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Time-decay solutions of the initial-boundary value problem of rotating magnetohydrodynamic fluids

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Abstract

We have investigated an initial-boundary problem for the perturbation equations of rotating, incompressible, and viscous magnetohydrodynamic (MHD) fluids with zero resistivity in a horizontally periodic domain. The velocity of the fluid in the domain is non-slip on both upper and lower flat boundaries. We switch the analysis of the initial-boundary problem from Euler coordinates to Lagrangian coordinates under proper initial data, and get a so-called transformed MHD problem. Then, we exploit the two-tiers energy method. We deduce the time-decay estimates for the transformed MHD problem which, together with a local well-posedness result, implies that there exists a unique time-decay solution to the transformed MHD problem. By an inverse transformation of coordinates, we also obtain the existence of a unique time-decay solution to the original initial-boundary problem with proper initial data.

Keywords: magnetohydrodynamic fluid; equilibrium state; magnetic field; decay estimates; rotation

1 Introduction

The three-dimensional (3D) rotating, incompressible and viscous magnetohydrodynamic (MHD) equations with zero resistivity in a domain $\Omega \subset \mathbb{R}^3$ read as follows:

$$\begin{cases} \rho v_t + \rho v \cdot \nabla v + \nabla(p + \lambda_0 |M|^2/2) + 2\rho(\vec{\omega} \times v) = \mu \Delta v + \lambda_0 M \cdot \nabla M, \\ M_t = M \cdot \nabla v - v \cdot \nabla M, \\ \operatorname{div} v = \operatorname{div} M = 0. \end{cases} \tag{1.1}$$

Here the unknowns $v = v(x,t)$, $M := M(x,t)$ and $p = p(x,t)$ denote the velocity, the magnetic field, and the pressure of the incompressible MHD fluid respectively; $\mu > 0$, ρ and λ_0 stand for the coefficients of the shear viscosity, the density constant, and the permeability of vacuum, respectively. $2\rho(\vec{\omega} \times v)$ represents the Coriolis force, and $\vec{\omega} = (0,0,\omega)$ denotes the constant angular velocity in the vertical direction. In system (1.1), equation $(1.1)_1$ describes the balance law of momentum, while $(1.1)_2$ is called the induction equation. As for the constraint $\operatorname{div} M = 0$, it can be seen just as a restriction on the initial value of M since $(\operatorname{div} M)_t = 0$ due to $(1.1)_2$.

Let $\bar{M} := (m_1, m_2, m_3)$ be a constant vector with $m_3 \neq 0$, and $(0, \bar{M}, \bar{p})$ be a rest state of the system (1.1). We denote the perturbation to an equilibrium state $(0, \bar{M})$ by

$$v = v - 0, \qquad N = M - \bar{M}, \qquad \tilde{q} = p - \bar{p}.$$

Then, (v, N, q) satisfies the perturbation equations

$$\begin{cases} \rho v_t + \rho v \cdot \nabla v + \nabla(\tilde{q} + \lambda_0 |N + \bar{M}|^2/2) \\ \quad = \mu \Delta v + \lambda_0 (N + \bar{M}) \cdot \nabla (N + \bar{M}) + 2\rho\omega(v_2 e_1 - v_1 e_2), \\ N_t = (N + \bar{M}) \cdot \nabla v - v \cdot \nabla (N + \bar{M}), \\ \operatorname{div} v = \operatorname{div} N = 0, \end{cases} \tag{1.2}$$

where we have used the relation $\vec{\omega} \times v = \omega(v_1 e_2 - v_2 e_1)$. For system (1.2), we impose the initial and the boundary conditions:

$$(v, N)|_{t=0} = (v_0, N_0) \quad \text{in } \Omega, \tag{1.3}$$

$$v(\cdot, t)|_{\partial\Omega} = 0 \quad \text{for any } t > 0, \tag{1.4}$$

where v_0 and N_0 should satisfy the compatibility conditions $\operatorname{div} v_0 = \operatorname{div} N_0 = 0$. We call the initial-boundary value problem (1.2)-(1.4) the MHD problem (with rotation) for simplicity. In this article, we always assume that the domain is horizontally periodic with finite height, *i.e.*,

$$\Omega := \left\{ x := (x', x_3) \in \mathbb{R}^3 \mid x' \in \mathcal{T}, 0 < x_3 < h \right\} \quad \text{with } h > 0,$$

where $\mathcal{T} := (2\pi L_1 \mathbb{T}) \times (2\pi L_2 \mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $2\pi L_1, 2\pi L_2 > 0$ are the periodicity lengths.

The effects of magnetic fields and rotation on the motion of pure fluids were widely investigated; see [1–10] and the references cited therein. In particular, Tan and Wang [11] showed that the well-posedness problem of the initial-boundary problem (1.2)-(1.4) for $\omega = 0$ (*i.e.*, without the effect of rotation). In this article, we further consider $\omega \neq 0$, and show that there also exists a unique time-decay solution to the initial-boundary problem (1.2)-(1.4) in Lagrangian coordinates (see Theorem 2.1), which, together with the inverse transformation of coordinates, implies the existence of a time-decay solution to the original initial-boundary problem (1.2)-(1.4) with proper initial data in $H^7(\Omega)$. Our result also holds for the case $\omega = 0$, thus improves Tan and Wang's result in [11], in which the sufficiently small initial data at least belong to $H^{16}(\Omega)$.

In the next section we introduce the form of the initial-boundary problem (1.2)-(1.4) in Lagrangian coordinates, and the details of our result.

2 Main results

2.1 Reformulation

In general, it is difficult to directly show the existence of a unique global-in-time solution to (1.2)-(1.4). Instead, we switch our analysis to Lagrangian coordinates as in [12, 13]. To this end, we assume that there is an invertible mapping $\zeta_0 := \zeta_0(y) : \Omega \to \Omega$, such that

$$\partial\Omega = \zeta_0(\partial\Omega) \quad \text{and} \quad \det \nabla \zeta_0 \equiv 1, \tag{2.1}$$

where ζ_3^0 denotes the third component of ζ_0. We define the flow map ζ as the solution to

$$\begin{cases} \zeta_t(y,t) = v(\zeta(y,t),t), \\ \zeta(y,0) = \zeta_0. \end{cases} \tag{2.2}$$

We denote the Eulerian coordinates by (x,t) with $x = \zeta(y,t)$, where $(y,t) \in \Omega \times \mathbb{R}^+$ stand for the Lagrangian coordinates. In order to switch back and forth from Lagrangian to Eulerian coordinates, we assume that $\zeta(\cdot,t)$ is invertible and $\Omega = \zeta(\Omega,t)$. In other words, the Eulerian domain of the fluid is the image of Ω under mapping ζ. In view of the non-slip boundary condition $v|_{\partial\Omega} = 0$, we have $\partial\Omega = \zeta(\partial\Omega,t)$. In addition, since $\det \nabla\zeta_0 = 1$, we have

$$\det(\nabla\zeta) = 1 \tag{2.3}$$

due to $\operatorname{div} v = 0$; see [14], Proposition 1.4.

Now, we further define the Lagrangian unknowns by

$$(u,\tilde{p},B)(y,t) = \bigl(v, p + \lambda_0|M|^2/2, M\bigr)\bigl(\zeta(y,t),t\bigr) \quad \text{for } (y,t) \in \Omega \times \mathbb{R}^+. \tag{2.4}$$

Thus in Lagrangian coordinates the evolution equations for u, $\tilde{p}$ and B read as

$$\begin{cases} \zeta_t = u, \\ \rho u_t - \mu\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}}\tilde{p} = \lambda_0 B \cdot \nabla_{\mathcal{A}} B + 2\rho\omega(u_2 e_1 - u_1 e_2), \\ B_t - B \cdot \nabla_{\mathcal{A}} u = 0, \\ \operatorname{div}_{\mathcal{A}} u = 0, \end{cases} \tag{2.5}$$

with initial and boundary conditions

$$(u, \zeta - y)|_{\partial\Omega} = 0 \quad \text{and} \quad (\zeta, u, B)|_{t=0} = (\zeta_0, u_0, B_0).$$

Moreover, $\operatorname{div}_{\mathcal{A}} B = 0$ if the initial data ζ_0 and B_0 satisfy

$$\operatorname{div}_{\mathcal{A}_0} B_0 = 0. \tag{2.6}$$

Here $\mathcal{A}_0$ denotes the initial value of $\mathcal{A}$, the matrix $\mathcal{A} := (\mathcal{A}_{ij})_{3\times 3}$ via $\mathcal{A}^T = (\nabla\zeta)^{-1} := (\partial_j\zeta_i)^{-1}_{3\times 3}$, and the differential operators $\nabla_{\mathcal{A}}$, $\operatorname{div}\mathcal{A}$ and $\Delta_{\mathcal{A}}$ are defined by $\nabla_{\mathcal{A}} f := (\mathcal{A}_{1k}\partial_k f, \mathcal{A}_{2k}\partial_k f, \mathcal{A}_{3k}\partial_k f)^T$, $\operatorname{div}_{\mathcal{A}}(X_1,X_2,X_3)^T := \mathcal{A}_{lk}\partial_k X_l$, and $\Delta_{\mathcal{A}} f := \operatorname{div}_{\mathcal{A}} \nabla_{\mathcal{A}} f$ for appropriate f and X. It should be noted that we have used the Einstein convention of summation over repeated indices, and $\partial_k = \partial_{y_k}$. In addition, in view of the definition of $\mathcal{A}$ and (2.3), we can see that $\mathcal{A} = (A^*_{ij})_{3\times 3}$, where A^*_{ij} is the algebraic complement minor of the (i,j)th entry $\partial_j\zeta_i$. Since $\partial_k A^*_{ik} = 0$, we can get an important relation

$$\operatorname{div}_{\mathcal{A}} u = \partial_l(\mathcal{A}_{kl} u_k) = 0, \tag{2.7}$$

which will be used in the derivation of temporal derivative estimates.

Our next goal is to eliminate B by expressing it in terms of ζ. This can be achieved in the same manner as in [12, 13]. For the reader's convenience, we give the derivation here. In view of the definition of $\mathcal{A}$, one has

$$\partial_i \zeta_k \mathcal{A}_{kj} = \mathcal{A}_{ik} \partial_k \zeta_j = \delta_{ij},$$

where $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ij} = 1$ for $i = j$. Thus, applying $\mathcal{A}_{jl}$ to $(2.5)_4$, we obtain

$$\mathcal{A}_{jl} \partial_t B_j = B_i \mathcal{A}_{ik} \partial_k u_j \mathcal{A}_{jl} = B_i \mathcal{A}_{ik} \partial_t (\partial_k \zeta_j) \mathcal{A}_{jl} = -B_i \mathcal{A}_{ik} \partial_k \zeta_j \partial_t \mathcal{A}_{jl} = -B_j \partial_t \mathcal{A}_{jl},$$

which implies that $\partial_t (\mathcal{A}_{jl} B_j) = 0$ (*i.e.*, $(\mathcal{A}^T B)_t = 0$). Hence,

$$\mathcal{A}_{jl} B_j = \mathcal{A}^0_{jl} B^0_j, \tag{2.8}$$

which yields $B_i = \partial_l \zeta_i \mathcal{A}^0_{jl} B^0_j$, *i.e.*,

$$B = \nabla \zeta \mathcal{A}^T_0 B_0. \tag{2.9}$$

Here and in what follows, the notation f^0 also denotes the initial data of the function f. To obtain the asymptotic stability in time, we naturally expect

$$(\zeta, B) \text{ converges to } (y, \bar{M}) \quad \text{as } t \to \infty. \tag{2.10}$$

Thus (2.9) formally implies

$$\mathcal{A}^T_0 B_0 = \bar{M}, \quad \textit{i.e.}, B_0 = \bar{M} \cdot \nabla \zeta_0. \tag{2.11}$$

Putting the above expression of B_0 into (2.9), we get

$$B = \bar{M} \cdot \nabla \zeta. \tag{2.12}$$

Moreover, in view of (2.8), (2.11) and (2.12), the Lorentz force term can be represented by

$$B \cdot \nabla_{\mathcal{A}} B = B_l \mathcal{A}_{lk} \partial_k B = \mathcal{A}^0_{lk} B^0_l \partial_k (\bar{M} \cdot \nabla \zeta) = \bar{M} \cdot \nabla (\bar{M} \cdot \nabla \zeta). \tag{2.13}$$

Summing up the above analyses, we can see that, under the initial conditions (2.1) and (2.11), one can use the relation (2.13) to change (2.5) into the following Navier-Stokes system:

$$\begin{cases} \zeta_t = u, \\ \rho u_t - \mu \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} \tilde{p} = \lambda_0 \bar{M} \cdot \nabla (\bar{M} \cdot \nabla \zeta) + 2\rho\omega (u_2 e_1 - u_1 e_2), \\ \operatorname{div}_{\mathcal{A}} u = 0, \end{cases}$$

and B is defined by (2.12). Now, we introduce the shift functions

$$\eta = \zeta - y \quad \text{and} \quad q = \tilde{p} - \bar{p}. \tag{2.14}$$

Then the evolution equations for the shift functions (η, q) and u read as

$$\begin{cases} \eta_t = u, \\ \rho u_t - \mu \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} q - \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla \zeta) = 2\rho\omega(u_2 e_1 - u_1 e_2), \\ \operatorname{div}_{\mathcal{A}} u = 0, \end{cases} \tag{2.15}$$

where $\mathcal{A} = (I + \nabla\eta)^{-1}$, $I = (\delta_{ij})_{3\times 3}$. The associated initial and boundary conditions read as follows:

$$(\eta, u)|_{t=0} = (\eta_0, u_0), \qquad (\eta, u)|_{\partial\Omega} = 0. \tag{2.16}$$

It should be noted that the shift function q is the sum of the perturbed fluid and the magnetic pressures in Lagrangian coordinates. Hence we still call q the perturbation pressure for the sake of simplicity. In this article, we call the initial-boundary value problem (2.15)-(2.16) the transformed MHD problem.

2.2 Main results

Before stating our first main result on the transformed MHD problem in detail, we introduce some simplified notations that shall be used throughout this paper:

$$\mathbb{R}_0^+ := [0, \infty), \qquad \int := \int_\Omega, \qquad L^p := L^p(\Omega) := W^{0,p}(\Omega) \quad \text{for } 1 < p \le \infty,$$

$$H_0^1 := W_0^{1,2}(\Omega), \qquad H^k := W^{k,2}(\Omega), \qquad \|\cdot\|_k := \|\cdot\|_{H^k(\Omega)} \quad \text{for } k \ge 1,$$

$$\partial_h^k \text{ denotes } \partial_1^{\alpha_1}\partial_2^{\alpha_2} \quad \text{for any } \alpha_1 + \alpha_2 = k, \qquad \|\cdot\|_{m,k} := \sum_{\alpha_1+\alpha_2=m} \|\partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdot\|_k^2,$$

$$a \lesssim b \text{ means that } a \le cb,$$

where, and in what follows, the letter c denotes a generic constant which may depend on the domain Ω and some physical parameters, such as λ_0, $\bar{M}$, g, μ and ρ in the MHD equations (2.15). It should be noted that a product space $(X)^n$ of vector functions is still denoted by X, for example, a vector function $u \in (H^2)^3$ is denoted by $u \in H^2$ with norm $\|u\|_{H^2} := (\sum_{k=1}^3 \|u_k\|_{H^2}^2)^{1/2}$. Finally, we define some functionals:

$$\mathcal{E}^L := \|\nabla\eta\|_{3,0}^2 + \|(\eta, u)\|_3^2 + \|(u_t, \nabla q)\|_1^2,$$

$$\mathcal{D}^L := \|(\eta, \bar{M}\cdot\nabla\eta, \nabla u)\|_{3,0}^2 + \|(\eta, u)\|_3^2 + \sum_{k=1}^2 \|\partial_t^k u\|_{4-2k}^2 + \|\nabla q\|_1^2 + \|\nabla q_t\|_0^2,$$

$$\mathcal{E}^H := \|\nabla\eta\|_{6,0}^2 + \|\eta\|_6^2 + \sum_{k=0}^3 \|\partial_t^k u\|_{6-2k}^2 + \sum_{k=0}^2 \|\nabla\partial_t^k q\|_{4-2k}^2,$$

$$\mathcal{D}^H := \|(\eta, \bar{M}\cdot\nabla\eta, \nabla u)\|_{6,0}^2 + \|(\eta, u)\|_6^2 + \sum_{k=1}^3 \|\partial_t^k u\|_{7-2k}^2 + \sum_{k=1}^2 \|\nabla\partial_t^k q\|_{5-2k}^2 + \|\nabla q\|_4^2,$$

$$\mathcal{G}_1(t) = \sup_{0\le\tau<t} \|\eta(\tau)\|_7^2, \qquad \mathcal{G}_2(t) = \int_0^t \frac{\|(\eta, u)(\tau)\|_7^2}{(1+\tau)^{3/2}}\,\mathrm{d}\tau,$$

$$\mathcal{G}_3(t) = \sup_{0\le\tau<t} \mathcal{E}^H(\tau) + \int_0^t \mathcal{D}^H(\tau)\,\mathrm{d}\tau, \qquad \mathcal{G}_4(t) = \sup_{0\le\tau<t} (1+\tau)^3 \mathcal{E}^L(\tau).$$

Next, we introduce our main result.

Theorem 2.1 *Let* Ω *be a horizontally periodic domain with finite height,* ω *be an arbitrary real number, and* $m_3 \neq 0$. *Then there is a sufficiently small constant* $\delta > 0$, *such that for any* $(\eta_0, u_0) \in H^7 \times H^6$ *satisfying the following conditions*:

(1) $\|\eta_0\|_7^2 + \|u_0\|_6^2 \leq \delta$;
(2) $\zeta_0 := y + \eta_0$ *satisfies* (2.1);
(3) (η_0, u_0) *satisfies necessary compatibility conditions* (*i.e.*, $\partial_t^j u(x,0)|_{\partial\Omega} = 0$ *for* $j = 1$ *and* 2),

there exists a unique global solution $(\eta, u) \in C^0(\mathbb{R}_0^+, H^7 \times H^6)$ *to the transformed MHD problem* (2.15)-(2.16) *with an associated perturbation pressure* q. *Moreover,* (η, u, q) *enjoys the following stability estimate*:

$$\mathcal{G}(\infty) := \sum_{k=1}^{4} \mathcal{G}_k(\infty) \leq c\big(\|\eta_0\|_7^2 + \|u_0\|_6^2\big). \tag{2.17}$$

Here the positive constants δ *and* c *depend on the domain* Ω *and other known physical parameters* λ_0, $\bar{M}$, μ *and* ρ.

Remark 2.1 Exploiting the inverse transformation of ζ, we can easily deduce from Theorem 2.1 the global well-posedness of the original MHD problem (1.2)-(1.4). More precisely, there is a sufficiently small constant $\delta_0 > 0$, such that, for any $(v_0, N_0) \in H^6$ satisfying the following conditions:

(1) there exists an invertible mapping $\zeta_0 := \zeta_0(x) : \Omega \to \Omega$, such that (2.1) holds, where $\mathcal{A}_0^T = (\nabla\zeta_0)^{-1}$;
(2) $(\bar{M} + N_0)(\zeta_0) = \bar{M} \cdot \nabla\zeta_0$;
(3) $\|\zeta_0 - x\|_7^2 + \|v_0\|_6^2 \leq \delta_0$;
(4) the initial data ϱ_0, v_0, N_0 satisfy necessary compatibility conditions (*i.e.*, $\partial_t^j v(x,0)|_{\partial\Omega} = 0$ for $j = 1$ and 2),

there exists a unique global solution $(v, N) \in C^0(\mathbb{R}_0^+, H^6)$ to the original MHD problem (1.2)-(1.4) with an associated perturbation pressure $\tilde{q}$. Moreover, $(v, N, \tilde{q})$ enjoys the following stability estimate:

$$\begin{aligned} &\sup_{0\leq t<\infty}\left(\|N\|_6^2 + \sum_{k=0}^{3}\big\|\partial_t^k v(t)\big\|_{6-2k}^2 + \sum_{k=0}^{2}\big\|\nabla\partial_t^k\tilde{q}(t)\big\|_{4-2k}^2\right) \\ &\quad + \sup_{0\leq t<\infty}(1+t)^3\Big(\big\|(v,N)\big\|_3^2 + \big\|(v_t,\nabla\tilde{q})\big\|_1^2\Big) \leq c\big(\|\zeta_0 - x\|_7^2 + \|v_0\|_6^2\big). \end{aligned} \tag{2.18}$$

Now we briefly describe the basic idea in the proof of Theorem 2.1. By the standard energy method, there are two functionals $\tilde{\mathcal{E}}^L$ and Q of (η, u) satisfying the lower-order energy inequality (see Proposition 3.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^L + \mathcal{D}^L \leq \mathcal{Q}\mathcal{D}^L, \tag{2.19}$$

where the functional $\tilde{\mathcal{E}}^L$ is equivalent to $\mathcal{E}^L$. Unfortunately, we can not close the energy estimates only based on (2.19), since $\mathcal{Q}$ can not be controlled by $\tilde{\mathcal{E}}^L$. However, we observe that the structure of the energy inequality above is very similar to that of the surface wave problem [15], for which Guo and Tice developed a two-tier energy method to overcome this difficulty. In the spirit of the two-tier energy method, we look after a higher-order energy inequality to match the lower-order energy inequality (2.19). Since $\tilde{\mathcal{E}}^L$ contains $\|\eta\|_3$, we find that the higher-order energy at least includes $\|\eta\|_6$. Thus, similar to (2.19), we establish the higher-order energy inequality (see Proposition 4.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^H + \mathcal{D}^H \leq \sqrt{\mathcal{E}^L}\big\|(\eta,u)\big\|_7^2, \tag{2.20}$$

where the functional $\tilde{\mathcal{E}}^H$ is equivalent to $\mathcal{E}^H$. Moreover, the highest-order norm $\|(\eta,u)\|_7^2$ enjoys the highest-order energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{7,*}^2 + \big\|(\eta,u)\big\|_7^2 \lesssim \mathcal{E}^H + \mathcal{D}^H, \tag{2.21}$$

where the norm $\|\eta\|_{7,*}$ is equivalent to $\|\eta\|_7$. In the derivation of the *a priori* estimates, we have $\mathcal{Q} \lesssim \mathcal{E}^H$, and thus (2.19) implies (see Proposition 3.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^L + \mathcal{D}^L \leq 0. \tag{2.22}$$

Consequently, by the two-tier energy method, we can deduce the global-in-time stability estimate (2.17) based on (2.20)-(2.22).

The rest of the sections are mainly devoted to the proof of Theorem 2.1. In Section 3, we first derive the lower-order energy inequality (2.22) for the transformed MHD problem. Then in Section 4 we derive the higher-order energy inequality (2.20) and the highest-order energy inequality (2.21). Based on these three energy inequalities, we prove Theorem 2.1 by adapting the two-tier energy method in Section 5.

3 Lower-order energy inequality

In this section, we start to derive the lower-order energy inequality in the *a priori* estimates for the transformed MHD problem. To this end, let (η,u) be a solution of the transformed MHD problem with perturbed pressure q, such that

$$\sqrt{\mathcal{G}_1(T) + \sup_{0\leq\tau\leq T}\mathcal{E}^H(\tau)} \leq \delta \in (0,1) \quad \text{for some } T > 0, \tag{3.1}$$

where δ is sufficiently small. It should be noted that the smallness depends on the known physical parameters in (1.4), and will be repeatedly used in what follows. Moreover, we assume that the solution (η,u,q) possesses proper regularity, so that the formal calculation makes sense. We remind the reader that in the calculations, we shall repeatedly use Cauchy-Schwarz's inequality, Hölder's inequality, the embedding inequalities (see [16], 4.12 Theorem)

$$\|f\|_{L^p} \lesssim \|f\|_1 \quad \text{for } 2 \leq p \leq 6 \quad \text{and} \quad \|f\|_{L^\infty} \lesssim \|f\|_2, \tag{3.2}$$

and the interpolation inequalities (see [16], 5.2 Theorem)

$$\|f\|_j \lesssim \|f\|_0^{1-\frac{j}{i}}\|f\|_i^{\frac{j}{i}} \leq C_\epsilon\|f\|_0 + \epsilon\|f\|_{H^i} \tag{3.3}$$

for any $0 \leq j < i$ and any constant $\epsilon > 0$, where the constant C_ϵ depends on the domain Ω and ϵ. In addition, we shall also frequently use the following two estimates:

$$\|fg\|_j \lesssim \|f\|_j \|g\|_{\kappa(j)} \quad \text{for } j \geq 0 \tag{3.4}$$

and

$$\|f\|_0 \lesssim \|\partial_3 f\|_0 \quad \text{for } f \in H_0^1, \tag{3.5}$$

where $\kappa(j) = j$ for $j \geq 2$ and $\kappa(j) = 2$ for $j \leq 1$. We also introduce the following inequality, see (3.4) in [17]:

$$\|f\|_0 \leq h\|\bar{M} \cdot \nabla f\|_0 / \pi. \tag{3.6}$$

Before deriving the lower-order energy inequality defined on $(0, T]$, we first give some preliminary estimates, temporal derivative estimates, horizontal spatial estimates and Stokes estimates in sequence.

3.1 Preliminary estimates

In this subsection, we derive some preliminary estimates for $\mathcal{A}$. To begin with, we give an expression of $\mathcal{A}$. Using $(2.15)_1$, we have

$$\partial_t \det(I + \nabla\eta) = \sum_{1\leq i,j\leq 3} \partial_t \partial_j \eta_i A_{ij}^* = \sum_{1\leq i,j\leq 3} A_{ij}^* \partial_j u_i, \tag{3.7}$$

where A_{ij}^* is the algebraic complement minor of the (i,j)th entry in the matrix $I + \nabla\eta$. Recalling the definition of $\mathcal{A}$, we see that

$$\mathcal{A} = \left(A_{ij}^*\right)_{3\times 3} / \det(I + \nabla\eta).$$

Inserting this relation into (3.7), we get

$$\partial_t \det(I + \nabla\eta) = \det(I + \nabla\eta) \sum_{1\leq i,j,\leq 3} \mathcal{A}_{ij} \partial_j u_i = 0,$$

which, together with initial condition $\det(I + \nabla\eta_0) = 1$, implies

$$\det(I + \nabla\eta) = 1.$$

Thus we obtain

$$\mathcal{A} = \left(A_{ij}^*\right)_{3\times 3}. \tag{3.8}$$

Now, exploiting $(2.15)_1$, (3.1), (3.4) and (3.8), we easily see that

$$\begin{aligned} &\|\mathcal{A}\|_j \lesssim 1 + \|\eta\|_{j+1}\left(1 + \|\eta\|_{j+1}\right) \lesssim 1 \quad \text{for } 0 \leq j \leq 6, \\ &\|\nabla\mathcal{A}\|_j \leq \|\eta\|_{j+2}\left(1 + \|\eta\|_{j+1}\right) \lesssim \|\eta\|_{j+2} \quad \text{for } 0 \leq j \leq 5. \end{aligned} \tag{3.9}$$

Similarly, we further deduce that

$$\left\|\partial_t^i \mathcal{A}\right\|_j \lesssim \sum_{k=0}^{i-1}\left\|\partial_t^k \nabla u\right\|_j \quad \text{for any } 1 \le i \le 4 \text{ and } 0 \le j \le 8-2i.$$

Letting $\tilde{\mathcal{A}} := \mathcal{A} - I$, we next bound $\tilde{\mathcal{A}}$. To this end, we assume that δ is so small that the following expansion holds:

$$\mathcal{A}^T = I - \nabla\eta + (\nabla\eta)^2 \sum_{i=0}^{\infty}(-\nabla\eta)^i = I - \nabla\eta + (\nabla\eta)^2\mathcal{A}^T,$$

whence

$$\tilde{\mathcal{A}}^T = (\nabla\eta)^2\mathcal{A}^T - \nabla\eta.$$

Using (3.1), (3.4) and (3.9), we find that

$$\|\tilde{\mathcal{A}}\|_j \lesssim \|\nabla\eta\|_j \quad \text{for } 0 \le j \le 6.$$

3.2 Temporal derivative estimates

In this subsection, we try to control temporal derivatives. For this purpose, we apply ∂_t^j to (2.15) to get

$$\begin{cases} \partial_t^{j+1}\eta = \partial_t^j u, \\ \rho\partial_t^{j+1}u - \mu\Delta_{\mathcal{A}}\partial_t^j u + \nabla_{\mathcal{A}}\partial_t^j q \\ \quad = \lambda_0\partial_t^j\bar{M}\cdot\nabla(\bar{M}\cdot\nabla\zeta) + 2\rho\omega\partial_t^j(u_2e_1 - u_1e_2) + \mu N_u^{t,j} + N_q^{t,j}, \\ \operatorname{div}_{\mathcal{A}}\partial_t^j u = \operatorname{div} D_u^{t,j}, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} N_u^{t,j} &:= \sum_{0\le m<j, 0\le n\le j} \partial_t^{j-m-n}\mathcal{A}_{il}\partial_l\left(\partial_t^n\mathcal{A}_{ik}\partial_t^m\partial_k u\right), \\ N_q^{t,j} &:= -\sum_{0\le l<j}\left(\partial_t^{j-l}\mathcal{A}_{ik}\partial_t^l\partial_k q\right)_{3\times1}, \\ D_u^{t,j} &:= \left(-\sum_{0\le l<j} C_j^{j-l}\partial_t^{j-l}\mathcal{A}_{ki}\partial_t^l u_k\right)_{3\times1}, \end{aligned} \tag{3.11}$$

C_j^{j-l} denotes the number of $(j-l)$-combinations from a given set S of j elements,

and we have used relation (2.7) in (3.11). Then from (3.10) we show the following estimates:

Lemma 3.1 *It holds that for $j = 0$ and 1,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\sqrt{\rho}\partial_t^j u\right\|_0^2 + \lambda_0\left\|\bar{M}\cdot\nabla\partial_t^j\eta\right\|_0^2\right) + \mu\left\|\nabla_{\mathcal{A}}\partial_t^j u\right\|_0^2 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathcal{A}}u_t\|_0^2 + \frac{1}{\mu}\|\sqrt{\rho}u_{tt}\|_0^2 \lesssim \|u\|_2^2 + \|u_t\|_0^2 + \sqrt{\mathcal{E}^H}\mathcal{D}^L. \tag{3.13}$$

Proof (1) We only prove (3.12) for $j = 1$, since the derivation of the case $j = 0$ is similar. Multiplying $(3.10)_2$ with $j = 1$ by u_t, integrating (by parts) the resulting equality over Ω, and using $(3.10)_1$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\sqrt{\rho}u_t\|_0^2 + \lambda_0\|\bar{M}\cdot\eta_t\|_0^2\big) + \mu\|\nabla_{\mathcal{A}}u_t\|_0^2 \\
&\quad = 2\rho\omega\int \partial_t(u_2e_1 - u_1e_2)\cdot u_t\,\mathrm{d}y + \int q_t \,\mathrm{div}_{\mathcal{A}}\, u_t\,\mathrm{d}y \\
&\qquad + \mu\int N_u^{t,1}\cdot u_t\,\mathrm{d}y + \int N_q^{t,1}\cdot u_t\,\mathrm{d}y \\
&\quad := \sum_{k=1}^{4} I_k^L.
\end{aligned} \tag{3.14}$$

The last three integrals $I_2^L,\dots,I_4^L$ can be estimated as follows:

$$\begin{aligned}
I_2^L &:= -\int \nabla q_t\cdot D_u^{t,1}\,\mathrm{d}y \lesssim \|\nabla q_t\|_0\|\mathcal{A}_t\|_{L^4}\|u\|_{L^4} \lesssim \|\nabla q_t\|_0\|\mathcal{A}_t\|_1\|u\|_1 \\
&\lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L,
\end{aligned} \tag{3.15}$$

$$I_3^L \lesssim \|\mathcal{A}\|_2\|\mathcal{A}_t\|_2\|u\|_2\|u_t\|_0 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.16}$$

$$I_4^L \lesssim \|\mathcal{A}_t\|_0\|\nabla q\|_{L^\infty}\|u_t\|_0 \lesssim \|\mathcal{A}_t\|_0\|\nabla q\|_2\|u_t\|_0 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.17}$$

where we have used $(3.10)_3$ in (3.15). Consequently, the desired estimate (3.12) follows from (3.15)-(3.17).

(2) Now we turn to the proof of (3.13). Multiplying $(3.10)_2$ with $j = 1$ by u_{tt}, integrating (by parts) the resulting equality over Ω, and using $(3.10)_1$, we conclude

$$\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathcal{A}}u_t\|_0^2 + \|\sqrt{\rho}u_{tt}\|_0^2 \\
&\quad = \int \big(2\rho\omega\partial_t(u_2e_1 - u_1e_2)\cdot u_{tt} + \lambda_0\bar{M}\cdot\nabla(\bar{M}\cdot\nabla u)\cdot u_{tt}\big)\,\mathrm{d}y \\
&\qquad + \int q_t\,\mathrm{div}_{\mathcal{A}}\, u_{tt}\,\mathrm{d}y + \mu\int N_u^{t,1}\cdot u_{tt}\,\mathrm{d}y + \int N_q^{t,1}\cdot u_{tt}\,\mathrm{d}y \\
&\qquad + \mu\int \nabla_{\mathcal{A}}u_t : \nabla_{\mathcal{A}_t}u_t\,\mathrm{d}y \\
&\quad := \sum_{k=1}^{5} J_k^H.
\end{aligned} \tag{3.18}$$

On the other hand, the five integrals $J_1^H,\dots,J_5^H$ can be bounded as follows:

$$J_1^H \lesssim \big(\|u_t\|_0 + \|u\|_2\big)\|u_{tt}\|_0, \tag{3.19}$$

$$\begin{aligned}
J_2^H &= -\int \nabla q_t\cdot D_u^{t,2}\,\mathrm{d}x \lesssim \|\nabla q_t\|_0\big(\|\mathcal{A}_{tt}\|_0\|u\|_2 + \|\mathcal{A}_t\|_2\|u_t\|_0\big) \\
&\lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L,
\end{aligned} \tag{3.20}$$

$$J_3^H \lesssim \|\mathcal{A}\|_2\|\mathcal{A}_t\|_2\|u\|_2\|u_{tt}\|_0 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.21}$$

$$J_4^H \lesssim \|\mathcal{A}_t\|_0 \|\nabla q\|_2 \|u_{tt}\|_0 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.22}$$

$$J_5^H \lesssim \|\mathcal{A}\|_2 \|\mathcal{A}_t\|_2 \|u_t\|_1^2 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L. \tag{3.23}$$

Thus, substituting (3.19)-(3.23) into (3.18) and using Cauchy-Schwarz's inequality, we immediately get (3.13). □

3.3 Horizontal spatial estimates

In this subsection, we establish the estimates of horizontal spatial derivatives. For this purpose, we rewrite (2.15) as the following non-homogeneous linear form:

$$\begin{cases} \eta_t = u, \\ \rho u_t - \mu\Delta u + \nabla q - \lambda_0 \bar{M}\cdot\nabla(\bar{M}\cdot\nabla\eta) = 2\rho\omega(u_2e_1 - u_1e_2) + \mu N_u^h + N_q^h, \\ \operatorname{div} u = D_u^h, \end{cases} \tag{3.24}$$

where

$$N_u^h := \partial_l\big[(\tilde{\mathcal{A}}_{jl}\tilde{\mathcal{A}}_{jk} + \tilde{\mathcal{A}}_{lk} + \tilde{\mathcal{A}}_{kl})\partial_k u_i\big]_{3\times 1},$$

$$N_q^h := -(\tilde{\mathcal{A}}_{ik}\partial_k q)_{3\times 1} \quad \text{and} \quad D_u^h := \tilde{\mathcal{A}}_{lk}\partial_k u_l.$$

Then we have the following estimate on horizontal spatial derivatives of η.

Lemma 3.2 *It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho\partial_h^j\eta\cdot\partial_h^j u\,\mathrm{d}y + \frac{\mu}{2}\big\|\nabla\partial_h^j\eta\big\|_0^2\right) + \lambda_0\big\|\bar{M}\cdot\nabla\partial_h^j\eta\big\|_0^2$$
$$\lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L + \|u\|_{j,0}^2, \quad 0 \le j \le 3.$$

Proof We only show the case $j = 3$; the remaining three cases can be verified similarly. Applying ∂_h^3 to $(3.24)_2$, multiplying the resulting equation by $\partial_h^3\eta$, and then using $(3.24)_1$, we get

$$\rho\partial_t\big(\partial_h^3\eta\cdot\partial_h^3 u\big) - \big(\mu\Delta\partial_h^3\eta_t + \lambda_0\bar{M}\cdot\nabla\big(\bar{M}\cdot\nabla\partial_h^3\eta\big)\big)\cdot\partial_h^3\eta$$
$$= \big(2\rho\omega\partial_h^3(u_2e_1 - u_1e_2) + \mu\partial_h^3N_u^h + \partial_h^3N_q^h - \nabla\partial_h^3 q\big)\cdot\partial_h^3\eta + \rho\big|\partial_h^3 u\big|^2.$$

If we integrate (by parts) the above identity over Ω, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho\partial_h^3\eta\cdot\partial_h^3 u\,\mathrm{d}y + \frac{\mu}{2}\big\|\nabla\partial_h^3\eta\big\|_0^2\right) + \lambda_0\big\|\bar{M}\cdot\nabla\partial_h^3\eta\big\|_0^2 \\ &\quad = 2\rho\omega\int \partial_h^3(u_2e_1 - u_1e_2)\cdot\partial_h^3\eta\,\mathrm{d}y + \int\big(\mu\partial_h^3N_u^h + \partial_h^3N_q^h\big)\cdot\partial_h^3\eta\,\mathrm{d}y \\ &\qquad + \int\partial_h^3 q\operatorname{div}\partial_h^3\eta\,\mathrm{d}y + \big\|\sqrt{\rho}\partial_h^3 u\big\|_0^2 \\ &\quad \lesssim \sum_{k=1}^3 K_k^L + \|u\|_{3,0}^2, \end{aligned} \tag{3.25}$$

where the first three integrals on the right-hand side of the first equality in (3.25) are denoted by K_1^L, K_2^L and K_3^L, respectively.

We have the boundedness

$$K_1^L \leq \left\| \partial_h^3 u \right\|_0 \left\| \partial_h^3 \eta \right\|_0. \tag{3.26}$$

Noting that

$$\begin{aligned} \left\| \mu \partial_h^3 N_u^h + \partial_h^3 N_q^h \right\|_0 &\lesssim \|\tilde{\mathcal{A}}\|_4 \|u\|_3 + \|\tilde{\mathcal{A}}\|_2 \|u\|_5 + \|\tilde{\mathcal{A}}\|_2 \|\nabla q\|_3 + \|\tilde{\mathcal{A}}\|_4 \|\nabla q\|_1 \\ &\lesssim \sqrt{\mathcal{E}^H \mathcal{D}^L}, \end{aligned}$$

we find that

$$K_2^L \lesssim \left\| \mu \partial_h^3 N_u^h + \partial_h^3 N_q^h \right\|_0 \left\| \partial_h^3 \eta \right\|_0 \lesssim \sqrt{\mathcal{E}^H} \mathcal{D}^L. \tag{3.27}$$

Next we estimate the third integral K_3^L. To start with, we analyze the property of $\operatorname{div} \eta$. Since

$$\det(I + \nabla \eta) = 1,$$

we have by Sarrus' rule

$$\begin{aligned} \operatorname{div} \eta = {} & \partial_1 \eta_2 \partial_2 \eta_1 + \partial_2 \eta_3 \partial_3 \eta_2 + \partial_3 \eta_1 \partial_1 \eta_3 - \partial_1 \eta_1 \partial_2 \eta_2 - \partial_1 \eta_1 \partial_3 \eta_3 - \partial_2 \eta_2 \partial_3 \eta_3 \\ & + \partial_1 \eta_1 (\partial_2 \eta_3 \partial_3 \eta_2 - \partial_2 \eta_2 \partial_3 \eta_3) + \partial_2 \eta_1 (\partial_1 \eta_2 \partial_3 \eta_3 - \partial_1 \eta_3 \partial_3 \eta_2) \\ & + \partial_3 \eta_1 (\partial_1 \eta_3 \partial_2 \eta_2 - \partial_1 \eta_2 \partial_2 \eta_3). \end{aligned}$$

Multiplying the above identity by a smooth test function ϕ, and then integrating (by parts) the resulting identity over Ω, we derive that

$$\int \phi \operatorname{div} \eta \, \mathrm{d}y = -\int \nabla \phi \cdot \psi \, \mathrm{d}y,$$

where

$$\psi := \begin{pmatrix} \eta_1 (\partial_2 \eta_2 + \partial_3 \eta_3) + \eta_1 (\partial_2 \eta_3 \partial_3 \eta_2 - \partial_2 \eta_2 \partial_3 \eta_3) \\ \eta_2 \partial_3 \eta_3 - \eta_1 \partial_1 \eta_2 + \eta_1 (\partial_1 \eta_2 \partial_3 \eta_3 - \partial_1 \eta_3 \partial_3 \eta_2) \\ -\eta_1 \partial_1 \eta_3 - \eta_2 \partial_2 \eta_3 + \eta_1 (\partial_1 \eta_3 \partial_2 \eta_2 - \partial_1 \eta_2 \partial_2 \eta_3) \end{pmatrix}.$$

This means that

$$\operatorname{div} \eta = \operatorname{div} \psi.$$

Thus, it follows immediately that

$$\begin{aligned} K_3^L &= -\int \partial_h^3 \nabla q \cdot \partial_h^3 \psi \, \mathrm{d}y = -\int \partial_h \nabla q \cdot \partial_h^5 \psi \, \mathrm{d}y \\ &\lesssim \|\nabla q\|_1 \left\| \partial_h^5 \psi \right\|_0 \lesssim \|\eta\|_6 \|\nabla q\|_1 \|\eta\|_3 \lesssim \sqrt{\mathcal{E}^H} \mathcal{D}^L. \end{aligned} \tag{3.28}$$

Now, substituting (3.26), (3.27) and (3.28) into (3.25), we immediately obtain the desired estimate for the case $j = 3$. □

Similarly, we also establish the following estimates of horizontal spatial derivatives of u:

Lemma 3.3 *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\sqrt{\rho}\partial_h^j u\right\|_0^2 + \lambda_0\left\|\bar{M}\cdot\nabla\partial_h^j\eta\right\|_0^2\right) + \mu\left\|\nabla\partial_h^j u\right\|_0^2 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \quad j = 1,2,3.$$

Proof We only prove the case $j = 3$; the remaining two cases can be shown similarly. Applying ∂_h^3 to $(3.24)_2$, and multiplying the resulting equality by $\partial_h^3 u$, we make use of $(3.24)_1$ to get

$$\begin{aligned}&\rho\partial_h^3 u_t\cdot\partial_h^3 u - \mu\Delta\partial_h^3 u\cdot\partial_h^3 u - \lambda_0\bar{M}\cdot\nabla\left(\bar{M}\cdot\nabla\partial_h^3\eta\right)\cdot\partial_h^3\eta_t\\ &\quad = \left(2\rho\omega\partial_h^3(u_2e_1 - u_1e_2) + \mu\partial_h^3 N_u^h + \partial_h^3 N_q^h - \nabla\partial_h^3 q\right)\cdot\partial_h^3 u.\end{aligned}$$

Integrating (by parts) the above identity over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\rho\left|\partial_h^3 u\right|^2\mathrm{d}y + \lambda_0\left\|\bar{M}\cdot\nabla\partial_h^3\eta\right\|_0^2\right) + \frac{\mu}{2}\left\|\nabla\partial_h^3 u\right\|_0^2\\ &\quad = \int\left(\mu\partial_h^3 N_u^h + \partial_h^3 N_q^h\right)\cdot\partial_h^3 u\,\mathrm{d}y + \int\partial_h^3 q\,\mathrm{div}\,\partial_h^3 u\,\mathrm{d}y =: M_1^L + M_2^L.\end{aligned} \tag{3.29}$$

On the other hand, similarly to (3.27) and (3.28), the two integrals M_1^L and M_2^L can be estimated as follows:

$$M_1^L \lesssim \left\|\mu\partial_h^3 N_u^h + \partial_h^3 N_q^h\right\|_0\left\|\partial_h^3 u\right\|_0 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.30}$$

$$M_2^L = -\int\partial_h^4 q\,\partial_h^2 D_u^h\,\mathrm{d}y \lesssim \|\nabla q\|_3\|\tilde{\mathcal{A}}\|_2\|u\|_3 \lesssim \sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.31}$$

where we have used $(3.24)_3$ in (3.31). Consequently, putting the above two estimates into (3.29), we obtain Lemma 3.3 for the case $j = 3$. □

3.4 Stokes problem and stability condition

In this subsection, we use the regularity theory of the Stokes problem to derive more estimates of (η, u). To this end, we rewrite $(3.24)_2$ and $(3.24)_3$ as the following Stokes problem:

$$\begin{cases}-\Delta w + \nabla q\\ \quad = \lambda_0\bar{M}\cdot\nabla(\bar{M}\cdot\nabla\eta) - \lambda_0 m_3^2\Delta\eta + 2\rho\omega(u_2e_1 - u_1e_2) - \rho u_t + \mu N_u^h + N_q^h,\\ \mathrm{div}\,w = \mu D_u^h + \lambda_0 m_3^2\,\mathrm{div}\,\eta,\end{cases} \tag{3.32}$$

coupled with boundary condition

$$\omega|_{\partial\Omega} = 0, \tag{3.33}$$

where $\omega = \lambda_0 m_3^2\eta + \mu u$.

Now, applying ∂_h^k to (3.32) and (3.33), we get

$$\begin{cases} -\Delta \partial_h^k w + \nabla \partial_h^k q \\ \quad = \partial_h^k(\lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla\eta) - \lambda_0 m_3^2 \Delta\eta + 2\rho\omega(u_2e_1 - u_1e_2)) - \rho\partial_h^k u_t + \mu\partial_h^k N_u^h + N_q^h, \\ \operatorname{div} \partial_h^k w = \mu\partial_h^k D_u^h + \lambda_0 m_3^2 \operatorname{div} \partial_h^k \eta, \\ \partial_h^k \omega|_{\partial\Omega} = 0. \end{cases}$$

Then we apply the classical regularity theory to the Stokes problem as in [18], Proposition 2.3, to deduce that

$$\|\omega\|_{k,i-k+2}^2 + \|\nabla q\|_{k,i-k}^2 \lesssim \|\nabla\eta\|_{k+1,i-k}^2 + \|(u,u_t)\|_{k,i-k}^2 + S_{k,i}^{\omega}, \tag{3.34}$$

where

$$S_{k,i}^{\omega} := \|(N_u^h, N_q^h)\|_{k,i-k}^2 + \|(D_u^h, \operatorname{div}\eta)\|_{k,i-k+1}^2.$$

In addition, applying ∂_t^k to $(3.24)_2$-$(3.24)_3$, we see that

$$\begin{cases} -\mu\Delta\partial_t^k u + \nabla\partial_t^k q \\ \quad = \lambda_0 \bar{M} \cdot \partial_t^k \nabla(\bar{M} \cdot \nabla\eta) + 2\rho\omega\partial_t^k(u_2e_1 - u_1e_2) - \rho\partial_t^{k+1} u + \mu\partial_t^k N_u^h + \partial_t^k N_q^h, \\ \operatorname{div} \partial_t^k u = \partial_t^k D_u^h, \\ \partial_t^k u|_{\partial\Omega} = 0. \end{cases}$$

Hence, we apply again the classical regularity theory to the Stokes problem to get

$$\|\partial_t^k u\|_{i-2k+2}^2 + \|\nabla\partial_t^k q\|_{i-2k}^2 \lesssim \|\partial_t^k(\nabla^2\eta, u, u_t)\|_{i-2k}^2 + S_{k,i}^u, \tag{3.35}$$

where $S_{k,i}^u := \|\partial_t^k(N_u^h, N_q^h)\|_{i-2k}^2 + \|\partial_t^k D_u^h\|_{i-2k+1}^2$. As a result of (3.34) and (3.35), one has the following estimates.

Lemma 3.4 *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{3,*}^2 + c(\|(\eta,u)\|_3^2 + \|\nabla q\|_1^2) \lesssim \|(u,u_t)\|_1^2 + \|\eta\|_{2,1}^2 + \mathcal{E}^H\mathcal{D}^L, \tag{3.36}$$

$$\|u\|_3^2 + \|\nabla q\|_1^2 \lesssim \|\eta\|_3^2 + \|(u,u_t)\|_1^2 + \mathcal{E}^H E^L, \tag{3.37}$$

$$\|u_t\|_2^2 + \|\nabla q_t\|_0^2 \lesssim \|u\|_2^2 + \|(u_t,u_{tt})\|_0^2 + \mathcal{E}^H\mathcal{D}^L, \tag{3.38}$$

where $E^L := \mathcal{E}^L - \|\nabla\eta\|_{3,0}^2$ *and* $\|\eta\|_{3,*}$ *is equivalent to* $\|\eta\|_3$.

Proof Noting that, by virtue of $(2.15)_1$,

$$\|\omega\|_{k,i-k+2}^2 = \|(\lambda_0 m_3^2\eta, \mu u)\|_{k,i-k+2}^2 + \frac{\lambda_0 m_3^2\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{k,i-k+2}^2,$$

we deduce from (3.34) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{k,i-k+2}^2 + c(\|(\eta,u)\|_{k,i-k+2}^2 + \|\nabla q\|_{k,i-k}^2) \\ &\quad \lesssim \|u\|_{k,i-k}^2 + \|\eta\|_{k+1,i-k+1}^2 + \|u_t\|_i^2 + S_{k,i}^{\omega}. \end{aligned} \tag{3.39}$$

In particular, we take $(i,k)=(1,0)$ and $(i,k)=(1,1)$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_3^2+c\big(\|(\eta,u)\|_3^2+\|\nabla q\|_1^2\big)\lesssim\|(u,u_t)\|_1^2+\|\eta\|_{1,2}^2+S_{0,1}^{\omega}\tag{3.40}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{1,2}^2+c\big(\|(\eta,u)\|_{1,2}^2+\|\nabla q\|_{1,0}^2\big)\lesssim\|(u,u_t)\|_1^2+\|\eta\|_{2,1}^2+S_{1,1}^{\omega}.\tag{3.41}$$

On the other hand, it is easy to show that

$$\begin{aligned}S_{0,1}^{\omega}+S_{1,1}^{\omega}&\le\big\|\big(N_u^h,N_q^h\big)\big\|_1^2+\big\|\big(D_u^h,\operatorname{div}\eta\big)\big\|_2^2\lesssim\|\tilde{\mathcal{A}}\|_2^2\big(\|u\|_3^2+\|\nabla q\|_1^2\big)+\|\eta\|_3^4\\&\lesssim\mathcal{E}^H E^L\lesssim\mathcal{E}^H\mathcal{D}^L.\end{aligned}\tag{3.42}$$

Thus we immediately obtain (3.36) from (3.40)-(3.42).

Now we turn to the derivation of (3.37). In view of (3.35) with $(i,k)=(1,0)$, we have

$$\|u\|_3^2+\|\nabla q\|_1^2\lesssim\|\eta\|_3^2+\|(u,u_t)\|_1^2+S_{0,1}^u.$$

On the other hand, we can use (3.42) to infer that

$$S_{0,1}^u=\big\|\big(N_u^h,N_q^h\big)\big\|_1^2+\big\|D_u^h\big\|_2^2\lesssim\mathcal{E}^H E^L.$$

Hence, (3.37) follows from the above two estimates.

Finally, to show (3.38), we take $(i,k)=(2,1)$ in (3.35) to deduce that

$$\|u_t\|_2^2+\|\nabla q_t\|_0^2\lesssim\big\|\big(\nabla^2 u,u_t,u_{tt}\big)\big\|_0^2+S_{1,2}^u\lesssim\|u\|_2^2+\|(u_t,u_{tt})\|_0^2+S_{1,2}^u.$$

Keeping in mind that

$$\begin{aligned}S_{1,2}^u&\lesssim\big\|\partial_t\big(N_u^h,N_q^h\big)\big\|_0^2+\big\|\partial_t D_u^h\big\|_1^2\\&\lesssim\|\tilde{\mathcal{A}}_t\|_2^2\big(\|u\|_2^2+\|\nabla q\|_0^2\big)+\|\tilde{\mathcal{A}}\|_2^2\big(\|u_t\|_2^2+\|\nabla q_t\|_0^2\big)\lesssim\mathcal{E}^H\mathcal{D}^L,\end{aligned}$$

we get (3.38) from the above two estimates. □

3.5 Lower-order energy inequality

Now, we are able to build the lower-order energy inequality. In what follows, the letters c_i^L and $i=1,\dots,7$ will denote generic positive constants which may depend on the domain Ω and some physical parameters in the transformed MHD equations (2.15).

Proposition 3.1 *Under the assumption* (3.1), *if δ is sufficiently small, then there is an energy functional $\tilde{\mathcal{E}}^L$ which is equivalent to $\mathcal{E}^L$, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^L+\mathcal{D}^L\le 0\quad\text{on }(0,T].\tag{3.43}$$

Proof We choose δ so small that

$$\left\|\nabla\partial_t^j u\right\|_0^2 \lesssim \left\|\nabla_{\mathcal{A}}\partial_t^j u\right\|_0^2, \quad 1 \le j \le 3. \tag{3.44}$$

Then, thanks to (3.5), (3.6) and (3.44), we deduce from (3.12) and Lemmas 3.2-3.3 that there are constants c_1^L, c_2^L and $\sigma_0 \ge 1$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_1^L + \tilde{\mathcal{D}}_1^L \le c_1^L(1+\sigma)\sqrt{\mathcal{E}^H}\mathcal{D}^L \quad \text{for any } \sigma \ge \sigma_0, \tag{3.45}$$

where σ_0 depends on the domain and the known physical parameters, and

$$\begin{aligned}
\tilde{\mathcal{E}}_1^L &:= \|\sqrt{\rho}u_t\|_0^2 + \lambda_0\|\bar{M}\cdot\nabla u\|_0^2 + \sigma\sum_{k=0}^{3}\left(\|\sqrt{\rho}u\|_{k,0}^2 + \lambda_0\|\bar{M}\cdot\nabla\eta\|_{k,0}^2\right) \\
&\quad + \sum_{k=0}^{3}\left(\sum_{\alpha_1+\alpha_2=k}\int\rho\partial_1^{\alpha_1}\partial_2^{\alpha_2}\eta\cdot\partial_1^{\alpha_1}\partial_2^{\alpha_2}u\,\mathrm{d}y + \frac{\mu}{2}\|\nabla\eta\|_{k,0}^2\right), \\
\tilde{\mathcal{D}}_1^L &:= c_2^L\left(\|u_t\|_1^2 + \sum_{k=0}^{3}\left(\left\|(\eta,\bar{M}\cdot\nabla\eta)\right\|_{k,0}^2 + \|u\|_{k,1}^2\right)\right).
\end{aligned}$$

Utilizing (3.13), (3.36), the interpolation inequality, and the estimate

$$\|\eta\|_{k,1}^2 \lesssim \|\eta\|_{k+1,0}^2 + \|\bar{M}\cdot\nabla\eta\|_{k,0}^2 \quad \text{for any } 0 \le k \le 6, \tag{3.46}$$

we find that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\eta\|_{3,*}^2 + c_3^L\|\nabla_{\mathcal{A}}u_t\|_0^2\right) + c_4^L\left(\left\|(\eta,u)\right\|_3^2 + \|\nabla q\|_1^2 + \|u_{tt}\|_0^2\right) \\
&\quad \le c_5^L\left(\left\|(u,u_t)\right\|_1^2 + \|\eta\|_{3,0}^2 + \|\bar{M}\cdot\nabla\eta\|_{2,0}^2 + \sqrt{\mathcal{E}^H}\mathcal{D}^L\right).
\end{aligned} \tag{3.47}$$

Now, multiplying (3.47) by $c_2^L/(2c_5^L)$ and adding the resulting inequality to (3.45), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_2^L + \tilde{\mathcal{D}}^L \le c_6^L(1+\sigma)\sqrt{\mathcal{E}^H}\mathcal{D}^L, \tag{3.48}$$

where $\tilde{\mathcal{E}}_2^L$ and $\tilde{\mathcal{D}}^L$ are defined by

$$\begin{aligned}
\tilde{\mathcal{E}}_2^L &:= \tilde{\mathcal{E}}_1^L + c_2^L\left(\|\eta\|_{3,*}^2 + c_3^L\|\nabla_{\mathcal{A}}u_t\|_0^2\right)/\left(2c_5^L\right), \\
\tilde{\mathcal{D}}^L &:= \tilde{\mathcal{D}}_1^L/2 + c_2^Lc_4^L\left(\left\|(\eta,u)\right\|_3^2 + \|\nabla q\|_1^2 + \|u_{tt}\|_0^2\right)/\left(2c_5^L\right).
\end{aligned}$$

On the other hand, from (3.38) we get $\mathcal{D}^L \lesssim \tilde{\mathcal{D}}^L + \sqrt{\mathcal{E}^H}\mathcal{D}^L$, which implies $\mathcal{D}^L \lesssim \tilde{\mathcal{D}}^L$ for sufficiently small δ. Therefore, (3.48) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_2^L + c_7^L\mathcal{D}^L \le c_6^L(1+\sigma)\sqrt{\mathcal{E}^H}\mathcal{D}^L. \tag{3.49}$$

Next we show that $\mathcal{E}^L$ can be controlled by $\tilde{\mathcal{E}}_2^L$. By virtue of Cauchy-Schwarz's inequality, (3.5) and (3.44), there is an appropriately large constant σ, depending on ρ, μ and Ω, such that

$$\|\nabla\eta\|_{3,0}^2 + \|\eta\|_3^2 + \|u_t\|_1^2 \lesssim \tilde{\mathcal{E}}_2^L.$$

In view of (3.37), we further have

$$\mathcal{E}^L \lesssim \tilde{\mathcal{E}}_2^L + \sqrt{\mathcal{E}^H}\mathcal{E}^L,$$

which implies $\mathcal{E}^L \lesssim \tilde{\mathcal{E}}_2^L$ for sufficiently small δ. In addition, obviously $\tilde{\mathcal{E}}_2^L \lesssim \mathcal{E}^L$. Hence, the energy functional $\tilde{\mathcal{E}}_2^L$ is equivalent to $\mathcal{E}^L$. Finally, letting $\delta \le c_7^L/2c_6^L(1+\sigma)$ and noting $\tilde{\mathcal{E}}^L = 2\tilde{\mathcal{E}}_2^L/c_7^L$, we see that (3.49) immediately implies (3.43). Obviously, the energy functional $\tilde{\mathcal{E}}^L$ is still equivalent to $\mathcal{E}^L$. This completes the proof. □

4 Higher-order and highest-order energy inequalities

In this section, we derive the estimates (2.20) and (2.21) for the transformed magnetic RT problem. First we shall establish the higher-order version of Lemmas 3.1-3.4 in the following:

Lemma 4.1 *We have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\big\|\sqrt{\rho}\partial_t^j u\big\|_0^2 + \lambda_0\big\|\bar{M}\cdot\nabla\partial_t^{j-1}u\big\|_0^2 + \int \nabla\partial_t^{j-1}q\cdot D_u^{t,j}\,\mathrm{d}y\right) + \mu\big\|\nabla_{\mathcal{A}}\partial_t^j u\big\|_0^2\\&\quad\lesssim \sqrt{\mathcal{E}^L}\mathcal{D}^H, \quad j=2,3.\end{aligned}$$

Proof We only prove the case $j=3$, and the case $j=2$ can be shown similarly. Multiplying $(3.10)_2$ with $j=3$ by $\partial_t^3 u$, integrating (by parts) the resulting equation over Ω, and using $(3.10)_1$ and $(3.10)_3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\sqrt{\rho}\partial_t^3 u\big\|_0^2 + \lambda_0\|\bar{M}\cdot\nabla u_{tt}\|_0^2\big) + \mu\big\|\nabla_{\mathcal{A}}\partial_t^3 u\big\|_0^2\\&\quad= \int \partial_t^3 q\,\mathrm{div}\,D_u^{t,3}\,\mathrm{d}y + \mu\int N_u^{t,3}\cdot\partial_t^3 u\,\mathrm{d}y + \int N_q^{t,3}\cdot\partial_t^3 u\,\mathrm{d}y := \sum_{k=1}^3 I_k^H. \qquad (4.1)\end{aligned}$$

On the other hand, the integrals I_1^H, I_2 and I_3^H can be estimated as follows:

$$\begin{aligned}I_1^H &= -\int \nabla\partial_t^3 q\cdot D_u^{t,3}\,\mathrm{d}y = -\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla\partial_t^2 q\cdot D_u^{t,3}\,\mathrm{d}y + \int \nabla\partial_t^2 q\cdot\partial_t D_u^{t,3}\,\mathrm{d}y\\&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla\partial_t^2 q\cdot D_u^{t,3}\,\mathrm{d}y\\&\quad + c\big\|\nabla\partial_t^2 q\big\|_0\big(\big\|\partial_t^4\mathcal{A}\big\|_0\|u\|_2 + \big\|\partial_t^3\mathcal{A}\big\|_1\|u_t\|_1 + \|\mathcal{A}_{tt}\|_0\|u_{tt}\|_2 + \|\mathcal{A}_t\|_1\big\|\partial_t^3 u\big\|_1\big)\\&\le -\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla\partial_t^2 q\cdot D_u^{t,3}\,\mathrm{d}y + c\sqrt{\mathcal{E}^L}\mathcal{D}^H,\\I_2^H &\lesssim \big[\big\|\partial_t^3\mathcal{A}\big\|_0\|\mathcal{A}\|_2\|u\|_3 + \|\mathcal{A}_{tt}\|_0\big(\|\mathcal{A}_t\|_2\|u\|_3 + \|\mathcal{A}\|_2\|u_t\|_3\big)\\&\quad + \|\mathcal{A}_t\|_2\big(\|\mathcal{A}_{tt}\|_2\|u\|_2 + \|\mathcal{A}_t\|_2\|u_t\|_2 + \|\mathcal{A}\|_2\|u_{tt}\|_2\big)\\&\quad + \|\mathcal{A}\|_2\big(\big\|\partial_t^3\mathcal{A}\big\|_1\|u\|_2 + \|\mathcal{A}_{tt}\|_2\|u_t\|_1 + \|\mathcal{A}_{tt}\|_0\|u_t\|_3 + \|\mathcal{A}_t\|_1\|u_{tt}\|_2\big)\big]\big\|\partial_t^3 u\big\|_1\\&\lesssim \sqrt{\mathcal{E}^L}\mathcal{D}^H,\end{aligned}$$

$$I_3^H \lesssim \left(\left\|\partial_t^3 \mathcal{A}\right\|_0 \|\nabla q\|_1 + \|\mathcal{A}_{tt}\|_0 \|\nabla q_t\|_1 + \|\mathcal{A}_t\|_1 \|\nabla q_{tt}\|_0\right) \left\|\partial_t^3 u\right\|_1 \lesssim \sqrt{\mathcal{E}^L} \mathcal{D}^H.$$

Inserting the above three inequalities into (4.1), we get the desired conclusion immediately. □

Lemma 4.2 *We have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho \partial_h^k \eta \cdot \partial_h^k u \,\mathrm{d}y + \frac{\mu}{2}\left\|\nabla \partial_h^k \eta\right\|_0^2\right) + \lambda_0 \left\|\bar{M} \cdot \nabla \partial_h^k \eta\right\|_0^2 \\
&\quad \lesssim \sqrt{\mathcal{E}^L}\left(\left\|(\eta, u)\right\|_7^2 + \mathcal{D}^H\right) + \left\|\partial_h^k u\right\|_0^2 \quad \textit{for any } k = 5 \textit{ and } 6.
\end{aligned}$$

Proof We only show the case $k = 6$, since the derivation of the case $k = 5$ is similar. Since the derivation involves the norm $\|\nabla q\|_5$, we first estimate $\|\nabla q\|_5$. It follows from (3.35) that

$$\|u\|_7 + \|\nabla q\|_5 \lesssim \|\eta\|_7 + \left\|(u, u_t)\right\|_5 + \left\|\left(N_u^h, N_q^h\right)\right\|_5 + \left\|D_u^h\right\|_6.$$

The last two terms on the right-hand side of the above inequality can be bounded as follows:

$$\left\|\left(N_u^h, N_q^h\right)\right\|_5 + \left\|D_u^h\right\|_6 \lesssim \|\tilde{\mathcal{A}}\|_6 \|u\|_7 + \|\tilde{\mathcal{A}}\|_5 \|\nabla q\|_5.$$

Hence, if δ is sufficiently small, then one gets from the above two estimates that

$$\|u\|_7 + \|\nabla q\|_5 \lesssim \|\eta\|_7 + \sqrt{\mathcal{D}^H}. \tag{4.2}$$

Now, applying ∂_h^6 to $(3.24)_2$, multiplying the resulting equality by $\partial_h^6 \eta$, we utilize $(3.24)_1$ to have

$$\begin{aligned}
&\rho \partial_t\left(\partial_h^6 \eta \cdot \partial_h^6 u\right) - \left(\mu \Delta \partial_h^6 \eta_t + \lambda_0 \bar{M} \cdot \nabla\left(\bar{M} \cdot \nabla \partial_h^6 \eta\right)\right) \cdot \partial_h^6 \eta \\
&\quad = \partial_h^6\left(2\rho\omega(u_2 e_1 - u_1 e_2) + \mu N_u^h + N_q^h - \nabla q\right) \cdot \partial_h^6 \eta + \rho\left|\partial_h^6 u\right|^2.
\end{aligned}$$

Integrating (by parts) the above identity over Ω, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \rho \partial_h^6 \eta \cdot \partial_h^6 u \,\mathrm{d}y + \frac{\mu}{2}\left\|\nabla \partial_h^6 \eta\right\|_0^2\right) + \lambda_0 \left\|\bar{M} \cdot \nabla \partial_h^6 \eta\right\|_0^2 \\
&\quad = 2\rho\omega \int \partial_h^6 (u_2 e_1 - u_1 e_2) \cdot \partial_h^6 \eta \,\mathrm{d}y + \mu \int \partial_h^6 N_u^h \cdot \partial_h^6 \eta \,\mathrm{d}y - \int \partial_h^5 N_q^h \cdot \partial_h^7 \eta \,\mathrm{d}y \\
&\qquad - \int \partial_h^6 q \,\mathrm{div}\, \partial_h^6 \eta \,\mathrm{d}y + \left\|\sqrt{\rho} \partial_h^6 u\right\|_0^2 \\
&\quad \lesssim \sum_{k=1}^4 J_k^H + \left\|\partial_h^6 u\right\|_0^2,
\end{aligned} \tag{4.3}$$

where the first four integrals on the right-hand side are denoted by $J_1^H, \ldots, J_4^H$, respectively.

On the other hand, the four integrals $J_1^H, \ldots, J_4^H$ can be bounded as follows:

$$J_1^H \lesssim \left\|\partial_h^6 \eta\right\|_0 \left\|\partial_h^6 u\right\|_0, \tag{4.4}$$

$$\begin{aligned}
J_2^H &= -\mu \int \partial_h^6\big((\tilde{\mathcal{A}}_{jl}\tilde{\mathcal{A}}_{jk} + \tilde{\mathcal{A}}_{lk} + \tilde{\mathcal{A}}_{kl})\partial_k u_i\big)\partial_l \partial_h^6 \eta_i \,\mathrm{d}y \\
&\lesssim \big(\|\tilde{\mathcal{A}}\|_6\|u\|_3 + \|\tilde{\mathcal{A}}\|_4\|u\|_5 + \|\tilde{\mathcal{A}}\|_2\|u\|_7\big)\|\eta\|_{6,1} \\
&\lesssim \big(\|\tilde{\mathcal{A}}\|_6\|u\|_3 + \|\tilde{\mathcal{A}}\|_2^{\frac12}\|\tilde{\mathcal{A}}\|_6^{\frac12}\|u\|_3^{\frac12}\|u\|_7^{\frac12} + \|\tilde{\mathcal{A}}\|_2\|u\|_7\big)\|\eta\|_{6,1} \\
&\lesssim \sqrt{\mathcal{E}^L}\big\|(\eta,u)\big\|_7^2, \qquad (4.5) \\
J_3^H &\lesssim \big(\|\tilde{\mathcal{A}}\|_6\|\nabla q\|_1 + \|\tilde{\mathcal{A}}\|_3\|\nabla q\|_4 + \|\tilde{\mathcal{A}}\|_2\|\nabla q\|_5\big)\|\eta\|_7 \\
&\lesssim \big(\|\tilde{\mathcal{A}}\|_6\|\nabla q\|_1 + \|\tilde{\mathcal{A}}\|_2^{\frac34}\|\tilde{\mathcal{A}}\|_6^{\frac14}\|\nabla q\|_1^{\frac14}\|\nabla q\|_5^{\frac34} + \|\tilde{\mathcal{A}}\|_2\|\nabla q\|_5\big)\|\eta\|_7 \\
&\lesssim \sqrt{\mathcal{E}^L}\big(\|\eta\|_7^2 + \mathcal{D}^H\big), \qquad (4.6) \\
J_4^H &= \int \partial_h^6 q\cdot \partial_h^6 \operatorname{div}\eta \,\mathrm{d}y \lesssim \|\nabla q\|_5\big(\|\eta\|_3\|\eta\|_7 + \|\eta\|_5^2\big) \\
&\lesssim \|\eta\|_3\|\nabla q\|_5\big(\|\eta\|_5 + \|\eta\|_7\big) \lesssim \sqrt{\mathcal{E}^L}\big(\|\eta\|_7^2 + \mathcal{D}^H\big), \qquad (4.7)
\end{aligned}$$

where the interpolation inequality (3.3) has been employed in (4.6) and (4.7). Consequently, putting the above four estimates into (4.3), we obtain Lemma 4.2. □

Lemma 4.3 *We have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\sqrt{\rho}\partial_h^k u\big\|_0^2 + \lambda_0\big\|\bar{M}\cdot\nabla\partial_h^k\eta\big\|_0^2\big) + \mu\big\|\nabla\partial_h^k u\big\|_0^2 \\
&\quad\lesssim \sqrt{\mathcal{E}^L}\big(\big\|(\eta,u)\big\|_7^2 + \mathcal{D}^H\big) \quad \textit{for } k = 5 \textit{ and } 6.
\end{aligned}$$

Proof We only show the case $k = 6$, since the derivation of the case $k = 5$ is similar. Applying ∂_h^6 to $(3.24)_2$, multiplying the resulting equality by $\partial_h^6 u$, we make use of $(3.24)_1$ to have

$$\begin{aligned}
&\rho\partial_h^6 u_t\cdot\partial_h^6 u - \mu\Delta\partial_h^6 u\cdot\partial_h^6 u - \lambda_0 m^2\partial_3^2\partial_h^6\eta\cdot\partial_h^6\eta_t \\
&\quad= \big(2\rho\omega\partial_h^6(u_2e_1 - u_1e_2) + \mu\partial_h^6 N_u^h + \partial_h^6 N_q^h - \nabla\partial_h^6 q\big)\cdot\partial_h^6 u.
\end{aligned}$$

Integrating (by parts) the above identity over Ω, we get

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\rho\big|\partial_h^6 u\big|^2\,\mathrm{d}y + \lambda_0\big\|\bar{M}\cdot\nabla\partial_h^6\eta\big\|_0^2\right) + \frac{\mu}{2}\big\|\nabla\partial_h^6 u\big\|_0^2 \\
&\quad= \mu\int\partial_h^6 N_u^h\cdot\partial_h^6 u\,\mathrm{d}y - \int\partial_h^5 N_q^h\cdot\partial_h^7 u\,\mathrm{d}y + \int\partial_h^6 q\operatorname{div}\partial_h^6 u\,\mathrm{d}y =: \sum_{k=1}^3 M_k^H. \qquad (4.8)
\end{aligned}$$

Analogously to (4.4)-(4.7), the three integrals M_1^H, M_2^H and M_3^H can be bounded as follows:

$$\begin{aligned}
M_1^H &= -\int\partial_h^6\big[(\tilde{\mathcal{A}}_{jl}\tilde{\mathcal{A}}_{jk} + \tilde{\mathcal{A}}_{lk} + \tilde{\mathcal{A}}_{kl})\partial_k u_i\big]\cdot\partial_l\partial_h^6 u\,\mathrm{d}y \lesssim \sqrt{\mathcal{E}^L}\big\|(\eta,u)\big\|_7^2, \\
M_2^H &\lesssim \sqrt{\mathcal{E}^L}\big(\big\|(\eta,u)\big\|_7^2 + \mathcal{D}^H\big), \\
M_3^H &= \int\partial_h^6 q\cdot\partial_h^6 D_u^h\,\mathrm{d}y \lesssim \|\nabla q\|_5\big(\|\tilde{\mathcal{A}}\|_6\|u\|_3 + \|\tilde{\mathcal{A}}\|_4\|u\|_5 + \|\tilde{\mathcal{A}}\|_2\|u\|_7\big) \\
&\lesssim \|\nabla q\|_5\big(\|\eta\|_7\|u\|_3 + \|\tilde{\mathcal{A}}\|_2^{\frac12}\|\tilde{\mathcal{A}}\|_6^{\frac12}\|u\|_3^{\frac12}\|u\|_7^{\frac12} + \|\tilde{\mathcal{A}}\|_2\|u\|_7\big) \lesssim \sqrt{\mathcal{E}^L}\big(\|\eta\|_7^2 + \mathcal{D}^H\big).
\end{aligned}$$

Substituting the above three inequalities into (4.8), we obtain Lemma 4.3. □

Lemma 4.4 *The following estimates hold*:

$$\sum_{k=0}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\lesssim\|\eta\|_6^2+\left\|\left(u,u_t,u_{tt},\partial_t^3 u\right)\right\|_0^2+\mathcal{E}^L E^H, \tag{4.9}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{6,*}^2+\|\eta\|_6^2+\sum_{k=0}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\\&\quad\lesssim\|\eta\|_{5,1}^2+\left\|\left(\eta,u,u_t,u_{tt},\partial_t^3 u\right)\right\|_0^2+\mathcal{E}^L\mathcal{D}^H,\end{aligned} \tag{4.10}$$

$$\sum_{k=1}^{2}\left(\left\|\partial_t^k u\right\|_{7-2k}^2+\left\|\nabla\partial_t^k q\right\|_{5-2k}^2\right)\lesssim\|u\|_5^2+\left\|\left(u_t,u_{tt},\partial_t^3 u\right)\right\|_1^2+\mathcal{E}^H\mathcal{D}^H, \tag{4.11}$$

where the norm $\|\eta\|_{6,*}^2$ *is equivalent to* $\|\eta\|_6^2$ *and* $E^H:=\mathcal{E}^H-\|\nabla\eta\|_{6,0}^2$.

Proof (1) We begin with the derivation of (4.9). Taking $(i,k)=(4,0)$ in the Stokes estimate (3.35), we have

$$\|u\|_6^2+\|\nabla q\|_4^2\lesssim\|\eta\|_6^2+\left\|(u,u_t)\right\|_4^2+S_{0,4}^u.$$

Noting that

$$\begin{aligned}S_{0,4}^u&=\left\|\left(N_u^h,N_q^h\right)\right\|_4^2+\left\|D_u^h\right\|_5^2\\&\lesssim\|\tilde{\mathcal{A}}\|_5^2\|u\|_3^2+\|\tilde{\mathcal{A}}\|_2^2\|u\|_6^2+\|\tilde{\mathcal{A}}\|_5^2\|\nabla q\|_1^2+\|\tilde{\mathcal{A}}\|_2^2\|\nabla q\|_4^2\lesssim\mathcal{E}^L E^H,\end{aligned}$$

we get

$$\|u\|_6^2+\|\nabla q\|_4^2\lesssim\|\eta\|_6^2+\left\|(u,u_t)\right\|_4^2+\mathcal{E}^L E^H. \tag{4.12}$$

Using the recursion formula (3.35) for $i=4$ from $k=1$ to 2, we obtain

$$\sum_{k=1}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\lesssim\left\|\partial_t^2(u,u_t)\right\|_0^2+\sum_{k=1}^{2}\left(\left\|\partial_t^{k-1}u\right\|_{6-2k}^2+S_{k,4}^u\right).$$

On the other hand, $S_{k,4}^u$ can be bounded from above by

$$\begin{aligned}\sum_{k=1}^{2}S_{k,4}^u&=\sum_{k=1}^{2}\left(\left\|\partial_t^k N_u^h\right\|_{4-2k}^2+\left\|\partial_t^k D_u^h\right\|_{5-2k}^2+\left\|\partial_t^k N_q^h\right\|_{4-2k}^2\right)\\&\lesssim\|\tilde{\mathcal{A}}\|_2^2\left(\|u_t\|_4^2+\|\nabla q_t\|_2^2\right)+\|\tilde{\mathcal{A}}\|_2^2\left(\|u_{tt}\|_2^2+\|\nabla q_{tt}\|_0^2\right)\\&\quad+\|\tilde{\mathcal{A}}\|_5^2\|u_t\|_1^2+\|\tilde{\mathcal{A}}_t\|_2^2\left(\|u\|_4^2+\|\nabla q\|_2^2\right)+\|\tilde{\mathcal{A}}_t\|_2^2\left(\|u_t\|_2^2+\|\nabla q_t\|_0^2\right)\\&\quad+\|\tilde{\mathcal{A}}_t\|_3^2\|u\|_3^2+\|\tilde{\mathcal{A}}_{tt}\|_1^2\left(\|u\|_3^2+\|\nabla q\|_1^2\right)\\&\lesssim\mathcal{E}^L E^H.\end{aligned}$$

Therefore,

$$\sum_{k=1}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\lesssim\left\|\partial_t^2(u,u_t)\right\|_0^2+\sum_{k=1}^{2}\left\|\partial_t^{k-1}u\right\|_{6-2k}^2+\mathcal{E}^L E^H, \tag{4.13}$$

which, together with (4.12), yields

$$\sum_{k=0}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\lesssim\|\eta\|_6^2+\left\|\partial_t^2(u,u_t)\right\|_0^2+\sum_{k=1}^{2}\left\|\partial_t^{k-1}u\right\|_{6-2k}^2+\mathcal{E}^L E^H. \tag{4.14}$$

In view of the interpolation inequality (3.3), we have

$$\|u\|_4\le C_\epsilon\|u\|_0+\epsilon\|u\|_6\quad\text{and}\quad\|u_t\|_2\lesssim C_\epsilon\|u_t\|_0+\epsilon\|u_t\|_4. \tag{4.15}$$

Thus, inserting (4.15) into (4.14), one gets (4.9).

(2) We proceed to prove the estimate (4.10). Exploiting the recursion formula (3.39) for $i=4$ from $k=0$ to 4, we see that there are positive constants c_k, $k=0,\dots,4$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{4}c_k\|\eta\|_{k,6-k}^2+\left\|(\eta,u)\right\|_6^2+\|\nabla q\|_4^2\lesssim\|u_t\|_4^2+\|\eta\|_{5,1}^2+\sum_{k=0}^{4}S_{k,4}^{\omega},$$

which combined with (4.13) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{4}c_k\|\eta\|_{k,6-k}^2+c\left(\|\eta\|_6^2+\sum_{k=0}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\right)\\&\quad\lesssim\|\eta\|_{5,1}^2+\left\|\partial_t^2(u,u_t)\right\|_0^2+\sum_{k=1}^{2}\left\|\partial_t^{k-1}u\right\|_{6-2k}^2+\mathcal{E}^L E^H+S_{0,4}^{\omega},\end{aligned}$$

where we have used the fact that $S_{k,4}^{\omega}\le S_{0,4}^{\omega}$ for $0\le k\le 4$. Since

$$\begin{aligned}S_{0,4}^{\omega}&=\left\|\left(N_u^h,N_q^h\right)\right\|_4^2+\left\|\left(D_u^h,\operatorname{div}\eta\right)\right\|_5^2\\&\lesssim\left(\|\tilde{\mathcal{A}}\|_5^2\|u\|_3^2+\|\tilde{\mathcal{A}}\|_2^2\|u\|_6^2+\|\eta\|_3^2\|\eta\|_6^2+\|\tilde{\mathcal{A}}\|_5^2\|\nabla q\|_1^2+\|\tilde{\mathcal{A}}\|_2^2\|\nabla q\|_4^2\right)\\&\lesssim\mathcal{E}^L E^H\end{aligned}$$

and $E^H\le\mathcal{D}^H$, we further infer that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{4}c_k\|\eta\|_{k,6-k}^2+c\left(\|\eta\|_6^2+\sum_{k=0}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2+\left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\right)\\&\quad\lesssim\|\eta\|_{5,1}^2+\left\|\partial_t^2(u,u_t)\right\|_0^2+\sum_{k=1}^{2}\left\|\partial_t^{k-1}u\right\|_{6-2k}^2+\mathcal{E}^L\mathcal{D}^H.\end{aligned} \tag{4.16}$$

Using the interpolation inequality (3.3), we get (4.10) from (4.16), where $\|\eta\|_{6,*}^2$ equals to $\sum_{k=0}^{4}c_k\|\eta\|_{k,6-k}^2$ multiplied by some positive constant.

(3) Finally, we derive the estimate (4.11) for higher-order dissipation estimates. We use the recursion formula (3.35) for $i = 5$ from $k = 1$ to 2 to deduce that

$$\sum_{k=1}^{2}\left(\left\|\partial_t^k u\right\|_{7-2k}^2 + \left\|\nabla\partial_t^k q\right\|_{5-2k}^2\right) \lesssim \left\|\partial_t^2(u,u_t)\right\|_1^2 + \sum_{k=1}^{2}\left(\left\|\partial_t^{k-1}u\right\|_{7-2k}^2 + S_{k,5}^u\right). \tag{4.17}$$

Noting that

$$\begin{aligned}\sum_{k=1}^{2} S_{k,5}^u &= \sum_{k=1}^{2}\left(\left\|\partial_t^k\left(N_u^h,N_q^h\right)\right\|_{5-2k}^2 + \left\|\partial_t^k D_u^h\right\|_{6-2k}^2\right)\\
&\lesssim \|\tilde{\mathcal{A}}_t\|_4^2\|u\|_5^2 + \|\tilde{\mathcal{A}}\|_4^2\|u_t\|_5^2 + \|\tilde{\mathcal{A}}_t\|_3^2\|\nabla q\|_3^2 + \|\tilde{\mathcal{A}}\|_3^2\|\nabla q_t\|_3^2\\
&\quad + \|\tilde{\mathcal{A}}_{tt}\|_2^2\|u\|_3^2 + \|\tilde{\mathcal{A}}_t\|_2^2\|u_t\|_3^2 + \|\tilde{\mathcal{A}}\|_2^2\|u_{tt}\|_3^2\\
&\quad + \|\tilde{\mathcal{A}}_{tt}\|_2^2\|\nabla q\|_1^2 + \|\tilde{\mathcal{A}}_t\|_2^2\|\nabla q_t\|_1^2 + \|\tilde{\mathcal{A}}\|_2^2\|\nabla q_{tt}\|_1^2\\
&\lesssim \mathcal{E}^H\mathcal{D}^H,\end{aligned}$$

we obtain (4.11) from (4.17). □

Now we are in a position to build the higher-order and highest-order energy inequalities. In what follows, the letters c_i^H and $i = 1,\dots,6$ will denote generic constants which may depend on the domain Ω and some physical parameters in the transformed MHD equations (2.15).

Proposition 4.1 *Under the assumption* (3.1), *if δ is sufficiently small, then there are two norms $\tilde{\mathcal{E}}^H$ and $\|\eta\|_{7,*}^2$ which are equivalent to $\mathcal{E}^H$ and $\|\eta\|_7^2$ respectively, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^H + \mathcal{D}^H \lesssim \sqrt{\mathcal{E}^L}\left\|(\eta,u)\right\|_7^2, \tag{4.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{7,*}^2 + \left\|(\eta,u)\right\|_7^2 \lesssim \mathcal{E}^H + \mathcal{D}^H \quad \text{on } (0,T]. \tag{4.19}$$

Proof (1) We first prove (4.18). Similarly to (3.48), we make use of (3.5), (3.6), (3.44) and (3.46) to deduce from Lemmas 4.1-4.3 and (4.10) that there are constants c_1^H, c_2^H, c_3^H and $\alpha_0 \geq 1$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_1^H + \tilde{\mathcal{D}}_1^H \leq c_1^H\left[(1+\alpha)\sqrt{\mathcal{E}^L}\left(\left\|(\eta,u)\right\|_7^2 + \mathcal{D}^H\right) + \left\|(\eta,u,u_t)\right\|_0^2\right] \quad \text{for any } \alpha \geq \alpha_0, \tag{4.20}$$

where α_0 depends on the domain and the known physical parameters, and

$$\begin{aligned}\tilde{\mathcal{E}}_1^H := &\sum_{k=2}^{3}\left(\left\|\sqrt{\rho}\partial_t^k u\right\|_0^2 + \lambda_0\left\|\bar{M}\cdot\nabla\partial_t^{k-1}u\right\|_0^2 + \int\nabla\partial_t^{k-1}q\cdot D_u^{t,k}\,\mathrm{d}y\right)\\
&+ \sum_{5\leq\alpha_1+\alpha_2\leq 6}\int\rho\partial_1^{\alpha_1}\partial_2^{\alpha_2}\eta\cdot\partial_1^{\alpha_1}\partial_2^{\alpha_2}u\,\mathrm{d}y + \sum_{k=5}^{6}\frac{\mu}{2}\|\nabla\eta\|_{k,0}^2\\
&+ \alpha\left(\|\sqrt{\rho}u\|_{6,0}^2 + \lambda_0\|\bar{M}\cdot\nabla\eta\|_{6,0}^2\right) + c_2^H\|\eta\|_{6,*}^2,\end{aligned}$$

$$\tilde{\mathcal{D}}_1^H := c_3^H\left(\sum_{k=2}^{3}\left\|\partial_t^k u\right\|_1^2 + \sum_{k=5}^{6}\left\|(\eta, \bar{M}\cdot\nabla\eta, \nabla u)\right\|_{k,0}^2 + \|\eta\|_6^2 + \sum_{k=0}^{2}\left(\left\|\partial_t^k u\right\|_{6-2k}^2 + \left\|\nabla\partial_t^k q\right\|_{4-2k}^2\right)\right).$$

Moreover, by (4.20) and Proposition 3.1 we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_2^H + \tilde{\mathcal{D}}^H \leq c_1^H(1+\sigma)\sqrt{\mathcal{E}^L}\left(\left\|(\eta,u)\right\|_7^2 + \mathcal{D}^H\right), \tag{4.21}$$

where $\tilde{\mathcal{E}}_2^H := \tilde{\mathcal{E}}_1^H + c_4^H\tilde{\mathcal{E}}^L$ and $\tilde{\mathcal{D}}^H := \tilde{\mathcal{D}}_1^H + c_5^H\mathcal{D}^L$. By (4.11), we have $\mathcal{D}^H \lesssim \tilde{\mathcal{D}}^H + \mathcal{E}^H\mathcal{D}^H$. So, (4.21) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_2^H + c_6^H\mathcal{D}^H \leq c_1^H(1+\sigma)\sqrt{\mathcal{E}^L}\left\|(\eta,u)\right\|_7^2 \quad \text{for sufficiently small } \delta. \tag{4.22}$$

Next, we show that $\mathcal{E}^H$ can be controlled by $\tilde{\mathcal{E}}_2^H$. Keeping in mind that

$$\begin{aligned}\sum_{k=2}^{3}\int \nabla\partial_t^{k-1}q\cdot D_u^{t,k}\,\mathrm{d}y &\lesssim \sum_{k=2}^{3}\left\|\nabla\partial_t^{k-1}q\right\|_0\left\|D_u^{t,k}\right\|_0 \\ &\lesssim \sqrt{\mathcal{E}^H}\left(\left\|(u,u_t)\right\|_2\sum_{k=2}^{3}\left\|\partial_t^k\mathcal{A}\right\|_0 + \|\mathcal{A}_t\|_2\|u_{tt}\|_0\right) \\ &\lesssim \left(\mathcal{E}^H\right)^{\frac{3}{2}},\end{aligned} \tag{4.23}$$

we use Cauchy-Schwarz's inequality, (3.5) and (4.23) to infer that there is an appropriately large constant α, depending on ρ, μ and Ω, such that

$$\|\nabla\eta\|_{6,0}^2 + \|\eta\|_6^2 + \sum_{k=0}^{3}\left\|\partial_t^k u\right\|_0^2 \lesssim \tilde{\mathcal{E}}_2^H + \left(\mathcal{E}^H\right)^{\frac{3}{2}}.$$

Recalling (4.9) and the fact that $E^H \leq \mathcal{E}^H$, we have

$$\mathcal{E}^H \lesssim \tilde{\mathcal{E}}_2^H + \mathcal{E}^L\mathcal{E}^H + \left(\mathcal{E}^H\right)^2 \leq \tilde{\mathcal{E}}_2^H + 2\left(\mathcal{E}^H\right)^{\frac{3}{2}},$$

which implies $\mathcal{E}^H \lesssim \tilde{\mathcal{E}}_2^H$ for sufficiently small δ.

On the other hand, $\tilde{\mathcal{E}}_2^H \lesssim \mathcal{E}^H$ obviously. Hence, the energy functional $\tilde{\mathcal{E}}_2^H$ is equivalent to $\mathcal{E}^H$. Finally, letting $\delta \leq c_6^H/2c_3^H(1+\sigma)$ and denoting $\tilde{\mathcal{E}}^H = 2\tilde{\mathcal{E}}_2^H/c_6^H$, we see that (4.22) implies (4.18).

(2) We proceed to derive the highest-order estimate (4.19). We start with employing the recursion formula (3.39) with $i = 5$ from $k = 0$ to 4 to find that there are positive constants d_k ($k = 0,1,2$), such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{5}d_k\|\eta\|_{k,7-k}^2 + \left\|(\eta,u)\right\|_7^2 \lesssim \|u\|_5^2 + \|\eta\|_{6,1}^2 + \|u_t\|_5^2 + S_{0,5}^{\omega}.$$

On the other hand, by virtue of (4.2),

$$\begin{aligned} S^{\omega}_{0,5} &= \left\| \left(N^h_u, N^h_q\right) \right\|_5^2 + \left\| \left(D^h_u, \operatorname{div} \eta\right) \right\|_6^2 \\ &\lesssim \|\tilde{\mathcal{A}}\|_6^2 \|u\|_3^2 + \|\tilde{\mathcal{A}}\|_3^2 \|u\|_7^2 + \|\tilde{\mathcal{A}}\|_5^2 \|\nabla q\|_5^2 + \|\eta\|_4^2 \|\eta\|_7^2 \lesssim \mathcal{D}^H + \mathcal{E}^H \|\eta\|_7^2. \end{aligned}$$

Thus, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{5} d_k \|\eta\|_{k,7-k}^2 + \left\|(\eta, u)\right\|_7^2 \lesssim \mathcal{E}^H + \mathcal{D}^H + \mathcal{E}^H \|\eta\|_7^2.$$

Hence, (4.19) holds by defining $\|\eta\|_{7,*}^2 := 2\sum_{k=0}^5 d_k \|\eta\|_{k,7-k}^2$, provided δ is sufficiently small. □

The following lemma will be needed in the next section.

Lemma 4.5 *Under the assumption* (3.1), *if* δ *is sufficiently small, then*

$$\mathcal{E}^H \lesssim \mathcal{E} := \|\eta\|_7^2 + \|u\|_6^2. \tag{4.24}$$

Proof In view of (4.9), we have

$$\mathcal{E}^H \lesssim \|\eta\|_7^2 + \left\| \left(u, u_t, u_{tt}, \partial_t^3 u\right) \right\|_0^2 + \mathcal{E}^L \mathcal{E}^H,$$

which results in

$$\mathcal{E}^H \lesssim \|\eta\|_7^2 + \left\| \left(u, u_t, u_{tt}, \partial_t^3 u\right) \right\|_0^2 \tag{4.25}$$

for sufficiently small δ. Below we show that the L^2-norm of u_t and $\partial_t^3 u$ can be controlled by $\sqrt{\mathcal{E}}$.

(1) First, to bound u_t, we multiply $(3.10)_2$ with $j = 0$ by u_t and integrate the resulting equation over Ω to obtain

$$\begin{aligned} \|\sqrt{\rho} u_t\|_0^2 &= \int \left(\mu \Delta_{\mathcal{A}} u + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla u) + 2\rho\omega \partial_t (u_2 e_1 - u_1 e_2)\right) \cdot u_t \,\mathrm{d}y \\ &\quad - \int \nabla q \cdot D_u^{t,1} \,\mathrm{d}y \\ &\lesssim \mathcal{E} + \|\nabla q\|_0 \left\| D_u^{t,1} \right\|_0, \end{aligned}$$

where the last term on the right-hand side can be bounded from above by $\mathcal{E}$. Hence,

$$\|u_t\|_0^2 \lesssim \mathcal{E}. \tag{4.26}$$

(2) Then we control the term u_{tt}. Multiplying $(3.10)_2$ with $j = 1$ by u_{tt} and integrating the resulting equation over Ω, we infer that

$$\begin{aligned} \|\sqrt{\rho} u_{tt}\|_0^2 &= \int \left(\mu \Delta_{\mathcal{A}} \partial_t u + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla u) + 2\rho\omega (u_2 e_1 - u_1 e_2) + \mu N_u^{t,1} + N_q^{t,1}\right) \\ &\quad \cdot \partial_t^2 u \,\mathrm{d}y \\ &\quad - \int \nabla q_t \cdot D_u^{t,2} \,\mathrm{d}y, \end{aligned}$$

which yields

$$\|u_{tt}\|_0^2 \lesssim \|(u,u_t)\|_2^2 + \|(N_u^{t,1}, N_q^{t,1})\|_0^2 + \|\nabla q_t\|_0 \|D_u^{t,2}\|_0.$$

Noting that

$$\|(N_u^{t,1}, N_q^{t,1})\|_0^2 \lesssim \mathcal{E}$$

and

$$\|\nabla q_t\|_0 \|D_u^{t,2}\|_0 \lesssim \|\mathcal{A}_t\|_1 \|u_t\|_1 + \|\mathcal{A}_{tt}\|_0 \|u\|_2 \lesssim \|u_t\|_1^2 + \mathcal{E},$$

we see that

$$\|u_{tt}\|_0^2 \lesssim \|u_t\|_2^2 + \mathcal{E}. \tag{4.27}$$

(3) Finally, we estimate the term $\partial_t^3 u$. If we multiply $(3.10)_2$ with $j = 2$ by $\partial_t^3 u$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}\|\sqrt{\rho}\partial_t^3 u\|_0^2 = {} & \int \big(\mu \Delta_{\mathcal{A}} \partial_t^2 u + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla u_t) \\ & + 2\rho\omega \partial_t^2 (u_2 e_1 - u_1 e_2) + \mu N_u^{t,2} + N_q^{t,2}\big) \cdot \partial_t^3 u \,\mathrm{d}y \\ & - \int \nabla q_{tt} \cdot D_u^{t,3} \,\mathrm{d}y,\end{aligned}$$

whence

$$\|\partial_t^3 u\|_0^2 \lesssim \|u\|_1^2 + \|(u_t, u_{tt})\|_2^2 + \|(N_u^{t,2}, N_q^{t,2})\|_0^2 + \|\nabla q_{tt}\|_0 \|D_u^{t,3}\|_0.$$

On the other hand, it is easy to show that

$$\|(N_u^{t,2}, N_q^{t,2})\|_0^2 \lesssim \|u_t\|_1^2 + \mathcal{E}$$

and

$$\|\nabla q_{tt}\|_0 \|D_u^{t,3}\|_0 \lesssim \|u_t\|_2^2 + \|u_{tt}\|_1^2 + \mathcal{E}.$$

Therefore,

$$\|\partial_t^3 u\|_0^2 \lesssim \|u_t\|_2^2 + \|u_{tt}\|_2^2 + \mathcal{E}. \tag{4.28}$$

Now, we control the term $\|u_{tt}\|_2$ in (4.28). By $(3.10)_2$ with $j = 1$, we deduce that

$$\|u_{tt}\|_2^2 \lesssim \|u_t\|_4^2 + \|\nabla q_t\|_2^2 + \|(\nabla^2 u, N_u^{t,1}, N_q^{t,1})\|_2^2,$$

where the last term on the right-hand side can be bounded by $\mathcal{E}$. Consequently,

$$\|u_{tt}\|_2^2 \lesssim \|u_t\|_4^2 + \|\nabla q_t\|_2^2 + \mathcal{E}. \tag{4.29}$$

To estimate ∇q_t, we rewrite $(3.24)_2$ as follows:

$$\Delta q_t = f_1,$$

with boundary condition

$$\nabla q_t \cdot \nu = f_2 \cdot \nu \quad \text{on } \partial\Omega,$$

where

$$f_1 := \operatorname{div}\big(\mu \partial_t N_u^h + \partial_t N_q^h - \rho u_{tt} + \mu \Delta u_t + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla u) + 2\rho\omega\partial_t(u_2 e_1 - u_1 e_2)\big),$$
$$f_2 := \mu \partial_t N_u^h + \partial_t N_q^h + \mu \Delta u_t + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla u) + 2\rho\omega\partial_t(u_2 e_1 - u_1 e_2).$$

In view of the standard elliptic regularity estimates [19], we find that

$$\|\nabla q_t\|_2^2 \lesssim \|f_1\|_1^2 + \|f_2\|_2^2 \le \|u_t\|_4^2 + \|u_{tt}\|_1^2 + \mathcal{E}.$$

Putting the above inequality into (4.29), we obtain

$$\|u_{tt}\|_2^2 \lesssim \|u_t\|_4^2 + \|u_{tt}\|_1^2 + \mathcal{E},$$

which, together with the interpolation inequality (3.3), yields

$$\|u_{tt}\|_2^2 \lesssim \|u_t\|_4^2 + \|u_{tt}\|_0^2 + \mathcal{E}. \tag{4.30}$$

Consequently, from (4.28) and (4.30) it follows that

$$\big\|\partial_t^3 u\big\|_0^2 \lesssim \|u_{tt}\|_0^2 + \|u_t\|_4^2 + \mathcal{E}. \tag{4.31}$$

Next, we control $\|u_t\|_4$ in (4.31). Using $(2.15)_2$, we derive that

$$\|u_t\|_4^2 \lesssim \big\|\big(\Delta_{\mathcal{A}} u, \nabla^2\eta, u, \nabla_{\mathcal{A}} q\big)\big\|_4^2 \lesssim \|\nabla q\|_4^2 + \mathcal{E}, \tag{4.32}$$

where, by virtue of $(3.24)_2$, ∇q satisfies

$$\Delta q = f_3$$

with boundary condition

$$\nabla q \cdot \nu = f_4 \cdot \nu \quad \text{on } \partial\Omega,$$

and

$$f_3 := \operatorname{div}\big(\mu N_u^h + N_q^h - \rho u_t + \mu \Delta u + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla \eta) + 2\rho\omega(u_2 e_1 - u_1 e_2)\big),$$
$$f_4 := \big(\mu N_u^h + N_q^h + \mu \Delta u + \lambda_0 \bar{M} \cdot \nabla(\bar{M} \cdot \nabla \eta)\big).$$

Thus, we can use the elliptic regularity estimate to get

$$\|\nabla q\|_4^2 \lesssim \|f_3\|_3^2 + \|f_4\|_4^2 \lesssim \|u_t\|_3^2 + \mathcal{E}.$$

Hence, from (4.32) we get

$$\|u_t\|_4^2 \lesssim \|u_t\|_3^2 + \mathcal{E},$$

which, together with the interpolation inequality (3.3), implies that

$$\|u_t\|_4^2 \lesssim \|u_t\|_0^2 + \mathcal{E}. \tag{4.33}$$

Inserting (4.33) into (4.31), we conclude

$$\left\|\partial_t^3 u\right\|_0^2 \lesssim \left\|(u_t, u_{tt})\right\|_0^2 + \mathcal{E}. \tag{4.34}$$

(4) Now we are able to show (4.24). Summing up the estimates (4.26), (4.27), (4.33) and (4.34), we arrive at

$$\sum_{k=1}^{3} \left\|\partial_t^k u_t\right\|_0^2 \lesssim \mathcal{E},$$

which, combined with (4.25), gives (4.24). □

5 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. Roughly speaking, Theorem 2.1 is shown by combining the *a priori* stability estimate (2.17) and the local well-posedness of the transformed MHD problem. Before we derive the *a priori* stability estimate (2.17), we begin with estimating the terms $\mathcal{G}_1, \ldots, \mathcal{G}_4$.

Using (4.19), and recalling the equivalence of $\|\eta\|_{7,*}^2$ and $\|\eta\|_7^2$, we deduce that

$$\begin{aligned}
\|\eta\|_7^2 &\lesssim \|\eta_0\|_7^2 e^{-t} + \int_0^t e^{-(t-\tau)}\big(\mathcal{E}^H(\tau) + \mathcal{D}^H(\tau)\big)\,\mathrm{d}\tau \\
&\lesssim \|\eta_0\|_7^2 e^{-t} + \sup_{0\le\tau\le t} \mathcal{E}^H(\tau) \int_0^t e^{-(t-\tau)}\,\mathrm{d}\tau + \int_0^t \mathcal{D}^H(\tau)\,\mathrm{d}\tau \\
&\lesssim \|\eta_0\|_7^2 e^{-t} + \mathcal{G}_3(t),
\end{aligned}$$

which yields

$$\mathcal{G}_1(t) \lesssim \|\eta_0\|_7^2 + \mathcal{G}_3(t). \tag{5.1}$$

Multiplying (4.19) by $(1+t)^{-3/2}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|\eta\|_{7,*}^2}{(1+t)^{3/2}} + \frac{3}{2}\frac{\|\eta\|_{7,*}^2}{(1+t)^{5/2}} + \frac{\|(\eta,u)\|_7^2}{(1+t)^{3/2}} \lesssim \frac{\mathcal{E}^H}{(1+t)^{3/2}} + \frac{\mathcal{D}^H}{(1+t)^{3/2}},$$

which implies that

$$\mathcal{G}_2(t) \lesssim \|\eta_0\|_7^2 + \mathcal{G}_3(t). \tag{5.2}$$

An integration of (4.18) with respect to t gives

$$\mathcal{G}_3(t) \lesssim \mathcal{E}^H(0) + \int_0^t \sqrt{\mathcal{E}^L(\tau)} \left\|(\eta, u)(\tau)\right\|_7^2 \, \mathrm{d}\tau.$$

Let

$$\mathcal{G}_5(t) := \mathcal{G}_1(t) + \sup_{0 \leq \tau \leq t} \mathcal{E}^H(\tau) + \mathcal{G}_4(t).$$

From now on, we further assume $\sqrt{\mathcal{G}_5(T)} \leq \delta$, which is a stronger requirement than (3.1). Thus, we make use of (5.2) to find that

$$\begin{aligned}\mathcal{G}_3(t) &\lesssim \mathcal{E}^H(0) + \int_0^t \delta(1+\tau)^{-3/2} \left\|(\eta, u)(\tau)\right\|_7^2 \, \mathrm{d}\tau \\ &\lesssim \mathcal{E}^H(0) + \delta\left(\|\eta_0\|_7^2 + \mathcal{G}_3(t)\right),\end{aligned}$$

which implies

$$\mathcal{G}_3(t) \lesssim \|\eta_0\|_7^2 + \mathcal{E}^H(0). \tag{5.3}$$

Finally, we show the time decay behavior of $\mathcal{G}_4(t)$, noting that $\mathcal{E}^L$ can be controlled by $\mathcal{D}^L$, except the term $\|\nabla\eta\|_{3,0}$ in $\mathcal{E}^L$. To deal with $\|\nabla\eta\|_{3,0}$, we use (3.3) to get

$$\|\nabla\eta\|_{3,0} \lesssim \|\eta\|_{3,0}^{\frac{3}{4}} \|\eta\|_{3,4}^{\frac{1}{4}}.$$

On the other hand, we combine (5.1) with (5.3) to get

$$\mathcal{E}^L + \|\eta\|_7^2 \lesssim \tilde{\mathcal{E}}^L + \|\eta\|_7^2 \lesssim \|\eta_0\|_7^2 + \mathcal{E}^H(0).$$

Thus,

$$\tilde{\mathcal{E}}^L \lesssim \mathcal{E}^L \lesssim \left(\mathcal{D}^L\right)^{\frac{3}{4}} \left(\mathcal{E}^L + \|\eta\|_7^2\right)^{\frac{1}{4}} \lesssim \left(\mathcal{D}^L\right)^{\frac{3}{4}} \left(\|\eta_0\|_7^2 + \mathcal{E}^H(0)\right)^{\frac{1}{4}}.$$

Putting the above estimate into the lower-order energy inequality (3.43), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^L + \frac{(\tilde{\mathcal{E}}^L)^{\frac{4}{3}}}{\mathcal{I}_0^{1/3}} \lesssim 0,$$

which yields

$$\mathcal{E}^L \lesssim \tilde{\mathcal{E}}^L \lesssim \frac{\mathcal{I}_0}{((\mathcal{I}_0/\mathcal{E}^L(0))^{1/3} + t/3)^3} \lesssim \frac{\|\eta_0\|_7^2 + \mathcal{E}^H(0)}{1 + t^3}$$

with $\mathcal{I}_0 := c(\mathcal{E}^H(0) + \|\eta_0\|_7^2)$ for some positive constant c. Therefore,

$$\mathcal{G}_4(t) \lesssim \|\eta_0\|_7^2 + \mathcal{E}^H(0). \tag{5.4}$$

Now we sum up the estimates (5.1)-(5.4) to conclude that

$$\mathcal{G}(t) := \sum_{k=1}^{4} \mathcal{G}_k(t) \lesssim \|\eta_0\|_7^2 + \mathcal{E}^H(0) \lesssim \|\eta_0\|_7^2 + \|u_0\|_6^2,$$

where (4.24) has been also used. Consequently, we have proved the following *a priori* stability estimate.

Proposition 5.1 *Let (η, u) be a solution of the transformed MHD problem with an associated perturbation pressure q. Then there is a sufficiently small δ_1, such that (η, u, q) enjoys the following* a priori *stability estimate*:

$$\mathcal{G}(T_1) \le C_1\big(\|\eta_0\|_7^2 + \|u_0\|_6^2\big), \tag{5.5}$$

provided that $\sqrt{\mathcal{G}_5(T_1)} \le \delta_1$ for some $T_1 > 0$. Here $C_1 \ge 1$ denotes a constant depending on the domain Ω and other physical parameters in the transformed MHD equations.

In view of the *a priori* stability estimate in Proposition 5.1 and the following result of local existence of a small solution to the transformed MHD problem, we immediately obtain Theorem 2.1.

Proposition 5.2 *There is a sufficiently small δ_2, such that for any given initial data $(\eta_0, u_0) \in H^7 \times H^6$ satisfying*

$$\sqrt{\|\eta_0\|_7^2 + \|u_0\|_6^2} \le \delta_2$$

and the compatibility conditions (i.e., $\partial_t^j u(x,0)|_{\partial\Omega} = 0, j = 1, 2$), there exist a $T_2 := T_2(\delta_2) > 0$ which depends on δ_2, the domain Ω and other known physical parameters, and a unique classical solution $(\eta, u) \in C^0([0, T_2], H^7 \times H^6)$ to the transformed MHD problem (2.15), (2.16) *with an associated perturbation pressure q. Moreover, $\partial_t^i u \in C^0([0, T_2], H^{6-2i})$ for $1 \le i \le 3$, $q \in C^0([0, T_2], H^5)$, $\mathcal{E}^H(0) \lesssim \|\eta_0\|_7^2 + \|u_0\|_6^2$, and*

$$\sup_{0\le\tau\le T_2} \big(\|\eta(\tau)\|_7^2 + \mathcal{E}^H(\tau)\big) + \int_0^{T_2} \big(\mathcal{D}^H(\tau) + \|u(\tau)\|_7^2 + \|q(\tau)\|_6^2\big)\,\mathrm{d}\tau < \infty,$$

and $\mathcal{G}(t)$ is continuous on $[0, T_2]$.

Proof The transformed MHD problem is very similar to the surface wave problem (1.4) in [20]. Moreover, the current problem is indeed simpler than the surface wave problem due to the non-slip boundary condition $u|_{\partial\Omega} = 0$. Using the standard method in [20], one can easily establish Proposition 5.2, hence we omit its proof here. In addition, the continuity, such as $(\eta, u, q) \in C^0([0, T], H^7 \times H^6 \times H^6)$, $\mathcal{G}(t)$ and so on, can be verified by using the regularity of (η, u, q), the transformed MHD equations and a standard regularized method. □

6 Conclusion

We have proved the existence of a unique time-decay solution to the initial-boundary problem (1.2)-(1.4) of rotating MHD fluids in Lagrangian coordinates, which, together with the inverse transformation of coordinates, implies the existence of a time-decay solution to the original initial-boundary problem (1.2)-(1.4) with proper initial data in $H^7(\Omega)$. Our result also holds for the case $\omega = 0$ (*i.e.*, the absence of rotation), thus it improves Tan and Wang's result in [11], in which the sufficiently small initial data at least belongs to $H^{16}(\Omega)$. Hence our result reveals that rotation does not affect the existence of solutions of rotating MHD fluids. We mention that the phenomenon of rotating MHD fluids widely exists in nature, so our result has potential applications. In addition, based on Theorem 2.1, we will further study the Rayleigh-Taylor problem of rotating MHD fluids in the future; please refer to [21–28] for relevant results on the Rayleigh-Taylor problem.

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References
1. Chen, Q, Tan, ZYW: Strong solutions to the incompressible magnetohydrodynamic equations. Math. Methods Appl. Sci. **34**(1), 94-107 (2011)
2. Tan, Z, Wang, YJ: Global analysis for strong solutions to the equations of a ferrofluid flow model. J. Math. Anal. Appl. **364**(2), 424-436 (2010)
3. Tan, Z, Wang, YJ: Global existence and asymptotic analysis of weak solutions to the equations of ferrohydrodynamics. Nonlinear Anal., Real World Appl. **11**(5), 4254-4268 (2010)
4. Tan, Z, Wang, YJ: Global existence and large-time behavior of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force. Nonlinear Anal. **71**(11), 5866-5884 (2009)
5. Duan, R, Jiang, F, Jiang, S: On the Rayleigh-Taylor instability for incompressible, inviscid magnetohydrodynamic flows. SIAM J. Appl. Math. **71**, 1990-2013 (2012)
6. Abdallah, MA, Fei, J, Zhong, T: Decay estimates for isentropic compressible magnetohydrodynamic equations in bounded domain. Acta Math. Sci. **32**(6), 2211-2220 (2012)
7. Duan, R, Jiang, F, Yin, J: Rayleigh-Taylor instability for compressible rotating flows. Acta Math. Sci. **35**(6), 1359-1385 (2015)
8. Jiang, F, Jiang, S: On instability and stability of three-dimensional gravity flows in a bounded domain. Adv. Math. **264**, 831-863 (2014)
9. Jiang, F, Tan, Z, Wang, H: A note on global existence of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force. J. Math. Anal. Appl. **379**(1), 316-324 (2011)
10. Jiang, F: On effects of viscosity and magnetic fields on the largest growth rate of linear Rayleigh-Taylor instability. J. Math. Phys. **57**, 111503 (2016)
11. Tan, Z, Wang, YJ: Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems (2015). arXiv:1509.08349v1
12. Jiang, F, Jiang, S, Wang, Y: On the Rayleigh-Taylor instability for the incompressible viscous magnetohydrodynamic equations. Commun. Partial Differ. Equ. **39**(3), 399-438 (2014)
13. Wang, Y: Critical magnetic number in the MHD Rayleigh-Taylor instability. J. Math. Phys. **53**, 073701 (2012)
14. Majda, AJ, Bertozzi, AL: Vorticity and Incompressible Flow. Cambridge university press, Cambridge (2002)
15. Guo, Y, Tice, I: Decay of viscous surface waves without surface tension in horizontally infinite domains. Anal. PDE **6**(6), 1429-1533 (2013)
16. Adams, RA, John, JFF: Sobolev Space. Academic Press, New York (2005)
17. Jiang, F, Jiang, S: Nonlinear stability and instablity in Rayleight-Taylor problem of stratisfied compressible MHD fluids (2017). arXiv:1702.07529
18. Temam, R: Navier-Stokes Equations: Theory and Numerical Analysis. AMS Chelsea Publishing, Providence (2001)

19. Agmon, S, Douglis, A, Nirenberg, L: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Commun. Pure Appl. Math. **12**, 623-727 (1959)
20. Guo, Y, Tice, I: Local well-posedness of the viscous surface wave problem without surface tension. Anal. PDE **6**(2), 287-369 (2013)
21. Jiang, F, Jiang, S, Wang, W: On the Rayleigh-Taylor instability for two uniform viscous incompressible flows. Chin. Ann. Math., Ser. B **35**(6), 907-940 (2014)
22. Jiang, F, Jiang, S, Wang, W: Nonlinear Rayleigh-Taylor instability in nonhomogeneous incompressible viscous magnetohydrodynamic fluids. Discrete Contin. Dyn. Syst., Ser. S **9**(6), 1853-1898 (2016)
23. Jiang, F, Wu, G, Zhong, X: On exponential stability of gravity driven viscoelastic flows. J. Differ. Equ. **260**, 7498-7534 (2016)
24. Jiang, F, Jiang, S, Ni, G: Nonlinear instability for nonhomogeneous incompressible viscous fluids. Sci. China Math. **56**, 665-686 (2013)
25. Huang, G, Jiang, F, Wang, W: On the nonlinear Rayleigh-Taylor instability of nonhomogeneous incompressible viscoelastic fluids under L^2-norm. J. Math. Anal. Appl. **455**, 873-904 (2017). doi:10.1016/j.jmaa.2017.06.022
26. Jiang, F, Jiang, S, Wu, G: On stabilizing effect of elasticity in the Rayleigh-Taylor problem of stratified viscoelastic fluids. J. Funct. Anal. **272**, 3763-3824 (2017)
27. Jiang, F, Jiang, S: On linear instability and stability of the Rayleigh-Taylor problem in magnetohydrodynamics. J. Math. Fluid Mech. **17**, 639-668 (2015)
28. Jiang, F: An improved result on Rayleigh-Taylor instability of nonhomogeneous incompressible viscous flows. Commun. Math. Sci. **14**, 1269-1281 (2016)