}\nabla\frac{\partial f_2}{\partial\varphi},$$

are gradient fields, which can be balanced by ∇p and $\nabla\Phi$ in (2.7). Hence, (2.7) are equivalent to the following equations:

$$\begin{cases} \frac{\partial u_\varphi}{\partial t} + (u\cdot\nabla)u_\varphi + \frac{u_r u_\varphi}{r_0} = \triangle u_\varphi - \frac{u_\varphi}{r_0^2} - \frac{\partial p}{\partial\varphi} + \frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0^2 H_r(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial\varphi}) \\ \qquad + \frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0(K_1 d r_0 + \frac{K_0}{d^2 r_0^2})(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial\varphi}), \\ \frac{\partial u_r}{\partial t} + (u\cdot\nabla)u_r - \frac{u_\varphi^2}{r} = \triangle u_r - \frac{2u_r}{r_0^2} - \frac{\partial p}{\partial r} + \sqrt{R}T, \\ \frac{\partial T}{\partial t} + (u\cdot\nabla)T = \triangle T + \sqrt{R}u_r, \\ \frac{\partial H_\varphi}{\partial t} = \frac{\partial}{\partial r}(u_\varphi H_r - u_r H_\varphi) + \triangle H_\varphi - \frac{H_\varphi}{r_0^2} + \frac{\partial\Phi}{\partial\varphi} + \frac{K_1 d}{H_0}u_\varphi \\ \qquad + \frac{1}{H_0}(K_1 d r_0 + \frac{K_0}{d^2 r_0^2})\frac{\partial u_\varphi}{\partial r}, \\ \frac{\partial H_r}{\partial t} = \frac{\partial}{\partial\varphi}(u_r H_\varphi - u_\varphi H_r) + \triangle H_r - \frac{2H_r}{r_0^2} + \frac{\partial\Phi}{\partial r} - \frac{1}{H_0}(K_1 d r_0 + \frac{K_0}{d^2 r_0^2})\frac{\partial u_\varphi}{\partial\varphi}, \\ \operatorname{div} u = 0, \\ \operatorname{div} H = 0. \end{cases} \tag{2.9}$$

To get the abstract form of (2.9), we define the following spaces:

$$H = \{(u,T,H)\in L^2(M,R^5) \mid \operatorname{div} u = \operatorname{div} H = 0, (u,T,H) \text{ are periodic in } \varphi\text{-direction}\},$$
$$H_1 = \{(u,T,H)\in H^2(M,R^5)\cap H \mid (u,T,H) \text{ satisfy (2.8)}\}.$$

Now, we define the operators $L = -A + B : H_1 \to H$ and $G : H_1 \to H$ by

$$\begin{aligned} AU = -P\bigg(&\triangle u_\varphi - \frac{u_\varphi}{r_0^2} + \widetilde{A}F\bigg(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial\varphi}\bigg), \triangle u_r - \frac{u_r}{r_0^2}, \\ &\triangle T, \triangle H_\varphi - \frac{1}{r_0^2}H_\varphi + \frac{1}{H_0}F\frac{\partial u_\varphi}{\partial r}, \triangle H_r - \frac{1}{r_0^2}H_r - \frac{1}{H_0}F\frac{\partial u_\varphi}{\partial\varphi}\bigg)^T, \\ BU = P\bigg(&0, \sqrt{R}T, \sqrt{R}u_r, \frac{K_1 d}{H_0}u_\varphi, 0\bigg)^T, \\ GU = P\bigg(&-(u\cdot\nabla)u_\varphi - \frac{u_r u_\varphi}{r_0} + \frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0^2 H_r\bigg(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial\varphi}\bigg), \\ &-(u\cdot\nabla)u_r - \frac{u_\varphi^2}{r_0}, -(u\cdot\nabla)T, \frac{\partial}{\partial r}(u_\varphi H_r - u_r H_\varphi), \frac{\partial}{\partial\varphi}(u_r H_\varphi - u_\varphi H_r)\bigg)^T, \end{aligned}$$

where $P : L^2(M,R^5) \to H$ is the Leray projection, $U = (u,T,H) \in H_1$ and

$$F = K_1 r_0 d + \frac{K_0}{r_0^2 d^2}, \qquad \widetilde{A} = \frac{H_0 d^2}{\rho_0\kappa^2}. \tag{2.10}$$

Therefore, the problem (2.9) with boundary conditions (2.8) are equivalent to the following abstract equation.

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = L_\lambda U + GU, \\ U(0) = U_0, \end{cases} \tag{2.11}$$

where $\lambda = (R,F) \in R^2$ is the parameter and U_0 is the initial value of (2.9).

3 Under the condition $K_1 = 0$

In this section, we study (2.9) the case that $K_1 = 0$. Hence, the boundary condition in (2.5) satisfies

$$H_1 = \frac{r_0^2}{r_1^2} H_0.$$

We choose the Rayleigh number and the F-number as the control parameters, which are given as follows:

$$R = \frac{g\alpha(T_0 - T_1)}{\kappa\nu} d^3, \qquad F = \frac{H_0}{d^2}. \tag{3.1}$$

Then we consider the following eigenvalue equations of (2.9):

$$\begin{cases} \triangle u_\varphi - \frac{u_\varphi}{r_0^2} - \frac{\partial p}{\partial \varphi} + \widetilde{A}F(\frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial \varphi}) = \beta u_\varphi, \\ \triangle u_r - \frac{2u_r}{r_0^2} - \frac{\partial p}{\partial r} + \sqrt{R}T = \beta u_r, \\ \triangle T + \sqrt{R} u_r = \beta T, \\ \triangle H_\varphi - \frac{H_\varphi}{r_0^2} + \frac{\partial \Phi}{\partial \varphi} + \frac{1}{H_0} F \frac{\partial u_\varphi}{\partial r} = \beta H_\varphi, \\ \triangle H_r - \frac{2H_r}{r_0^2} + \frac{\partial \Phi}{\partial r} - \frac{1}{H_0} F \frac{\partial u_\varphi}{\partial \varphi} = \beta H_r, \\ \operatorname{div} u = 0, \\ \operatorname{div} H = 0, \end{cases} \tag{3.2}$$

with the boundary conditions (2.8). We proceed with the separation of variables. Under the periodic boundary condition (2.8), the problem (3.2) possesses two eigenvectors: Ψ and $\tilde{\Psi}$ in the following forms:

$$\Psi: \quad \begin{cases} u_\varphi = \cos a_k\varphi \cdot h_k'(r), \\ u_r = a_k \sin a_k\varphi \cdot h_k(r), \\ T = \sin a_k\varphi \cdot T_k(r), \\ H_\varphi = \cos a_k\varphi \cdot g_k'(r), \\ H_r = a_k \sin a_k\varphi \cdot g_k(r), \\ p = p_k(r) \sin a_k\varphi, \\ \Phi = \Phi_k(r) \sin a_k\varphi, \end{cases} \tag{3.3}$$

$$\tilde{\Psi}: \quad \begin{cases} u_\varphi = \sin a_k\varphi \cdot h_k'(r), \\ u_r = -a_k \cos a_k\varphi \cdot h_k(r), \\ T = -\cos a_k\varphi \cdot T_k(r), \\ H_\varphi = \sin a_k\varphi \cdot g_k'(r), \\ H_r = -a_k \cos a_k\varphi \cdot g_k(r), \\ \tilde{p} = -p_k(r) \cos a_k\varphi, \\ \tilde{\Phi} = -\Phi_k(r) \cos a_k\varphi, \end{cases} \tag{3.4}$$

where $a_k = 2k\pi/L$. Putting (Ψ, p, Φ) and $(\tilde{\Psi}, \tilde{p}, \tilde{\Phi})$ into (3.2), respectively, we deduce from (3.2), (3.3) and (3.4) that $(h_k, g_k, T_k, p_k, \Phi_k)$ satisfy the eigenvalue problems,

$$\begin{cases} D^*Dh_k - \frac{1}{r_0^2}Dh_k - a_kp_k + \widetilde{A}FD^*g_k = \beta Dh_k, \\ a_kD^*h_k - \frac{2}{r_0^2}a_kh_k + \sqrt{R}T_k - Dp_k = \beta a_kh_k, \\ D^*T_k + \sqrt{R}a_kh_k = \beta T_k, \\ D^*Dg_k - \frac{1}{r_0^2}Dg_k + a_k\Phi_k + \frac{1}{H_0}K_1dDh_k + \frac{1}{H_0}FD^2h_k = \beta Dg_k, \\ a_kD^*g_k - \frac{2}{r_0^2}a_kg_k + D\Phi_k + \frac{1}{H_0}Fa_kDh_k = \beta a_kg_k, \end{cases} \tag{3.5}$$

for any $k \in Z$, with $k \neq 0$, where

$$D = \frac{d}{dr}, \qquad D^* = D^2 - a_k^2.$$

We infer from (3.5) and the boundary conditions (2.8) that the h_k satisfy the equations

$$\begin{aligned} &\Big[\Big(D^*\big(\beta - D^*\big)^2 - \frac{1}{r_0^2}\big(D^* - a^2\big)\big(D^* - \beta\big) + Ra^2\Big)\Big(D^*\big(\beta - D^*\big) + \frac{1}{r_0^2}\big(D^* - a^2\big)\Big) \\ &\quad + \widetilde{A}F\frac{1}{H_0}FD^*D\big(D^* - \beta\big)D^*D\Big]h_k = 0, \end{aligned} \tag{3.6}$$

$$h_k = D^2h_k = D^4h_k = 0 \quad \text{at } r = r_0, r_0 + 1. \tag{3.7}$$

By (3.7), the h_k are sine functions, *i.e.*,

$$h_k = \sin l\pi(r - r_0) \quad (l = 1, 2, 3, \ldots). \tag{3.8}$$

Putting (3.8) into (3.6), we see that

$$\begin{aligned} &\alpha_{kl}^4\big(\alpha_{kl}^2 + \beta\big)^3 + \frac{2}{r_0^2}\alpha_{kl}^2\big(\alpha_{kl}^2 + \beta\big)^2\big(\alpha_{kl}^2 + a_k^2\big) + \frac{1}{r_0^4}\big(\alpha_{kl}^2 + a_k^2\big)^2\big(\alpha_{kl}^2 + \beta\big) \\ &\quad - Ra^2\Big[\alpha_{kl}^2\big(\alpha_{kl}^2 + \beta\big) + \frac{1}{r_0^2}\big(\alpha_{kl}^2 + a_k^2\big)\Big] + \frac{d^2}{\rho_0\kappa^2}F^2\alpha_{kl}^4l^2\pi^2\big(\alpha_{kl}^2 + \beta\big) = 0, \end{aligned} \tag{3.9}$$

where

$$\alpha_{kl}^2 = a_k^2 + l^2\pi^2, \qquad a_k = 2k\pi/L. \tag{3.10}$$

It is well known that all solutions of the cubic equation (3.9) β_{kl} are eigenvalues of (3.2). Let β_{kl}^i $(1 \le i \le 3)$ be three zero points of (3.9). It is easy to see that $Re\beta_{kl}^1 \ge Re\beta_{kl}^2 \ge Re\beta_{kl}^3$. Furthermore, $\beta = 0$ is a zero point of (3.9) if and only if

$$b_{kl} - c_{kl}R + d_{kl}F^2 = 0,$$

where

$$\begin{cases} b_{kl} = \alpha_{kl}^{10} + \frac{2}{r_0^2}\alpha_{kl}^6(\alpha_{kl}^2 + a_k^2) + \frac{1}{r_0^4}(\alpha_{kl}^2 + a_k^2), \\ c_{kl} = a_k^2[\alpha_{kl}^4 + \frac{1}{r_0^2}(\alpha_{kl}^2 + a_k^2)], \\ d_{kl} = \frac{d^2}{\rho_0\kappa^2}\alpha_{kl}^6l^2\pi^2, \end{cases} \tag{3.11}$$

where α_{kl} and a_k are as defined in (3.10). When $F = 0$, we can get the critical Rayleigh number R_C

$$R_C = \min_{(k,l)} \frac{b_{kl}}{c_{kl}} = \min_{(k,l)} \frac{\alpha_{kl}^{10} + \frac{2}{r_0^2}\alpha_{kl}^6(\alpha_{kl}^2 + a_k^2) + \frac{1}{r_0^4}(\alpha_{kl}^2 + \alpha_{kl}^2)}{a_k^2[\alpha_{kl}^4 + \frac{1}{r_0^2}(\alpha_{kl}^2 + a_k^2)]} = \frac{b_{k_0l_0}}{c_{k_0l_0}}. \tag{3.12}$$

Then the critical parameter curve in Figure 2 is given by

$$L = \{(R,F) \in R_+^2 \mid b_{k_0l_0} - c_{k_0l_0}R + d_{k_0l_0}F^2 = 0\}, \tag{3.13}$$

where $b_{k_0l_0}$, $c_{k_0l_0}$ and $d_{k_0l_0}$ as in (3.11) and (3.12).

Lemma 3.1 *Let* $(k,l) = (k_0,l_0)$ *minimize the right hand side of* (3.12) *and the zero point* $\beta_{k_0l_0}^1$ *of* (3.9) *is a real single eigenvalue of* (3.2) *near curve* L. *Then* $\beta_{k_0l_0}^1$ *satisfies*

$$\beta_{k_0l_0}^1 \begin{cases} < 0, & (R,F) \in I, \\ = 0, & (R,F) \in L, \\ > 0, & (R,F) \in II, \end{cases}$$

$\beta_{kl}^i < 0, \forall(k,l,i) \neq (k_0,l_0,1)$, *and* (R,F) *near* L, *where the region I and II are as Figure* 2.

Proof We first prove that

$$\beta_{k_0l_0}^1 < 0, \quad (R,F) \in \mathrm{I}. \tag{3.14}$$

In fact, as $(R,F) = (0,0)$, the solutions of (3.9) are

$$\beta_{k_0l_0}^1 = -\alpha_{k_0l_0}^2, \qquad \beta_{k_0l_0}^2 = \beta_{k_0l_0}^3 = -\alpha_{k_0l_0}^2 - \frac{\alpha_{k_0l_0}^2 + a_k^2}{\alpha_{k_0l_0}^2 r_0^2}.$$

Namely, $\beta_{k_0l_0}^1(0,0) < 0$. Since $\beta_{k_0l_0}^1(R,F)$ are continuous on (R,F), then (3.14) holds true. Next, we are ready to prove that the first eigenvalue $\beta_{k_0l_0}^1(R,F) > 0$ in region II near the curve L. Let

$$\begin{aligned}
g_{k_0l_0} &= \alpha_{k_0l_0}^4(\alpha_{k_0l_0}^2 + \beta)^3 + \frac{2}{r_0^2}\alpha_{k_0l_0}^2(\alpha_{k_0l_0}^2 + \beta)^2(\alpha_{k_0l_0}^2 + a_{k_0}^2) \\
&\quad + \frac{1}{r_0^4}(\alpha_{k_0l_0}^2 + a_{k_0}^2)^2(\alpha_{k_0l_0}^2 + \beta) - Ra^2\left[\alpha_{k_0l_0}^2(\alpha_{k_0l_0}^2 + \beta) + \frac{1}{r_0^2}(\alpha_{k_0l_0}^2 + a_{k_0}^2)\right] \\
&\quad + \frac{d^2}{\rho_0\kappa^2}F^2\alpha_{k_0l_0}^4 l^2\pi^2(\alpha_{k_0l_0}^2 + \beta) \\
&= \alpha_{k_0l_0}^4\beta^3 + \left[3\alpha_{k_0l_0}^6 + \frac{2}{r_0^2}\alpha_{k_0l_0}^2(\alpha_{k_0l_0}^2 + a_k^2)\right]\beta^2 \\
&\quad + \left[3\alpha_{k_0l_0}^8 + \frac{4}{r_0^2}\alpha_{k_0l_0}^4(\alpha_{k_0l_0}^2 + a_k^2) \right.\\
&\quad \left. + \frac{1}{r_0^4}(\alpha_{k_0l_0}^2 + a_k^2)^2 - Ra_k^2\alpha_{k_0l_0}^2 + \frac{d^2}{\rho_0^2\kappa^2}F^2\alpha^4 l_0^2\pi^2\right]\beta \\
&\quad + L_{k_0l_0},
\end{aligned}$$

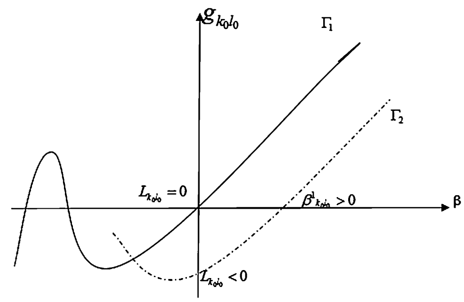

Figure 3 The graph of $g_{k_0l_0}$. As the figure shows: When $L_{k_0l_0} = 0$, Γ_1 is the graph of $g_{k_0l_0}$; when $L_{k_0l_0} < 0$, Γ_2 is the graph of $g_{k_0l_0}$,Then the intersecting point of Γ_i $(i = 1, 2)$ and β-axis is the first eigenvalue $\beta^1_{k_0l_0}$. When $L_{k_0l_0} = 0$, the intersecting point of Γ_1 and β-axis, $\beta^1_{k_0l_0} = 0$, and (R, F) is in curve L. When $L_{k_0l_0} < 0$, the intersecting point of Γ_2 and β-axis, $\beta^1_{k_0l_0} > 0$, and (R, F) is in region II.

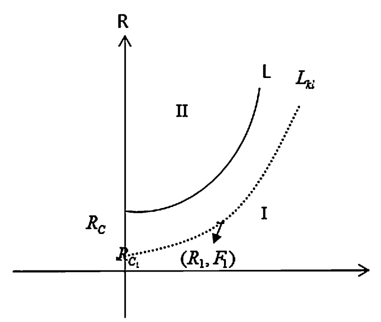

Figure 4 The graph of L_{kl}. The curl L_{kl} passes through the point (R_1, F_1) and intersects with the R-axis at R_{C_1}, it is easy to find that $R_C > R_{C_1}$.

where $L_{k_0l_0} = b_{k_0l_0} - c_{k_0l_0}R + d_{k_0l_0}F^2$. When $L_{k_0l_0} = 0$, Γ_1 is the graph of $g_{k_0l_0}$; when $L_{k_0l_0} < 0$, Γ_2 is the graph of $g_{k_0l_0}$, as shown in Figure 3. Then the intersecting point of Γ_i $(i = 1, 2)$ and β-axis is the first eigenvalue $\beta^1_{k_0l_0}$. When $L_{k_0l_0} = 0$, the intersecting point of Γ_1 and β-axis, $\beta^1_{k_0l_0} = 0$, and (R, F) is in curve L. When $L_{k_0l_0} < 0$, the intersecting point of Γ_2 and β-axis, $\beta^1_{k_0l_0} > 0$, and (R, F) is in region II. At last, we will show that $\beta^i_{kl} < 0$, $\forall (k, l) \neq (k_0, l_0)$, when (R, F) in region I or in region II near the curve L. We only need

$$\beta^i_{kl} \neq 0, \quad \forall (R, F) \in \mathrm{I}, (k, l) \neq (k_0, l_0). \tag{3.15}$$

Otherwise, if (3.15) does not hold true, there exist a pair of (R_1, F_1), such that $\beta^i_{kl}(R_1, F_1) = 0$. That is to say, there exists a curl L_{kl}, which passes through the point (R_1, F_1) and intersects with R-axis at R_{C_1}, as shown in Figure 4. It is easy to find that $R_C > R_{C_1}$, which makes a contradiction that R_C is the minimum critical Rayleigh number. So, we prove (3.15). When $(R, F) = (0, 0)$, we have

$$\beta^1_{kl} = -\alpha^2_{kl} < 0, \qquad \beta^2_{kl} = \beta^3_{kl} = -\alpha^2_{kl} - \frac{\alpha^2_{kl} + a^2_k}{\alpha^2_{kl} r^2_0} < 0,$$

which implies $\beta^i_{kl} < 0$, $\forall (k, l) \neq (k_0, l_0)$. Because $\beta^i_{kl}(R, F)$ are continuous, $\beta^i_{kl} < 0$ in region I or in region II near the curve L. This proves Lemma 3.1. □

Based on Theorem 2.3.1 in [1], Theorem 2 in [20] and Lemma 3.1, we derive the following theorem.

Theorem 3.2 *The critical parameter curve L divides the RF-plane into two regions I and II* (*see Figure* 2), *such that the following conclusions hold true.*

1. *If* $(R,F)\in I$, *the problem* (2.8) *and* (2.9) *have no bifurcation, and the basic flow in* (2.4) *with* $K_1=0$ *is stable.*
2. *If* $(R,F)\in II$, *there exists a bifurcation solution.*
3. *The transition takes place as the control parameter* (R,F) *crosses the critical curve* L *from region I into II.*

4 For the condition that $K_1\neq 0$ and $F=0$

4.1 Eigenvalue problem

Consider the case that $F=0$, and F is given by (2.10). Then equations (2.9) become

$$\begin{cases}\frac{\partial u_\varphi}{\partial t}+(u\cdot\nabla)u_\varphi+\frac{u_ru_\varphi}{r_0}=\triangle u_\varphi-\frac{u_\varphi}{r_0^2}-\frac{\partial p}{\partial\varphi}+\frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0^2H_r(\frac{\partial H_\varphi}{\partial r}-\frac{\partial H_r}{\partial\varphi}),\\ \frac{\partial u_r}{\partial t}+(u\cdot\nabla)u_r-\frac{u_\varphi^2}{r}=\triangle u_r-\frac{2u_r}{r_0^2}-\frac{\partial p}{\partial r}+\sqrt{R}T,\\ \frac{\partial T}{\partial t}+(u\cdot\nabla)T=\triangle T+\sqrt{R}u_r,\\ \frac{\partial H_\varphi}{\partial t}=\frac{\partial}{\partial r}(u_\varphi H_r-u_rH_\varphi)+\triangle H_\varphi-\frac{H_\varphi}{r_0^2}+\frac{\partial\Phi}{\partial\varphi}+\frac{K_1d}{H_0}u_\varphi,\\ \frac{\partial H_r}{\partial t}=\frac{\partial}{\partial\varphi}(u_rH_\varphi-u_\varphi H_r)+\triangle H_r-\frac{2H_r}{r_0^2}+\frac{\partial\Phi}{\partial r},\\ \operatorname{div}u=0,\\ \operatorname{div}H=0.\end{cases}\tag{4.1}$$

Essentially, the control parameter in (4.1) is only the Rayleigh number R in (3.1). Then the eigenvalues and the eigenvectors Ψ, $\tilde{\Psi}$ are similar to those in Section 3, and we can get the corresponding eigenvalue problems

$$\begin{cases}D^*Dh_k-\frac{1}{r_0^2}Dh_k-a_kp_k=\beta Dh_k,\\ a_kD^*h_k-\frac{2}{r_0^2}a_kh_k+\sqrt{R}T_k-Dp_k=\beta a_kh_k,\\ D^*T_k+\sqrt{R}a_kh_k=\beta T_k,\\ D^*Dg_k-\frac{1}{r_0^2}Dg_k+a_k\Phi_k+\frac{1}{H_0}K_1dDh_k=\beta Dg_k,\\ a_kD^*g_k-\frac{2}{r_0^2}a_kg_k+D\Phi_k=\beta a_kg_k.\end{cases}\tag{4.2}$$

From (4.2), we see that g_k $(k=1,2,\ldots)$ satisfy the equations

$$\begin{aligned}\Big[&-(D^*)^2(D^*-\beta)^3+\frac{2}{r_0^2}(D^*-a_k^2)D^*(D^*-\beta)^2\\ &-\frac{1}{r_0^4}(D^*-a_k^2)^2(D^*-\beta)-Ra_k^2D^*(D^*-\beta)\\ &+\frac{1}{r_0^2}Ra_k^2(D^*-a_k^2)\Big]g_k=0,\quad k=1,2,\ldots.\end{aligned}\tag{4.3}$$

By (4.2) and the boundary conditions (2.8), g_k are the sine functions

$$g_{kl}=\sin l\pi(r-r_0),\quad \text{for } l=1,2,3,\ldots.$$

Moreover, h_k and T_k are determined by the following:

$$D^*(\beta-D^*)g_k+\frac{1}{r_0^2}(D^*-a_k^2)g_k=\frac{1}{H_0}K_1dD^2h_k,\tag{4.4}$$

$$\sqrt{R}a_k T_k = \big(D^*\big)\big(D^* - \beta\big)h_k - \frac{1}{r_0^2}\big(D^* - a_k^2\big)h_k. \tag{4.5}$$

Substituting g_{kl} into (4.3), we see that the eigenvalue β of (4.2) satisfies the cubic equation

$$\begin{aligned}&\alpha_{kl}^4\big(\alpha_{kl}^2 + \beta\big)^3 + \frac{2}{r_0^2}\big(\alpha_{kl}^2 + a_k^2\big)\big(\alpha_{kl}^2 + \beta\big)^2\alpha_{kl}^2 + \frac{1}{r_0^4}\big(\alpha_{kl}^2 + a_k^2\big)^2\big(\alpha_{kl}^2 + \beta\big)\\&\quad - Ra_k^2\alpha_{kl}^2\big(\alpha_{kl}^2 + \beta\big) - \frac{1}{r_0^2}Ra_k^2\big(\alpha_{kl}^2 + a_k^2\big) = 0.\end{aligned} \tag{4.6}$$

Hence, all eigenvalues and eigenvectors can be derived from the following cases.

1. For $(k, l) = (0, l)$, we have

$$\begin{aligned}&\beta_{0l}^1 = -l^2\pi^2, \qquad \beta_{0l}^2 = -l^2\pi^2 - \frac{1}{r_0^2},\\&\Psi_{0l}^1 = \big(0, 0, \sin l\pi(r - r_0), 0, 0\big), \qquad \tilde{\Psi}_{0l}^1 = \big(0, 0, \cos l\pi(r - r_0), 0, 0\big),\\&\Psi_{0l}^2 = \big(0, 0, 0, \cos l\pi(r - r_0), 0\big), \qquad \tilde{\Psi}_{0l}^2 = \big(0, 0, 0, \sin l\pi(r - r_0), 0\big).\end{aligned}$$

2. For $(k, l) = (k, 0)$, the eigenvalues and eigenvectors are given by

$$\begin{aligned}&\beta_{k0} = -a_k^2 - \frac{2}{r_0^2},\\&\Psi_{k0} = (0, 0, 0, 0, \sin a_k\varphi), \qquad \tilde{\Psi}_{k0} = (0, 0, 0, 0, \cos a_k\varphi).\end{aligned}$$

3. For $k \neq 0$ and $l \neq 0$, it is well known that all solutions of equation (4.6) are eigenvalues. Let $\beta_{kl}^1(R)$ $(1 \leq i \leq 3)$ be three zero points of (4.6) such that

$$Re\beta_{kl}^1 \geq Re\beta_{kl}^2 \geq Re\beta_{kl}^3.$$

Then, by (3.3), (3.4), (4.4) and (4.5), the eigenvectors Ψ_{kl}^i and $\tilde{\Psi}_{kl}^i$ corresponding to β_{kl}^1 can be written as

$$\Psi_{kl}^i = \begin{cases} l\pi h_{kl}^i \cos a_k\varphi \cdot \cos l\pi(r - r_0),\\ a_k h_{kl}^i \sin a_k\varphi \cdot \sin l\pi(r - r_0),\\ T_{kl}^i \sin a_k\varphi \cdot \sin l\pi(r - r_0),\\ l\pi \cos a_k\varphi \cdot \cos l\pi(r - r_0),\\ a_k \sin a_k\varphi \cdot \sin l\pi(r - r_0),\end{cases} \tag{4.7}$$

$$\tilde{\Psi}_{kl}^i = \begin{cases} l\pi h_{kl}^i \sin a_k\varphi \cdot \cos l\pi(r - r_0),\\ -a_k h_{kl}^i \cos a_k\varphi \cdot \sin l\pi(r - r_0),\\ -T_{kl}^i \cos a_k\varphi \cdot \sin l\pi(r - r_0),\\ l\pi \sin a_k\varphi \cdot \cos l\pi(r - r_0),\\ -a_k \cos a_k\varphi \cdot \sin l\pi(r - r_0),\end{cases} \tag{4.8}$$

where

$$h_{kl}^i = \frac{H_0[\alpha_{kl}^2(\alpha_{kl}^2 + \beta_{kj}^i) - \frac{1}{r_0^2}(\alpha_{kl}^2 + a_k^2)]}{l^2\pi^2 K_1 d},$$
$$T_{kl}^i = \frac{H_0[\alpha_{kl}^4(\alpha_{kl}^2 + \beta_{kj}^i)^2 - \frac{1}{r_0^4}(\alpha_{kl}^2 + a_k^2)^2]}{\sqrt{R}a_k l^2\pi^2 K_1 d}.$$

It is clear that the eigenvalues and eigenvectors have the following properties.

1. β_{0l}^i, β_{k0} and β_{kl}^i ($k \neq 0$, $l \neq 0$) consist of all eigenvalues of equation (4.2), and all eigenvectors form a basis of H;
2. $\beta_{0l}^i < 0$ ($i = 1,2$) and $\beta_{k0} < 0$;
3. the eigenvalues β_{kl}^i ($k \neq 0$, $l \neq 0$) only depend on the Rayleigh number.

Now, we are ready to determine the dual eigenvector. The dual eigenvectors Ψ_{kl}^{i*} and $\tilde{\Psi}_{kl}^{i*}$ are given by

$$\Psi_{kl}^{i*} = \begin{cases} l\pi h_{kl}^{i*}\cos a_k\varphi \cdot \cos l\pi(r - r_0), \\ a_k h_{kl}^{i*}\sin a_k\varphi \cdot \sin l\pi(r - r_0), \\ T_{kl}^{i*}\sin a_k\varphi \cdot \sin l\pi(r - r_0), \\ l\pi \cos a_k\varphi \cdot \cos l\pi(r - r_0), \\ a_k \sin a_k\varphi \cdot \sin l\pi(r - r_0), \end{cases} \tag{4.9}$$

$$\tilde{\Psi}_{kl}^{i*} = \begin{cases} l\pi h_{kl}^{i*}\sin a_k\varphi \cdot \sin l\pi(r - r_0), \\ -a_k h_{kl}^{i*}\cos a_k\varphi \cdot \cos l\pi(r - r_0), \\ -T_{kl}^{i*}\cos a_k\varphi \cdot \cos l\pi(r - r_0), \\ l\pi \sin a_k\varphi \cdot \sin l\pi(r - r_0), \\ -a_k \cos a_k\varphi \cdot \cos l\pi(r - r_0), \end{cases} \tag{4.10}$$

where

$$h_{kl}^{i*} = \frac{-\frac{K_1 d}{H_0}l^2\pi^2(\beta_{kl}^i + \alpha_{kl}^2)}{Ra_k^2 + (\beta_{kl}^i + \alpha_{kl}^2)(\alpha_{kl}^4 - \beta_{kl}^i\alpha_{kl}^2 - \frac{2}{r_0^2}a_k^2 - \frac{1}{r_0^2}l^2\pi^2)},$$
$$T_{kl}^{i*} = \frac{-\frac{K_1 d}{H_0}l^2\pi^2 a_k\sqrt{R}}{Ra_k^2 + (\beta_{kl}^i + \alpha_{kl}^2)(\alpha_{kl}^4 - \beta_{kl}^i\alpha_{kl}^2 - \frac{2}{r_0^2}a_k^2 - \frac{1}{r_0^2}l^2\pi^2)}.$$

Thus, all dual eigenvectors consist of $\Psi_{0l}^{1*} = (0, 0, \sin l\pi(r - r_0), 0, 0)$, $\Psi_{0l}^{2*} = (0, 0, 0, \sin l\pi(r - r_0), 0)$, $\Psi_{k0}^* = (0, 0, 0, 0, \sin a_k\varphi)$, $\tilde{\Psi}_{k0}^* = (0, 0, 0, 0, \cos a_k\varphi)$.

In this part, we are ready to study the critical-crossing of the first eigenvalue. Since the eigenvalues β_{kl}^i ($1 \leq i \leq 3$, $k \neq 0$, $l \neq 0$) only depend on the Rayleigh number, it suffices to focus on the eigenvalue problem (4.6). If $\beta = 0$ is a zero point of (4.6), we can get

$$\alpha_{kl}^{10} + \frac{2}{r_0^2}(\alpha_{kl}^2 + a_k^2)\alpha_{kl}^6 + \frac{1}{r_0^4}(\alpha_{kl}^2 + a_k^2)^2\alpha_{kl}^2 - Ra_k^2\left(\alpha_{kl}^4 + \frac{1}{r_0^2}\alpha_{kl}^2 + \frac{1}{r_0^2}a_k^2\right) = 0.$$

In this case, we have

$$R = \frac{\alpha_{kl}^{10} + \frac{2}{r_0^2}(\alpha_{kl}^2 + a_k^2)\alpha_{kl}^6 + \frac{1}{r_0^4}(\alpha_{kl}^2 + a_k^2)^2\alpha_{kl}^2}{a_k^2(\alpha_{kl}^4 + \frac{1}{r_0^2}\alpha_{kl}^2 + \frac{1}{r_0^2}a_k^2)}. \tag{4.11}$$

Hence the critical Rayleigh number R_C has a similar form to Section 3, which is given by

$$R_C = \min_{(k,l)} \frac{\alpha_{kl}^{10} + \frac{2}{r_0^2}\alpha_{kl}^6(\alpha_{kl}^2 + a_k^2) + \frac{1}{r_0^4}(\alpha_{kl}^2 + \alpha_{kl}^2)}{a_k^2[\alpha_{kl}^4 + \frac{1}{r_0^2}(\alpha_{kl}^2 + a_k^2)]} = \frac{b_{k_0 l_0}}{c_{k_0 l_0}}.$$

Thus, we have the following lemma.

Lemma 4.1 *Let* (k_0, l_0) *minimize the right hand side of* (4.11). *Assume the zero point* $\beta_{k_0 l_0}^1$ *of* (4.6) *is a real single eigenvalue of* (4.1) *near* R_C, *then* $\beta_{k_0 l_0}^1$ *satisfies*

$$\beta_{k_0 l_0}^1 \begin{cases} < 0, & \text{if } R < R_C, \\ = 0, & \text{if } R = R_C, \\ > 0, & \text{if } R > R_C, \end{cases}$$

$$Re\beta_{kl}^i(R_C) < 0, \quad \forall (k,l,i) \neq (k_0, l_0, 1).$$

The proof is similar to the proof of Lemma 3.1, so we omit it.

4.2 Transition theory with $F = 0$

In this section, we consider the transition theorem of problem (4.1) with boundary conditions (2.8) with $K_1 \neq 0$ and $F = 0$, where K_1, F as in (2.10). Then we have the following theorem.

Theorem 4.2 *Let* $K_1 \neq 0$ *and* $F = 0$, *we have the following conclusions for equations* (4.1) *with boundary conditions* (2.8).

1. *Equation* (4.1) *bifurcates from* $((u,T,H),R) = (0,R_C)$ *to an attractor* $\Sigma_R \in H_1$, *only consisting of a steady state solution.*
2. *The steady state solution* $(u,T,H) = (u_\varphi, u_r, T, H_\varphi, H_r)$ *can be expressed as*

$$(u_\varphi, u_r, T, H_\varphi, H_r) = C\big(\beta_{k_0 l_0}^1(R)\big)^{\frac{1}{2}}\big(x\Psi_{k_0 l_0}^1 + y\tilde{\Psi}_{k_0 l_0}^1\big) + o\big(\big|\beta_{k_0 l_0}^1\big|^{\frac{1}{2}}\big),$$

where $C > 0$ *is a constant,* $\Psi_{k_0 l_0}^1$ *and* $\tilde{\Psi}_{k_0 l_0}^1$ *are the first eigenvectors given by* (4.7) *and* (4.8), *and* $x^2 + y^2 = 1$.

Proof We first reduce the abstract equation (2.11) to the center manifold. Let $J_0 = (k_0, l_0, 1)$, then $\Psi_{k_0 l_0}^1 = \Psi_{J_0}$ and $\tilde{\Psi}_{k_0 l_0}^1 = \tilde{\Psi}_{J_0}$, the reduced equations read

$$\begin{cases} \frac{dx}{dt} = \beta_{J_0} x + \frac{1}{\langle \Psi_{J_0}, \Psi_{J_0}^* \rangle}\langle G(U,U), \Psi_{J_0}^* \rangle, \\ \frac{dy}{dt} = \beta_{J_0} y + \frac{1}{\langle \tilde{\Psi}_{J_0}, \tilde{\Psi}_{J_0}^* \rangle}\langle G(U,U), \tilde{\Psi}_{J_0}^* \rangle, \end{cases} \tag{4.12}$$

where $U = (u_\varphi, u_r, T, H_\varphi, H_r) \in H_1$ is written as

$$U = x\Psi_{J_0} + y\tilde{\Psi}_{J_0} + \sum_{J \neq J_0}(x_J \Psi_J + y_J \tilde{\Psi}_J) = x\Psi_{J_0} + y\tilde{\Psi}_{J_0} + \Phi. \tag{4.13}$$

Φ is the center manifold function, and G is a bilinear operator, which is

$$G(U_1,U_2)=P\bigg(-(u_1\cdot\nabla)u_{2\varphi}-\frac{u_{1r}u_{2\varphi}}{r_0}+\frac{1}{\rho_0}\frac{d^2}{\kappa^2}H_0^2H_{1r}\bigg(\frac{\partial H_{2\varphi}}{\partial r}-\frac{\partial H_{2r}}{\partial\varphi}\bigg),\\ -(u_1\cdot\nabla)u_{2r}-\frac{u_{1\varphi}u_{2\varphi}}{r_0},-(u_1\cdot\nabla)T_2,\frac{\partial}{\partial r}(u_{1\varphi}H_{2r}-u_{1r}H_{2\varphi}),\\ \frac{\partial}{\partial\varphi}(u_{1r}H_{2\varphi}-u_{1\varphi}H_{2r})\bigg)^T.$$

It is easy to verify that

$$\langle G(\Psi_{J_0},\Psi_{J_0}),\Psi_{J_0}^*\rangle=\langle G(\Psi_{J_0},\tilde{\Phi}_{J_0}),\Psi_{J_0}^*\rangle\\ =\langle G(\tilde{\Phi}_{J_0},\Psi_{J_0}),\Psi_{J_0}^*\rangle=\langle G(\tilde{\Phi}_{J_0},\tilde{\Phi}_{J_0}),\Psi_{J_0}^*\rangle=0.$$

Hence, (4.12) can be rewritten as

$$\begin{cases}\frac{dx}{dt}=\beta_{J_0}x+\frac{x}{\langle\Psi_{J_0},\Psi_{J_0}^*\rangle}[\langle G(\Psi_{J_0},\Phi),\Psi_{J_0}^*\rangle+\langle G(\Phi,\Psi_{J_0}),\Psi_{J_0}^*\rangle]\\ \quad+\frac{y}{\langle\Psi_{J_0},\Psi_{J_0}^*\rangle}[\langle G(\tilde{\Psi}_{J_0},\Phi),\Psi_{J_0}^*\rangle+\langle G(\Phi,\tilde{\Psi}_{J_0}),\Psi_{J_0}^*\rangle]+o(|x|^3+|y|^3),\\ \frac{dy}{dt}=\beta_{J_0}y+\frac{x}{\langle\tilde{\Psi}_{J_0},\tilde{\Psi}_{J_0}^*\rangle}[\langle G(\Psi_{J_0},\Phi),\tilde{\Psi}_{J_0}^*\rangle+\langle G(\Phi,\Psi_{J_0}),\tilde{\Psi}_{J_0}^*\rangle]\\ \quad+\frac{y}{\langle\tilde{\Psi}_{J_0},\tilde{\Psi}_{J_0}^*\rangle}[\langle G(\tilde{\Psi}_{J_0},\Phi),\tilde{\Psi}_{J_0}^*\rangle+\langle G(\Phi,\tilde{\Psi}_{J_0}),\tilde{\Psi}_{J_0}^*\rangle]+o(|x|^3+|y|^3).\end{cases}\tag{4.14}$$

Based on the approximation formula Theorem 6.1 in[18] for center manifold functions, we see that the manifold function Φ satisfies

$$-L\Phi=-\sum_{J\neq J_0}(\beta_J x_J\Psi_J+\tilde{\beta}_J y_J\tilde{\Psi}_J)\\ =x^2G(\Psi_{J_0},\Psi_{J_0})+y^2G(\tilde{\Psi}_{J_0},\tilde{\Psi}_{J_0})+xyG(\Psi_{J_0},\tilde{\Psi}_{J_0})+xyG(\Psi_{J_0},\tilde{\Psi}_{J_0})\\ +o(|x|^2+|y|^2).\tag{4.15}$$

By direct calculation, from (4.15), the center manifold function can be written

$$\Phi=-(x^2+y^2)\frac{a_{k_0}T_{k_0l_0}^*h_{k_0l_0}}{2l_0\pi}\Psi_{0,2l_0}^1+o(|x|^2+|y|^2).\tag{4.16}$$

Inserting (4.16) into (4.14), we have

$$\begin{cases}\frac{dx}{dt}=\beta_{J_0}x-\frac{\delta x}{\langle\Psi_{J_0},\Psi_{J_0}^*\rangle}(x^2+y^3)+o(|x|^3+|y|^3),\\ \frac{dy}{dt}=\beta_{J_0}y-\frac{\delta y}{\langle\tilde{\Psi}_{J_0},\tilde{\Psi}_{J_0}^*\rangle}(x^2+y^3)+o(|x|^3+|y|^3),\end{cases}\tag{4.17}$$

where

$$\delta=\frac{L}{8}a_{k_0}^2(T_{k_0l_0}^*)^2h_{k_0l_0}^2\pi>0,\\ \langle\Psi_{J_0},\Psi_{J_0}^*\rangle=\langle\tilde{\Psi}_{J_0},\tilde{\Psi}_{J_0}^*\rangle>0.\tag{4.18}$$

Therefore, the conclusions of Theorem 4.2 follows from (4.17), (4.18) and the attractor bifurcation theorem in [3]. The proof of the theorem is completed. □

5 Conclusions

In summary, the production of granulation convection is illustrated by Theorem 3.2 and Figure 2, where R_C is the critical Rayleigh number. As is shown in Figure 2, when the differences of temperature and magnetic are small enough, the points (R, F) are below the critical curve L and in region I, the gas is in a static state. When they become greater than the curve L, the gas suddenly breaks into regular circulation cells, which is granulation convection. The circulation cells can be expressed by the bifurcation solution in Theorem 4.2. From both the mathematical and the physical points of view, the results are valuable to understand the phase transition in fluid dynamics.

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