Ma et al. *Boundary Value Problems* (2017) 2017:119
DOI 10.1186/s13661-017-0850-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Global well-posedness for the 2D Cahn-Hilliard-Boussinesq and a related system on bounded domains

Caochuan Ma[1,2*], Weijiang Gu[3] and Jianzhu Sun[4]

*Correspondence:
ccma@zjnu.edu.cn
[1]Department of Mathematics, Tianshui Normal University, Tianshui, 741000, P.R. China
[2]Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, P.R. China
Full list of author information is available at the end of the article

Abstract
This paper proves the global well-posedness for the 2D Cahn-Hilliard-Boussinesq and a related system with partial viscous terms on bounded domains.

MSC: 35Q35; 76D05; 35M10

Keywords: Cahn-Hilliard; Boussinesq; global regularity; bounded domain

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and ν be the unit outward normal vector to $\partial\Omega$. First, we consider the following inviscid Cahn-Hilliard-Boussinesq system [1]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi = \mu \nabla \phi + \theta e_2, \tag{1.1}$$

$$\partial_t \theta + u \cdot \nabla \theta = \Delta \theta, \tag{1.2}$$

$$\operatorname{div} u = 0, \tag{1.3}$$

$$\partial_t \phi + u \cdot \nabla \phi = \Delta \mu, \tag{1.4}$$

$$\mu = -\Delta \phi + f'(\phi), \quad f(\phi) = \frac{1}{4}\left(1-\phi^2\right)^2, \tag{1.5}$$

in $\Omega \times (0,\infty)$ with the boundary and initial conditions

$$u \cdot \nu = 0, \qquad \frac{\partial \theta}{\partial \nu} = 0, \qquad \frac{\partial \phi}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.6}$$

$$(u,\theta,\phi)(\cdot,0) = (u_0,\theta_0,\phi_0) \quad \text{in } \Omega. \tag{1.7}$$

Here u, π, and θ denote the velocity, pressure and temperature of the fluid, respectively. ϕ is the order parameter and μ is a chemical potential and $e_2 := \binom{0}{1}$.

Zhao [2] proved the global existence and uniqueness of smooth solutions to problem (1.1)-(1.7) with smooth initial data $u_0, \theta_0 \in H^3$ and $\phi_0 \in H^5$. Zhou and Fan [3] considered the vanishing limit for a 2D Cahn-Hilliard-Navier-Stokes system with a slip boundary condition. We refer the readers to [2, 4, 5] and the references therein for more discussions in this direction.

When $\phi = 0$, the system reduces to the well-known Boussinesq system. Very recently, Zhou and Li [6] proved the global well-posedness of the 2D Boussinesq system with zero viscosity (1.1)-(1.3) and (1.6), (1.7) for rough initial data $u_0 \in L^2$, $\operatorname{rot} u_0 \in L^\infty$ and $\theta_0 \in B_{q,r}^{2-\frac{2}{r}}$ with $1 < r < \infty$ and $2 < q < \infty$, which improves the results in [7, 8] with smooth initial data $u_0, \theta_0 \in H^3$. Several results for the related models can be found in [9, 10].

The first aim of this paper is to prove a similar result for problem (1.1)-(1.7), we will prove the following.

Theorem 1.1 *Let $\phi_0 \in H^4$, $u_0 \in L^2$, $\operatorname{rot} u_0 \in L^\infty$ and $\theta_0 \in B_{q,r}^{2-\frac{2}{r}}$ with $1 < r < \infty$ and $2 < q < \infty$. Then problem* (1.1)-(1.7) *has a unique solution (u, θ, ϕ) satisfying*

$$\begin{aligned} & u \in L^\infty\bigl(0, T; L^2\bigr), \qquad \operatorname{rot} u \in L^\infty\bigl(0, T; L^\infty\bigr), \\ & \theta \in C\bigl([0, T]; B_{q,r}^{2-\frac{2}{r}}\bigr) \cap L^r\bigl(0, T; W^{2,q}\bigr), \qquad \theta_t \in L^r\bigl(0, T; L^q\bigr), \\ & \phi \in L^\infty\bigl(0, T; H^4\bigr) \cap L^2\bigl(0, T; H^5\bigr), \qquad \phi_t \in L^\infty\bigl(0, T; L^2\bigr) \cap L^2\bigl(0, T; H^2\bigr) \end{aligned} \tag{1.8}$$

for any fixed $T > 0$.

Next, we consider the following Cahn-Hilliard-Boussinesq system:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \mu \nabla \phi + \theta e_2, \tag{1.9}$$

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{1.10}$$

$$\operatorname{div} u = 0, \tag{1.11}$$

$$\partial_t \phi + u \cdot \nabla \phi = \Delta \mu, \tag{1.12}$$

$$\mu = -\Delta \phi + f'(\phi), \quad f(\phi) := \frac{1}{4}\bigl(1 - \phi^2\bigr)^2, \tag{1.13}$$

$$u = 0, \qquad \frac{\partial \phi}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.14}$$

$$(u, \theta, \phi)(\cdot, 0) = (u_0, \theta_0, \phi_0) \quad \text{in } \Omega. \tag{1.15}$$

When $\phi = 0$, Zhou [11] showed the global well-posedness of the problem with rough initial data

$$u_0 \in \mathcal{D}_{A_q}^{1-\frac{1}{r},r} \cap H_0^1 \quad \text{with } 1 < r < \infty, 2 < q < \infty \text{ and } \theta_0 \in H^1,$$

which improved the results in [12] for $(u_0, \theta_0) \in H^3 \times H^2$ and in [13] for $(u_0, \theta_0) \in H^2 \times H^1$.

Here the space $\mathcal{D}_{A_q}^{1-\frac{1}{r},r}$ denotes some fractional domain of the Stokes operator in L^q with $2 - \frac{2}{r}$ derivatives (see Danchin [14]); moreover, we have

$$\mathcal{D}_{A_q}^{1-\frac{1}{r},r} \hookrightarrow B_{q,r}^{2-\frac{2}{r}} \cap L^q. \tag{1.16}$$

The second aim of this paper is to prove a similar result to problem (1.9)-(1.15), we will prove the following.

Theorem 1.2 *Let* $u_0 \in \mathcal{D}_{A_q}^{1-\frac{1}{r},r} \cap H_0^1$ *with* $1 < r < \infty$, $2 < q < \infty$ *and* $\theta_0 \in L^q$, $\phi_0 \in H^4$. *Then problem* (1.9)-(1.15) *has a unique solution* (u, θ, ϕ) *satisfying*

$$\begin{aligned} & u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2), \\ & u \in C([0,T];\mathcal{D}_{A_q}^{1-\frac{1}{r},r}) \cap L^r(0,T;W^{2,q}), \qquad u_t \in L^r(0,T;L^q), \\ & \theta \in L^\infty(0,T;L^q), \qquad \phi \in L^\infty(0,T;H^4) \cap L^2(0,T;H^5), \\ & \phi_t \in L^\infty(0,T;L^2) \cap L^2(0,T;H^2) \end{aligned} \tag{1.17}$$

for any fixed $T > 0$.

Finally, we consider the following model in electrohydrodynamics [15]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi = (n - p)\nabla \psi, \tag{1.18}$$

$$\operatorname{div} u = 0, \tag{1.19}$$

$$\partial_t n + u \cdot \nabla n - \operatorname{div}(\nabla n - n\nabla \psi) = 0, \tag{1.20}$$

$$\partial_t p + u \cdot \nabla p - \operatorname{div}(\nabla p + p\nabla \psi) = 0, \tag{1.21}$$

$$-\Delta \psi = p - n + D(x) \tag{1.22}$$

in $\Omega \times (0,\infty)$ with the boundary and initial conditions

$$u \cdot \nu = 0, \qquad \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.23}$$

$$(u,n,p)(\cdot,0) = (u_0,n_0,p_0) \quad \text{in } \Omega. \tag{1.24}$$

Here n, p and ψ denote the anion concentration, cation concentration and electric potential, respectively. $D(x)$ is the doping profile.

Equations (1.20), (1.21) and (1.22) appear in the context as the Nernst-Plank equation in astronomy [16] and as the Van Roosbroeck system in semiconductor devices [17].

The third aim of this paper is to prove a similar result to problem (1.18)-(1.24), we will prove the following.

Theorem 1.3 *Let* $u_0 \in L^2$, $\operatorname{rot} u_0 \in L^\infty$ *and* $n_0, p_0 \in B_{q,r}^{2-\frac{2}{r}}$ *with* $1 < r < \infty$ *and* $2 < q < \infty$ *and* $n_0, p_0 \geq 0$ *in* Ω *and* $D \in L^\infty(\Omega)$. *Then problem* (1.18)-(1.24) *has a unique solution* (u, n, p, ψ) *satisfying*

$$\begin{aligned} & u \in L^\infty(0,T;L^2), \qquad \operatorname{rot} u \in L^\infty(0,T;L^\infty), \\ & 0 \leq n,p \in C([0,T];B_{q,r}^{2-\frac{2}{r}}) \cap L^r(0,T;W^{2,q}), \qquad n_t,p_t \in L^r(0,T;L^q), \\ & \psi \in C([0,T];W^{2,q}) \cap L^r(0,T;W^{4,q}), \qquad \psi_t \in L^r(0,T;W^{2,q}) \end{aligned} \tag{1.25}$$

for any fixed $T > 0$.

Since the proof of Theorem 1.3 is very similar to that of Theorem 1.1 and that of [6], we omit the details here.

Now we recall the maximal regularity for the heat equation [18] and the Stokes system [14], which are critical to the proof of our main theorems.

Lemma 1.1 ([18]) *Assume that* $\theta_0 \in B_{q,r}^{2-\frac{2}{r}}$ *and* $f \in L^r(0,T;L^q)$ *with* $1 < r, q < \infty$. *Then the problem*

$$\begin{cases} \partial_t \theta - \Delta\theta = f, \\ \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \theta(\cdot,0) = \theta_0 \quad \text{in } \Omega, \end{cases} \tag{1.26}$$

has a unique solution θ *satisfying the following inequality for any fixed* $T > 0$:

$$\begin{aligned} &\|\theta\|_{C([0,T];B_{q,r}^{2-\frac{2}{r}})} + \|\theta\|_{L^r(0,T;W^{2,q})} + \|\theta_t\|_{L^r(0,T;L^q)} \\ &\quad \le C\bigl(\|\theta_0\|_{B_{q,r}^{2-\frac{2}{r}}} + \|f\|_{L^r(0,T;L^q)}\bigr), \end{aligned} \tag{1.27}$$

with $C := C(r,q,\Omega)$.

Lemma 1.2 ([14]) *Assume that* $u_0 \in \mathcal{D}_{A_q}^{1-\frac{1}{r},r}$ *and* $g \in L^r(0,T;L^q)$ *with* $1 < r, q < \infty$. *Then the problem*

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi = g, \\ \operatorname{div} u = 0, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{1.28}$$

has a unique solution (u,π) *satisfying the following estimate for any fixed* $T > 0$:

$$\begin{aligned} &\|u\|_{C([0,T];\mathcal{D}_{A_q}^{1-\frac{1}{r},r})} + \|u\|_{L^r(0,T;W^{2,q})} + \|u_t\|_{L^r(0,T;L^q)} + \|\nabla\pi\|_{L^r(0,T;L^q)} \\ &\quad \le C\bigl(\|u_0\|_{\mathcal{D}_{A_q}^{1-\frac{1}{r},r}} + \|g\|_{L^r(0,T;L^q)}\bigr), \end{aligned} \tag{1.29}$$

with $C := C(r,q,\Omega)$.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To prove the existence part, we only need to show a priori estimates (1.8). The uniqueness can be proved by the standard energy method of Yudovich [19], and thus we omit the details here.

Testing (1.2) by θ and using (1.3), we see that

$$\|\theta\|_{L^2}^2 + 2\int_0^T \|\nabla\theta\|_{L^2}^2\,dt \le \|\theta_0\|_{L^2}^2. \tag{2.1}$$

Testing (1.1) by u and (1.4) by μ, respectively, summing up the resulting equations and using (1.5), (1.3) and (2.1), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \bigl(|u|^2 + |\nabla\phi|^2 + 2f(\phi)\bigr)\,dx + \int |\nabla\mu|^2\,dx \\
&\quad = \int \theta e_2 u\,dx \le \|\theta\|_{L^2}\|u\|_{L^2} \le C\|u\|_{L^2} \le C\|u\|_{L^2}^2 + C,
\end{aligned}$$

which gives

$$\sup_{0\le t\le 1}\int \bigl(|u|^2 + |\nabla\phi|^2 + f(\phi)\bigr)\,dx + \int_0^T\int |\nabla\mu|^2\,dx\,dt \le C. \tag{2.2}$$

Taking ∇ to (1.5) and testing by $\nabla\Delta\phi$, we infer that

$$\begin{aligned}
\int_0^T\int |\nabla\Delta\phi|^2\,dx\,dt &= -\int_0^T\int \nabla\mu\cdot\nabla\Delta\phi x\,dt - \int_0^T\int \nabla\bigl(\phi-\phi^3\bigr)\cdot\nabla\Delta\phi\,dx\,dt \\
&= -\int_0^T\int \nabla\mu\cdot\nabla\Delta\phi\,dx\,dt - \int_0^T\int \nabla\phi\cdot\nabla\Delta\phi\,dx\,dt \\
&\quad - 3\int_0^T\int \phi^2(\Delta\phi)^2\,dx\,dt - 6\int_0^T\int \phi|\nabla\phi|^2\Delta\phi\,dx\,dt \\
&\le -\int_0^T\int \nabla\mu\cdot\nabla\Delta\phi\,dx\,dt - \int_0^T\int \nabla\phi\cdot\nabla\Delta\phi\,dx\,dt \\
&\quad + C\int_0^T\int |\nabla\phi|^4\,dx\,dt \\
&\le -\int_0^T\int \nabla\mu\cdot\nabla\Delta\phi\,dx\,dt - \int_0^T\int \nabla\phi\cdot\nabla\Delta\phi\,dx\,dt \\
&\quad + C\int_0^T \|\nabla\phi\|_{L^2}^3\|\nabla\Delta\phi\|_{L^2}\,dt + C\int_0^T \|\nabla\phi\|_{L^2}^4\,dt \\
&\le \frac{1}{2}\int_0^T\int |\nabla\Delta\phi|^2\,dx\,dt + C\int_0^T\int |\nabla\mu|^2\,dx\,dt \\
&\quad + C\int_0^T\int |\nabla\phi|^2\,dx\,dt + C\int_0^T \|\nabla\phi\|_{L^2}^6\,dt + C\int_0^T \|\nabla\phi\|_{L^2}^4\,dt \\
&\le \frac{1}{2}\int_0^T\int |\nabla\Delta\phi|^2\,dx\,dt + C,
\end{aligned}$$

which leads to

$$\int_0^T\int |\nabla\Delta\phi|^2\,dx\,dt \le C. \tag{2.3}$$

Here we used the Gagliardo-Nirenberg inequality

$$\|\nabla\phi\|_{L^4} \le C\|\nabla\phi\|_{L^2}^{\frac{3}{4}}\|\nabla\Delta\phi\|_{L^2}^{\frac{1}{4}} + C\|\nabla\phi\|_{L^2}. \tag{2.4}$$

It follows from (2.2), (2.3), (1.6) and the H^3-regularity of the Poisson equation that

$$\int_0^T \|\phi\|_{H^3}^2\,dt \le C. \tag{2.5}$$

Denote the vorticity $\omega := \operatorname{rot} u := \partial_1 u_2 - \partial_2 u_1$ and $a \times b := a_1 b_2 - a_2 b_1$ for vectors $a := (a_1, a_2)$ and $b := (b_1, b_2)$.

Applying rot to (1.1), we deduce that

$$\partial_t \omega + u \cdot \nabla\omega = -\nabla\Delta\phi \times \nabla\phi + \partial_1\theta. \tag{2.6}$$

Testing (2.6) by ω and using (1.3), we get

$$\begin{aligned}\|\omega\|_{L^2}\frac{d}{dt}\|\omega\|_{L^2} &= \int(-\nabla\Delta\phi \times \nabla\phi + \partial_1\theta)\omega\,dx \\ &\le \|\omega\|_{L^2}\big(\|\nabla\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^2}\big),\end{aligned}$$

whence

$$\frac{d}{dt}\|\omega\|_{L^2} \le \|\nabla\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^2}.$$

Integrating the above inequality, we observe that

$$\sup_{0\le t\le T}\|\omega\|_{L^2} \le \|\omega_0\|_{L^2} + \int_0^T\big(\|\nabla\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^2}\big) \le C. \tag{2.7}$$

Similarly, testing (2.6) by $|\omega|^{s-2}\omega$ and using (1.3), we derive

$$\|\omega\|_{L^s}^{s-1}\frac{d}{dt}\|\omega\|_{L^s} \le \big(\|\nabla\Delta\phi\|_{L^s}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^s}\big)\|\omega\|_{L^s}^{s-1},$$

whence

$$\frac{d}{dt}\|\omega\|_{L^s} \le \|\nabla\Delta\phi\|_{L^s}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^s}.$$

Integrating the above inequality, one has

$$\sup_{0\le t\le T}\|\omega\|_{L^s} \le \|\omega_0\|_{L^s} + \int_0^T\big(\|\nabla\Delta\phi\|_{L^s}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^s}\big)\,dt. \tag{2.8}$$

Taking $s \to +\infty$, we have

$$\sup_{0\le t\le T}\|\omega\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^T\big(\|\nabla\Delta\phi\|_{L^\infty}\|\nabla\phi\|_{L^\infty} + \|\partial_1\theta\|_{L^\infty}\big)\,dt. \tag{2.9}$$

Using Lemma 1.1 with $f := -u \cdot \nabla\theta$, we have

$$\begin{aligned}
&\|\theta\|_{C([0,T];B_{q,r}^{2-\frac{2}{r}})} + \|\theta\|_{L^r(0,T;W^{2,q})} + \|\theta_t\|_{L^r(0,T;L^q)} \\
&\leq C\|\theta_0\|_{B_{q,r}^{2-\frac{2}{r}}} + C\|u \cdot \nabla\theta\|_{L^r(0,T;L^q)} \\
&\leq C + C\|u\|_{L^\infty(0,T;L^q)}\|\nabla\theta\|_{L^r(0,T;L^\infty)} \leq C + C\|\nabla\theta\|_{L^r(0,T;L^\infty)} \\
&\leq C + C\epsilon\|\nabla^2\theta\|_{L^r(0,T;L^q)} + C\|\theta\|_{L^r(0,T;L^2)},
\end{aligned}$$

which yields

$$\|\theta\|_{C([0,T];B_{q,r}^{2-\frac{2}{r}})} + \|\theta\|_{L^r(0,T;W^{2,q})} + \|\theta_t\|_{L^r(0,T;L^q)} \leq C. \tag{2.10}$$

Here we used the interpolation inequality

$$\|\nabla\theta\|_{L^\infty} \leq \epsilon\|\nabla^2\theta\|_{L^q} + C\|\theta\|_{L^2} \tag{2.11}$$

for any $0 < \epsilon < 1$.

It follows from (2.10) that

$$\|\nabla\theta\|_{L^1(0,T;L^\infty)} \leq C. \tag{2.12}$$

Testing (1.4) by $\Delta^2\phi$, using (2.2) and (2.7), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\Delta\phi)^2\,dx + \int\left(\Delta^2\phi\right)^2\,dx = -\int u \cdot \nabla\phi\Delta^2\phi\,dx + \int \Delta f'(\phi)\Delta^2\phi\,dx \\
&\leq \|u\|_{L^4}\|\nabla\phi\|_{L^4}\|\Delta^2\phi\|_{L^2} + \left\|\Delta f'(\phi)\right\|_{L^2}\|\Delta^2\phi\|_{L^2} \\
&\leq C\|\nabla\phi\|_{L^4}\|\Delta^2\phi\|_{L^2} + C\left(\|\phi\|_{L^\infty}^2\|\Delta\phi\|_{L^2} + \|\phi\|_{L^\infty}\|\nabla\phi\|_{L^4}^2 + \|\Delta\phi\|_{L^2}\right)\|\Delta^2\phi\|_{L^2} \\
&\leq C\|\nabla\phi\|_{L^4}\|\Delta^2\phi\|_{L^2} + C\left(\|\Delta\phi\|_{L^2}\|\Delta^2\phi\|_{L^2}^{\frac{1}{2}} + C\|\Delta\phi\|_{L^2} + 1\right)\|\Delta^2\phi\|_{L^2} \\
&\leq \frac{1}{2}\|\Delta^2\phi\|_{L^2}^2 + C\|\Delta\phi\|_{L^2}^2 + C\|\Delta\phi\|_{L^2}^4 + C,
\end{aligned}$$

which implies

$$\|\phi\|_{L^\infty(0,T;H^2)} \leq C, \qquad \|\phi\|_{L^2(0,T;H^4)} \leq C. \tag{2.13}$$

Here we used the Gagliardo-Nirenberg inequalities

$$\|\phi\|_{L^\infty} \leq C\|\phi\|_{L^2}^{\frac{3}{4}}\|\Delta^2\phi\|_{L^2}^{\frac{1}{4}} + C\|\phi\|_{L^2}, \tag{2.14}$$

$$\|\nabla\phi\|_{L^4}^2 \leq C\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2} + C\|\nabla\phi\|_{L^2}^2. \tag{2.15}$$

It follows from (2.8), (2.12) and (2.13) that

$$\|\omega\|_{L^\infty(0,T;L^s)} \leq C \quad \text{for any } 2 \leq s < \infty. \tag{2.16}$$

Testing (1.1) by u_t and using (1.3), (2.1), (2.13) and (2.16), we have

$$\begin{aligned}\|u_t\|_{L^2} &\le \|u\cdot\nabla u\|_{L^2} + \|\Delta\phi\nabla\phi\|_{L^2} + \|\theta\|_{L^2}\\ &\le \|u\|_{L^4}\|\nabla u\|_{L^4} + \|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^\infty} + \|\theta\|_{L^2}\\ &\le C + C\|\nabla\phi\|_{L^\infty}.\end{aligned} \tag{2.17}$$

Applying ∂_t to (1.4), testing by ϕ_t, using (1.3), (2.13), and (2.17), we reach

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \phi_t^2\,dx + \int (\Delta\phi_t)^2\,dx &= \int \partial_t f'(\phi)\Delta\phi_t\,dx - \int u_t\cdot\nabla\phi\phi_t\,dx\\ &= \int \bigl(3\phi^2\phi_t - \phi_t\bigr)\Delta\phi_t\,dx + \int u_t\phi\nabla\phi_t\,dx\\ &\le C\|\phi_t\|_{L^2}\|\Delta\phi_t\|_{L^2} + \|u_t\|_{L^2}\|\phi\|_{L^\infty}\|\nabla\phi_t\|_{L^2}\\ &\le C\|\phi_t\|_{L^2}\|\Delta\phi_t\|_{L^2} + C\bigl(1+\|\nabla\phi\|_{L^\infty}\bigr)\|\nabla\phi_t\|_{L^2}\\ &\le \frac{1}{2}\|\Delta\phi_t\|_{L^2}^2 + C\|\phi_t\|_{L^2}^2 + C + C\|\nabla\phi\|_{L^\infty}^2,\end{aligned}$$

which gives

$$\|\phi_t\|_{L^\infty(0,T;L^2)} \le C, \qquad \|\phi_t\|_{L^2(0,T;H^2)} \le C. \tag{2.18}$$

Here we used the inequality

$$\|\nabla\phi_t\|_{L^2} \le C\|\Delta\phi_t\|_{L^2}$$

due to the inequality

$$\|v\|_{L^2} \le C\|\operatorname{div} v\|_{L^2} + C\|\operatorname{rot} v\|_{L^2}$$

for $v = \nabla\phi_t$ and $v\cdot n = 0$ on $\partial\Omega$.

By the standard H^s-regularity theory of elliptic equations, it follows from (1.4), (1.5), (2.13), (2.16) and (2.18) that

$$\|\phi\|_{L^\infty(0,T;H^4)} + \|\phi\|_{L^2(0,T;H^5)} \le C, \tag{2.19}$$

whence

$$\|\nabla\Delta\phi\|_{L^2(0,T;L^\infty)} \le C. \tag{2.20}$$

It follows from (2.9), (2.12) and (2.20) that

$$\|\omega\|_{L^\infty(0,T;L^\infty)} \le C. \tag{2.21}$$

This completes the proof. □

3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. To prove the existence part, we only need to show a priori estimates (1.17).

First, testing (1.10) by $|\theta|^{q-2}\theta$ and using (1.11), we see that

$$\|\theta\|_{L^\infty(0,T;L^q)} \leq \|\theta_0\|_{L^q}. \tag{3.1}$$

Next, we still have (2.2) and (2.5).

In the following proofs, we will use the Gagliardo-Nirenberg inequalities

$$\|\nabla\phi\|_{L^4} \leq C\|\nabla\phi\|_{L^2}^{\frac{3}{4}}\|\phi\|_{H^3}^{\frac{1}{4}}, \tag{3.2}$$

$$\|\Delta\phi\|_{L^4} \leq C\|\nabla\phi\|_{L^2}^{\frac{1}{4}}\|\phi\|_{H^3}^{\frac{3}{4}}. \tag{3.3}$$

Denoting $\tilde{\pi} := \pi - f(\phi)$, testing (1.9) by $\nabla\tilde{\pi} - \Delta u$, using (3.2), (3.3), (2.2), (2.5) and (3.1), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx + \int |\nabla\tilde{\pi} - \Delta u|^2\,dx \\
&\quad = \int (\Delta\phi\nabla\phi + \theta e_2 - u\cdot\nabla u)(\nabla\tilde{\pi} - \Delta u)\,dx \\
&\quad \leq \big(\|\Delta\phi\|_{L^4}\|\nabla\phi\|_{L^4} + \|\theta\|_{L^2} + \|u\|_{L^4}\|\nabla u\|_{L^4}\big)\|\nabla\tilde{\pi} - \Delta u\|_{L^2} \\
&\quad \leq C\big(\|\phi\|_{H^3} + 1 + \|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\cdot\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{\pi} - \Delta u\|_{L^2}^{\frac{1}{2}}\big)\|\nabla\tilde{\pi} - \Delta u\|_{L^2} \\
&\quad \leq \frac{1}{2}\|\nabla\tilde{\pi} - \Delta u\|_{L^2}^2 + C\|\phi\|_{H^3}^2 + C + C\|\nabla u\|_{L^2}^4,
\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)} \leq C. \tag{3.4}$$

Here we used the H^2-estimates of the Stokes system

$$\|u\|_{H^2} \leq C\|\nabla\tilde{\pi} - \Delta u\|_{L^2}. \tag{3.5}$$

We still have (2.13).

It follows from (1.9), (3.1), (3.4) and (2.13) that

$$\|u_t\|_{L^2(0,T;L^2)} \leq C. \tag{3.6}$$

We still have (2.19).

Using Lemma 1.2 with $g := \theta e_2 + \Delta\phi\nabla\phi - u\cdot\nabla u$ and $\tilde{\pi} := \pi - f(\phi)$, we have

$$\begin{aligned}
&\|u\|_{C([0,T];\mathcal{D}_{A_q}^{1-\frac{1}{r},r})} + \|u\|_{L^r(0,T;W^{2,q})} + \|u_t\|_{L^r(0,T;L^q)} \\
&\quad\le C\big(\|u_0\|_{\mathcal{D}_{A_q}^{1-\frac{1}{r},r}} + \|u\cdot\nabla u\|_{L^r(0,T;L^q)} + \|\Delta\phi\cdot\nabla\phi\|_{L^r(0,T;L^q)} + \|\theta\|_{L^r(0,T;L^q)}\big) \\
&\quad\le C + C\|u\cdot\nabla u\|_{L^r(0,T;L^q)} \\
&\quad\le C + C\|u\|_{L^\infty(0,T;L^q)}\|\nabla u\|_{L^r(0,T;L^\infty)} \\
&\quad\le C + C\|\nabla u\|_{L^r(0,T;L^\infty)} \\
&\quad\le C + C\epsilon\|\nabla^2 u\|_{L^r(0,T;L^q)} + C\|u\|_{L^r(0,T;L^q)},
\end{aligned}$$

which gives

$$\|u\|_{C([0,T];\mathcal{D}_{A_q}^{1-\frac{1}{r},r})} + \|u\|_{L^r(0,T;W^{2,q})} + \|u_t\|_{L^r(0,T;L^q)} \le C. \tag{3.7}$$

Here we used inequality (2.11) for $\theta = u$.

This completes the proof of (1.17).

Now we are in a position to prove the uniqueness part. To this end, let $(u_i, \pi_i, \theta_i, \phi_i)$ $(i = 1, 2)$ be two solutions to problem (1.9)-(1.15), set

$$\begin{aligned}
&\delta u := u_1 - u_2, \qquad \delta\pi := \pi_1 - \pi_2, \qquad \delta\theta := \theta_1 - \theta_2, \\
&\delta\phi := \phi_1 - \phi_2, \qquad \tilde{\pi}_i = \pi_i + f(\phi_i), \qquad \delta\tilde{\pi} := \tilde{\pi}_1 - \tilde{\pi}_2
\end{aligned}$$

and define ξ satisfying

$$\begin{aligned}
&-\Delta\xi = \delta\theta, \\
&\xi = 0 \quad \text{on } \partial\Omega \times (0,\infty).
\end{aligned} \tag{3.8}$$

Then $(\delta u, \delta\theta, \delta\phi)$ satisfy

$$\partial_t\delta u + u_1\cdot\nabla\delta u + \delta u\nabla u_2 + \nabla\delta\tilde{\pi} - \Delta\delta u = \Delta\phi_1\nabla\delta\phi + \Delta\delta\phi\nabla\phi_2 + \delta\theta e_2, \tag{3.9}$$

$$\partial_t\delta\theta + u_1\cdot\nabla\delta\theta + \delta u\cdot\nabla\theta_2 = 0, \tag{3.10}$$

$$\partial_t\delta\phi + u_1\cdot\nabla\delta\phi + \delta u\cdot\nabla\phi_2 = -\Delta^2\delta\phi + \Delta\big(f'(\phi_1) - f'(\phi_2)\big). \tag{3.11}$$

Testing (3.9) by δu and using (1.17) and (1.11), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\delta u|^2\,dx + \int|\nabla\delta u|^2\,dx \\
&\quad= -\int\delta u\cdot\nabla u_2\cdot\delta u\,dx + \int\Delta\phi_1\cdot\nabla\delta\phi\cdot\delta u\,dx \\
&\qquad + \int\Delta\delta\phi\nabla\phi_2\cdot\delta u\,dx - \int\Delta\xi e_2\delta u\,dx \\
&\quad\le \|\nabla u_2\|_{L^2}\|\delta u\|_{L^4}^2 + \|\Delta\phi_1\|_{L^\infty}\|\nabla\delta\phi\|_{L^2}\|\delta u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+ \|\nabla\phi_2\|_{L^\infty}\|\Delta\delta\phi\|_{L^2}\|\delta u\|_{L^2} + \|\nabla\xi\|_{L^2}\|\nabla\delta u\|_{L^2} \\
&\le C\|\delta u\|_{L^4}^2 + C\|\nabla\delta\phi\|_{L^2}\|\delta u\|_{L^2} + C\|\Delta\delta\phi\|_{L^2}\|\delta u\|_{L^2} + \|\nabla\xi\|_{L^2}\|\nabla\delta u\|_{L^2} \\
&\le \frac{1}{8}\|\nabla\delta u\|_{L^2}^2 + C\|\delta u\|_{L^2}^2 + C\|\delta\phi\|_{L^2}^2 + \frac{1}{8}\|\Delta\delta\phi\|_{L^2}^2 + C\|\nabla\xi\|_{L^2}^2. \qquad (3.12)
\end{aligned}
$$

Testing (3.10) by ξ and using (1.17) and (1.11), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int|\nabla\xi|^2\,dx &= \int u_1\nabla\Delta\xi\cdot\xi\,dx - \int\delta u\nabla\theta_2\xi\,dx \\
&= -\int u_1\Delta\xi\nabla\xi\,dx + \int\delta u\theta_2\nabla\xi\,dx \\
&= -\sum_{i,j}\int\partial_j u_{1i}\partial_i\xi\partial_j\xi\,dx + \int\delta u\theta_2\nabla\xi\,dx \\
&\le C\|\nabla u_1\|_{L^\infty}\|\nabla\xi\|_{L^2}^2 + \|\theta_2\|_{L^q}\|\delta u\|_{L^{\frac{2q}{q-2}}}\|\nabla\xi\|_{L^2} \\
&\le C\|\nabla u_1\|_{L^\infty}\|\nabla\xi\|_{L^2}^2 + C\|\delta u\|_{L^2}^{1-\frac{3}{q}}\|\nabla\delta u\|_{L^2}^{\frac{2}{q}}\|\nabla\xi\|_{L^2} \\
&\le \frac{1}{8}\|\nabla\delta u\|_{L^2}^2 + C\|\nabla u_1\|_{L^\infty}\|\nabla\xi\|_{L^2}^2 + C\|\nabla\xi\|_{L^2}^2 + C\|\delta u\|_{L^2}^2. \qquad (3.13)
\end{aligned}
$$

Testing (3.11) by $\delta\phi$ and using (1.17) and (1.11), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\delta\phi)^2\,dx + \int(\Delta\delta\phi)^2\,dx \\
&= -\int\delta u\cdot\nabla\phi_2\cdot\delta\phi\,dx + \int\big(f'(\phi_1)-f'(\phi_2)\big)\Delta\delta\phi\,dx \\
&\le \|\nabla\phi_2\|_{L^\infty}\|\delta u\|_{L^2}\|\delta\phi\|_{L^2} + C\|\delta\phi\|_{L^2}\|\Delta\delta\phi\|_{L^2} \\
&\le C\|\delta u\|_{L^2}^2 + C\|\delta\phi\|_{L^2}^2 + \frac{1}{8}\|\Delta\delta\phi\|_{L^2}^2. \qquad (3.14)
\end{aligned}
$$

Summing up (3.12), (3.13) and (3.14), and using the Gronwall inequality, we conclude that

$$
\delta u = 0, \qquad \xi = 0 \quad \text{and} \quad \delta\phi = 0.
$$

This completes the proof. □

4 Concluding remarks

The Cahn-Hilliard-Boussinesq system and a related system play an important role in the mathematical study of multi-phase flows. The applications of these systems cover a very wide range of physical objects, such as complicated phenomena in fluid mechanics involving phase transition, two-phase flow under shear through an order parameter formulation, the spinodal decomposition of binary fluid in a Hele-Shaw cell, tumor growth, cell sorting, and two phase flows in porous media.

In this paper, we have obtained the following global well-posedness results:

(1) If initial data $\phi_0 \in H^4$, $u_0 \in L^2$, $\operatorname{rot} u_0 \in L^\infty$ and $\theta_0 \in B_{q,r}^{2-\frac{2}{r}}$ with $1 < r < \infty$ and $2 < q < \infty$, then problem (1.1)-(1.7) admits a unique global solution.

(2) If initial data $u_0 \in \mathcal{D}_{A_q}^{1-\frac{1}{r},r} \cap H_0^1$ with $1 < r < \infty$, $2 < q < \infty$ and $\theta_0 \in L^q$, $\phi_0 \in H^4$, then problem (1.9)-(1.15) admits a unique global solution.

(3) If initial data $u_0 \in L^2$, $\operatorname{rot} u_0 \in L^\infty$ and $n_0, p_0 \in B_{q,r}^{2-\frac{2}{r}}$ with $1 < r < \infty$ and $2 < q < \infty$ and $n_0, p_0 \geq 0$ in Ω and $D \in L^\infty(\Omega)$, then problem (1.18)-(1.24) admits a unique global solution.

Acknowledgements
CM is partially supported by NSFC (Grant No. 11661070) and the Scientific Research Foundation of the Higher Education Institutions of Gansu Province (Grant No. 2016B-077).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Tianshui Normal University, Tianshui, 741000, P.R. China. [2]Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, P.R. China. [3]Center for Information, Nanjing Forestry University, Nanjing, 210037, P.R. China. [4]Department of Applied Mathematics, Nanjing Forestry University, Nanjing, 210037, P.R. China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 April 2017 Accepted: 3 August 2017 Published online: 18 August 2017

References
1. Boyer, F: Mathematical study of multi-phase flow under shear through order parameter formulation. Asymptot. Anal. **20**, 175-212 (1999)
2. Zhao, K: Global regularity for a coupled Cahn-Hilliard-Boussinesq system on bounded domains. Q. Appl. Math. **69**(2), 331-356 (2011)
3. Zhou, Y, Fan, J: The vanishing viscosity limit for a 2D Cahn-Hilliard-Navier-Stokes system with a slip boundary condition. Nonlinear Anal., Real World Appl. **14**(2), 1130-1134 (2013)
4. Zhou, Y, Fan, J: Blow-up criteria of smooth solutions for the Cahn-Hilliard-Boussinesq system with zero viscosity in a bounded domain. Abstr. Appl. Anal. **2012**, Article ID 802876 (2012)
5. Jiang, Z, Fan, J: Vanishing heat conductivity limit for the 2D Cahn-Hilliard-Boussinesq system. Bound. Value Probl. **2011**, Article ID 54 (2011)
6. Zhou, D, Li, Z: Global well-posedness for the 2D Boussinesq equations with zero viscosity. arXiv:1603.08301v1
7. Zhao, K: 2D inviscid heat conductive Boussinesq equations on a bounded domain. Mich. Math. J. **59**(2), 329-352 (2010)
8. Jin, L, Fan, J: Uniform regularity for the 2D Boussinesq system with a slip boundary condition. J. Math. Anal. Appl. **400**(1), 96-99 (2013)
9. Fan, J, Alzahrani, F, Hayat, T, Nakamura, G, Zhou, Y: Global regularity for the 2D liquid crystal model with mixed partial viscosity. Anal. Appl. **13**(2), 185-200 (2015)
10. Jin, L, Fan, J, Nakamura, G, Zhou, Y: Partial vanishing viscosity limit for the 2D Boussinesq system with a slip boundary condition. Bound. Value Probl. **2012**, Article ID 20 (2012)
11. Zhou, D: Global regularity for the two-dimensional Boussinesq equations without diffusivity in bounded domains. arXiv:1510.01960v1
12. Hu, W, Kukavica, I, Ziane, M: On the regularity for the Boussinesq equations in a bounded domain. J. Math. Phys. **54**, Article ID 081507 (2013)
13. Lai, M, Pan, R, Zhao, K: Initial boundary value problem for two-dimensional viscous Boussinesq equations. Arch. Ration. Mech. Anal. **199**, 736-760 (2011)
14. Danchin, R: Density-dependent incompressible fluids in bounded domains. J. Math. Fluid Mech. **8**, 333-381 (2006)
15. Rubinstein, I: Electro-Diffusion of Ions. Society for Industrial and Applied Mathematics, Philadelphia (1990)
16. Biler, P, Dolbeault, J: Long time behavior of solutions to Nernst-Plank and Debye-Hünkel drift-diffusion systems. Ann. Henri Poincaré **1**, 461-472 (2000)
17. Selberharr, S: Analysis and Simulation of Semiconductor Devices. Springer, Berlin (1983)
18. Amann, H: Linear and Quasilinear Parabolic Problem. Vol. I. Abstract Linear Theory. Birkhauser Boston, Boston (1995)
19. Yudovich, V: Non-stationary flows of an ideal incompressible fluid. USSR Comput. Math. Math. Phys. **3**(6), 1407-1456 (1963)