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Marcinkiewicz integrals associated with Schrödinger operator and their commutators on vanishing generalized Morrey spaces

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^n$ and the non-negative potential V belongs to the reverse Hölder class RH_q for $q \geq n/2$. In this paper, we study the boundedness of the Marcinkiewicz integral operators μ_j^L and their commutators $[b, \mu_j^L]$ with $b \in BMO_\theta(\rho)$ on generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operators μ_j^L from one vanishing generalized Morrey space $VM_{p,\varphi_1}^{\alpha,V}$ to another $VM_{p,\varphi_2}^{\alpha,V}$, $1 < p < \infty$ and from the space $VM_{1,\varphi_1}^{\alpha,V}$ to the weak space $VWM_{1,\varphi_2}^{\alpha,V}$. When b belongs to $BMO_\theta(\rho)$ and (φ_1, φ_2) satisfies some conditions, we also show that $[b, \mu_j^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{p,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{p,\varphi_2}^{\alpha,V}$, $1 < p < \infty$.

MSC: 42B35; 35J10

Keywords: Schrödinger operator; Marcinkiewicz integral; vanishing generalized Morrey space; commutator; BMO

1 Introduction and results

In this paper, we consider the Schrödinger differential operator

$$L = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, n \geq 3,$$

where $V(x)$ is a non-negative potential belonging to the reverse Hölder class RH_q for $q \geq n/2$.

A non-negative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to RH_q, $1 < q \leq \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)\,dy\right)^{1/q} \leq \left(\frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy\right) \tag{1.1}$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a non-negative polynomial, then $V \in RH_\infty$. Obviously, $RH_{q_2} \subset RH_{q_1}$, if $q_1 < q_2$. It is worth pointing out that the RH_q class is such that, if $V \in RH_q$

for some $q > 1$, then there exists an $\epsilon > 0$, which depends only n and the constant C in (1.1), such that $V \in RH_{q+\epsilon}$. Throughout this paper, we always assume that $0 \not\equiv V \in RH_{n/2}$.

For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\,dy \le 1\right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \not\equiv 0$. In particular, $m_V(x) = 1$ when $V = 1$ and $m_V(x) \sim 1 + |x|$ when $V(x) = |x|^2$.

According to [1], the new BMO space $BMO_\theta(\rho)$ with $\theta \ge 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - b_B\right| dy \le C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)\,dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequality above. Clearly, $BMO \subset BMO_\theta(\rho)$.

The classical Morrey spaces were originally introduced by Morrey in [2] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the reader to [2–4]. The classical version of Morrey spaces is equipped with the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda < n$ and $1 \le p < \infty$. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [5–8]).

The vanishing Morrey space $VM_{p,\lambda}$ in its classical version was introduced in [9], where applications to PDE were considered. We also refer to [10] and [11] for some properties of such spaces. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n, 0<t<r} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

We now present the definition of generalized Morrey spaces (including weak version) associated with Schrödinger operator, which introduced by second author in [12].

Definition 1.1 Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. We denote by $M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))}.$$

Also $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in WL^p_{\mathrm{loc}}(\mathbb{R}^n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-n/p} \|f\|_{WL_p(B(x,r))} < \infty.$$

Remark 1.1

(i) When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [2].

(ii) When $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [13].

(iii) When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Mizuhara and Nakai in [7, 8].

(iv) The generalized Morrey space associated with Schrödinger operator $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ was introduced by the second author in [12].

For brevity, in the sequel we use the notations

$$\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))}.$$

Definition 1.2 The vanishing generalized Morrey space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ associated with Schrödinger operator is defined as the spaces of functions $f \in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) = 0. \tag{1.2}$$

The vanishing weak generalized Morrey space $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ associated with Schrödinger operator is defined as the spaces of functions $f \in WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r) = 0.$$

The vanishing spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r),$$

$$\|f\|_{VWM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{W,p,\varphi}(f;x,r),$$

respectively.

We define the Marcinkiewicz integral associated with the Schrödinger operator L by

$$\mu^L_j f(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} K^L_j(x,y)f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $K^L_j(x,y) = \widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R^L_j = \frac{\partial}{\partial x_j}L^{-1/2}$, $j = 1,\dots,n$.

Let b be a locally integrable function, the commutator generalized by μ^L_j and b be defined by

$$\left[b,\mu^L_j\right]f(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} K^L_j(x,y)\big(b(x)-b(y)\big)f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Let $\widetilde{K_j^{\triangle}}(x,y)$ denote the kernel of the classical Riesz transform $R_j = \frac{\partial}{\partial x_j}\triangle^{-1/2}$. When $V = 0$, then $K_j^{\triangle}(x,y) = \widetilde{K_j^{\triangle}}(x,y)|x-y| = \frac{(x_j-y_j)/|x-y|}{|x-y|^{n-1}}$. Obviously, $\mu_j^{\triangle}f(x)$ is the classical Marcinkiewicz integral. Therefore, it will be an interesting thing to study the property of μ_j^L.

The area of Marcinkiewicz integral associated with the Schrödinger operator has been under intensive research recently. Gao and Tang in [14] showed that μ_j^L is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$, and bounded from $L_1(\mathbb{R}^n)$ to weak $WL_1(\mathbb{R}^n)$. Chen and Zou in [15] proved that the commutator $[b, \mu_j^L]$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$, where b belongs to $BMO_\theta(\rho)$. In [16–18], Akbulut *et al.* obtained the boundedness of μ_j^L and $[b, \mu_j^L]$ on the generalized Morrey space $M_{p,\varphi}$, Chen and Jin in [19] showed the boundedness of μ_j^L and $[b, \mu_j^L]$ on the Morrey spaces $L_{p,\lambda}^{\alpha,V}$ associated with Schrödinger operator.

In this paper, we study the boundedness of the Marcinkiewicz integral operators μ_j^L on generalized Morrey space $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ associated with Schrödinger operator and vanishing generalized Morrey space $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ associated with Schrödinger operator. When b belongs to the new BMO function spaces $BMO_\theta(\rho)$, we also show that $[b, \mu_j^L]$ is bounded on $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$.

Definition 1.3 We denote by $\Omega_p^{\alpha,V}$ the set of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that, for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n}\left\| \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{R}^n}\left\| \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively.

For the non-triviality of the space $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_p^{\alpha,V}$. Our main results are as follows.

Theorem 1.1 *Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 \leq p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_p^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \leq c_0\varphi_2(x,r), \tag{1.3}$$

where c_0 does not depend on x and r. Then the operator μ_j^L is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{p,\varphi_2}^{\alpha,V}$ for $p > 1$ and from $M_{1,\varphi_1}^{\alpha,V}$ to $WM_{1,\varphi_2}^{\alpha,V}$. Moreover, for $p > 1$

$$\left\|\mu_j^L f\right\|_{M_{p,\varphi_2}^{\alpha,V}} \leq C\|f\|_{M_{p,\varphi_1}^{\alpha,V}}$$

and for $p = 1$

$$\left\|\mu_j^L f\right\|_{WM_{1,\varphi_2}^{\alpha,V}} \leq C\|f\|_{M_{1,\varphi_1}^{\alpha,V}}.$$

Theorem 1.2 *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < \infty$, and $\varphi_1, \varphi_2 \in \Omega_p^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \leq c_0\varphi_2(x,r), \tag{1.4}$$

where c_0 does not depend on x and r. Then the operator $[b,\mu_j^L]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{p,\varphi_2}$ and

$$\big\|[b,\mu_j^L]f\big\|_{M^{\alpha,V}_{p,\varphi_2}} \le C[b]_\theta \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.$$

Definition 1.4 We denote by $\Omega^{\alpha,V}_{p,1}$ the set of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n}\inf_{r>\delta}\left(1+\frac{r}{\rho(x)}\right)^{-\alpha}\varphi(x,r) > 0, \quad \text{for some } \delta > 0, \tag{1.5}$$

and

$$\lim_{r\to 0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\frac{r^{n/p}}{\varphi(x,r)} = 0.$$

For the non-triviality of the space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega^{\alpha,V}_{p,1}$.

Theorem 1.3 *Let $V \in RH_{n/2}$, $\alpha \ge 0$, $1 \le p < \infty$ and $\varphi_1, \varphi_2 \in \Omega^{\alpha,V}_{p,1}$ satisfy the condition*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\,\frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t)\,\frac{dt}{t} \le C_0\varphi_2(x,r), \tag{1.6}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator μ_j^L is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{p,\varphi_2}$ for $p > 1$ and from $VM^{\alpha,V}_{1,\varphi_1}$ to $VWM^{\alpha,V}_{1,\varphi_2}$.

Theorem 1.4 *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < \infty$, and $\varphi_1, \varphi_2 \in \Omega^{\alpha,V}_{p,1}$ satisfy the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\,\frac{dt}{t} \le c_0\varphi_2(x,r), \tag{1.7}$$

where c_0 does not depend on x and r,

$$\lim_{r\to 0}\frac{\ln\frac{1}{r}}{\inf_{x\in\mathbb{R}^n}\varphi_2(x,r)} = 0 \tag{1.8}$$

and

$$c_\delta := \int_\delta^\infty \big(1+|\ln t|\big)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\,\frac{dt}{t} < \infty \tag{1.9}$$

for every $\delta > 0$.

Then the operator $[b,\mu_j^L]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{p,\varphi_2}$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some preliminaries

We would like to recall the important properties concerning the function $\rho(x)$.

Lemma 2.1 ([20]) *Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}} \tag{2.1}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.2 *Let $x \in B(x_0, r)$. Then for $k \in \mathbb{N}$ we have*

$$\frac{1}{(1+\frac{2^k r}{\rho(x)})^N} \lesssim \frac{1}{(1+\frac{2^k r}{\rho(x_0)})^{N/(k_0+1)}}.$$

Proof By (2.1) we get

$$\begin{aligned}\frac{1}{(1+\frac{2^k r}{\rho(x)})^N} &\lesssim \frac{1}{(1+\frac{2^k r}{\rho(x_0)(1+\frac{|x-x_0|}{\rho(x_0)})^{\frac{k_0}{k_0+1}}})^N}\\ &\lesssim \frac{(1+\frac{|x-x_0|}{\rho(x_0)})^{\frac{k_0 N}{k_0+1}}}{(1+\frac{2^k r}{\rho(x_0)})^N} \lesssim \frac{1}{(1+\frac{2^k r}{\rho(x_0)})^{N/(k_0+1)}}.\end{aligned}$$

□

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

Lemma 2.3 ([1]) *Let $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\left(\frac{1}{|B|}\int_B \left|b(y)-b_B\right|^s dy\right)^{1/s} \leq [b]_\theta\left(1+\frac{r}{\rho(x)}\right)^{\theta'}$$

for all $B = B(x, r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in (2.1).

Lemma 2.4 ([1]) *Let $1 \leq s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x, r)$. Then*

$$\left(\frac{1}{|2^k B|}\int_{2^k B} \left|b(y)-b_B\right|^s dy\right)^{1/s} \leq [b]_\theta k\left(1+\frac{2^k r}{\rho(x)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$, with θ' as in Lemma 2.3.

The following results give the estimates of the kernel of μ_j^L the boundedness of μ_j^L and their commutators on L_p.

Lemma 2.5 ([20]) *If $V \in RH_{n/2}$, then, for every N, there exists a constant C such that*

$$\left|K_j^L(x,y)\right| \le \frac{C}{(1+\frac{|x-y|}{\rho(x)})^N} \frac{1}{|x-y|^{n-1}}. \tag{2.2}$$

Lemma 2.6 ([16]) *Let $V \in RH_{n/2}$. Then*

$$\left\|\mu_j^L(f)\right\|_{L_p(\mathbb{R}^n)} \le C\|f\|_{L_p(\mathbb{R}^n)}$$

holds for $1 < p < \infty$, and

$$\left\|\mu_j^L(f)\right\|_{WL_1(\mathbb{R}^n)} \le C\|f\|_{L_1(\mathbb{R}^n)}.$$

Lemma 2.7 ([15]) *Let $V \in RH_{n/2}$, $1 < p < \infty$ and $b \in BMO_\theta(\rho)$. Then*

$$\left\|\left[b,\mu_j^L\right](f)\right\|_{L_p(\mathbb{R}^n)} \le C[b]_\theta \|f\|_{L_p(\mathbb{R}^n)}.$$

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 2.8 ([21]) *Let f be a real-valued non-negative function and measurable on E. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E} f(x)\right)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

Lemma 2.9 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$.*

(i) *If*

$$\sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \textit{for some } t > 0 \textit{ and for all } x \in \mathbb{R}^n, \tag{2.3}$$

then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

(ii) *If*

$$\sup_{0<r<\tau} \left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} = \infty \quad \textit{for some } \tau > 0 \textit{ and for all } x \in \mathbb{R}^n, \tag{2.4}$$

then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n) = \Theta$.

Proof (i) Let (2.3) be satisfied and f be not equivalent to zero. Then $\sup_{x\in\mathbb{R}^n} \|f\|_{L_p(B(x,t))} > 0$, hence

$$\begin{aligned}\|f\|_{M_{p,\varphi}^{\alpha,V}} &\ge \sup_{x\in\mathbb{R}^n} \sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))} \\ &\ge \sup_{x\in\mathbb{R}^n} \|f\|_{L_p(B(x,t))} \sup_{t<r<\infty} \left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1} r^{-\frac{n}{p}}.\end{aligned}$$

Therefore $\|f\|_{M_{p,\varphi}^{\alpha,V}} = \infty$.

(ii) Let $f \in M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and (2.4) be satisfied. Then there are two possibilities:

Case 1: $\sup_{0<r<t}(1+\frac{r}{\rho(x)})^{\alpha}\varphi(x,r)^{-1}=\infty$ for all $t>0$.
Case 2: $\sup_{0<r<t}(1+\frac{r}{\rho(x)})^{\alpha}\varphi(x,r)^{-1}<\infty$ for some $t\in(0,\tau)$.
For Case 1, by the Lebesgue differentiation theorem, for almost all $x\in\mathbb{R}^n$,

$$\lim_{r\to 0+}\frac{\|f\chi_{B(x,r)}\|_{L_p}}{\|\chi_{B(x,r)}\|_{L_p}}=|f(x)|. \tag{2.5}$$

We claim that $f(x)=0$ for all those x. Indeed, fix x and assume $|f(x)|>0$. Then by (2.5) there exists $t_0>0$ such that

$$r^{-\frac{n}{p}}\|f\|_{L_p(B(x,r))}\geq 2^{-1}v_n^{\frac{1}{p}}|f(x)|$$

for all $0<r\leq t_0$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Consequently,

$$\begin{aligned}\|f\|_{M^{\alpha,V}_{p,\varphi}}&\geq\sup_{0<r<t_0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}r^{-\frac{n}{p}}\|f\|_{L_p(B(x,r))}\\&\geq 2^{-1}v_n^{\frac{1}{p}}|f(x)|\sup_{0<r<t_0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}.\end{aligned}$$

Hence $\|f\|_{M^{\alpha,V}_{p,\varphi}}=\infty$, so $f\notin M_{p,\varphi}(\mathbb{R}^n)$ and we arrive at a contradiction.
Note that Case 2 implies that $\sup_{t<r<\tau}(1+\frac{r}{\rho(x)})^{\alpha}\varphi(x,r)^{-1}=\infty$, hence

$$\begin{aligned}\sup_{s<r<\infty}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}r^{-\frac{n}{p}}&\geq\sup_{t<r<\tau}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}r^{-\frac{n}{p}}\\&\geq\tau^{-\frac{n}{p}}\sup_{t<r<\tau}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi(x,r)^{-1}=\infty,\end{aligned}$$

which is the case in (i). □

3 Proof of Theorem 1.1

We first prove the following conclusions.

Theorem 3.1 *Let $V\in RH_{n/2}$. If $1<p<\infty$, then the inequality*

$$\left\|\mu_j^L(f)\right\|_{L_p(B(x_0,r))}\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}$$

holds for any $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, for $p=1$ the inequality

$$\left\|\mu_j^L(f)\right\|_{WL_1(B(x_0,r))}\lesssim r^{n}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{n}}\frac{dt}{t}$$

holds for any $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof For arbitrary $x_0\in\mathbb{R}^n$, set $B=B(x_0,r)$ and $\lambda B=B(x_0,\lambda r)$ for any $\lambda>0$. We write f as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2r)}(y)$ and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\left\|\mu_j^L(f)\right\|_{L_p(B(x_0,r))}\leq\left\|\mu_j^L(f_1)\right\|_{L_p(B(x_0,r))}+\left\|\mu_j^L(f_2)\right\|_{L_p(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of μ_j^L on $L_p(\mathbb{R}^n)$, $p > 1$, it follows that

$$\begin{aligned}\left\|\mu_j^L(f_1)\right\|_{L_p(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,r))} \\ &\lesssim r^{\frac{n}{p}} \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}. \end{aligned} \tag{3.1}$$

To estimate $\|\mu_j^L(f_2)\|_{L_p(B(x_0,2r))}$ obverse that $x \in B$, $y \in (2B)^c$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. Then by (2.2) and Minkowski's inequality

$$\begin{aligned}\sup_{x \in B(x_0,r)} \mu_j^L(f_2)(x) &\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-1}} \left(\int_{|x_0-y|}^{\infty} \frac{dt}{t^3}\right)^{1/2} dy \\ &\lesssim \sum_{k=1}^{\infty} \left(2^{k+1}r\right)^{-n} \int_{2^{k+1}B} \left|f(y)\right| dy.\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}\sup_{x \in B(x_0,r)} \mu_j^L(f_2)(x) &\lesssim \sum_{k=1}^{\infty} \|f\|_{L_p(2^{k+1}B)} \left(2^{k+1}r\right)^{-1-\frac{n}{p}} \int_{2^k r}^{2^{k+1}r} dt \\ &\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \\ &\lesssim \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.\end{aligned} \tag{3.2}$$

Then

$$\left\|\mu_j^L(f_2)\right\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \tag{3.3}$$

holds for $1 \le p < \infty$. Therefore, by (3.1) and (3.3) we get

$$\left\|\mu_j^L(f)\right\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \tag{3.4}$$

holds for $1 < p < \infty$.

When $p = 1$, from the boundedness of μ_j^L from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$, we get

$$\left\|\mu_j^L(f_1)\right\|_{WL_1(B(x_0,r))} \lesssim \|f\|_{L_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^n} \frac{dt}{t}.$$

From (3.3) we have

$$\left\|\mu_j^L(f_2)\right\|_{WL_1(B(x_0,r))} \le \left\|\mu_j^L(f_2)\right\|_{L_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^n} \frac{dt}{t}.$$

Then

$$\left\|\mu_j^L(f)\right\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^n} \frac{dt}{t}. \qquad \square$$

Remark 3.1 Note that another proof of Theorem 3.1 is given in [16].

Proof of Theorem 1.1 From Lemma 2.8, we have

$$\frac{1}{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x,s)s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecreasing function of t, and $f \in M_{p,\varphi_1}^{\alpha,V}$, then

$$\begin{aligned}
\frac{(1+\frac{t}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}} &\lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{(1+\frac{t}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x,s)s^{\frac{n}{p}}} \\
&\le \sup_{0<s<\infty} \frac{(1+\frac{s}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,s))}}{\varphi_1(x,s)s^{\frac{n}{p}}} \le \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Since $\alpha \ge 0$, and (φ_1, φ_2) satisfies the condition (1.3), then

$$\begin{aligned}
&\int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\quad = \int_{2r}^{\infty} \frac{(1+\frac{t}{\rho(x_0)})^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{(1+\frac{t}{\rho(x_0)})^{\alpha} t^{\frac{n}{p}}} \frac{dt}{t} \\
&\quad \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \int_{2r}^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{(1+\frac{t}{\rho(x_0)})^{\alpha} t^{\frac{n}{p}}} \frac{dt}{t} \\
&\quad \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\quad \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r).
\end{aligned} \tag{3.5}$$

Then by Theorem 3.1 we have

$$\begin{aligned}
&\left\|\mu_j^L(f)\right\|_{M_{p,\varphi_2}^{\alpha,V}} \\
&\quad \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/p} \left\|\mu_j^L(f)\right\|_{L_p(B(x_0,r))} \\
&\quad \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\quad \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Let $p = 1$. Similar to (3.5) we get

$$\int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^n} \frac{dt}{t} \lesssim \|f\|_{M_{1,\varphi_1}^{\alpha,V}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r).$$

From Theorem 3.1 we have

$$\begin{aligned}\|\mu_j^L(f)\|_{WM_{1,\varphi_2}^{\alpha,V}} &\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/p}\|\mu_j^L(f)\|_{WL_1(B(x_0,r))} \\ &\lesssim \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \\ &\lesssim \|f\|_{M_{1,\varphi_1}^{\alpha,V}}. \end{aligned}$$

□

4 Proof of Theorem 1.2

Similar to the proof of Theorem 1.1, it suffices to prove the following result.

Theorem 4.1 *Let $V\in RH_{n/2}$, $b\in BMO_\theta(\rho)$. If $1<p<\infty$, then the inequality*

$$\|[b,\mu_j^L(f)]\|_{L_p(B(x_0,r))}\lesssim [b]_\theta r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \tag{4.1}$$

holds for any $f\in L_{\mathrm{loc}}^p(\mathbb{R}^n)$.

Proof We write f as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2r)}(y)$. Then

$$\|[b,\mu_j^L](f)\|_{L_p(B(x_0,r))}\le \|[b,\mu_j^L](f_1)\|_{L_p(B(x_0,r))}+\|[b,\mu_j^L](f_2)\|_{L_p(B(x_0,r))}.$$

From the boundedness of $[b,\mu_j^L]$ on $L_p(\mathbb{R}^n)$ and (3.1) we get

$$\begin{aligned}\|[b,\mu_j^L](f_1)\|_{L_p(B(x_0,r))} &\lesssim [b]_\theta\|f\|_{L_p(B(x_0,2r))} \\ &\lesssim [b]_\theta r^{\frac{n}{p}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \\ &\lesssim [b]_\theta r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}. \end{aligned}\tag{4.2}$$

We now turn to deal with the term $\|[b,\mu_j^L](f_2)\|_{L_p(B(x_0,r))}$. For any given $x\in B(x_0,2r)$ we have

$$\begin{aligned}[b,\mu_j^L](f_2)(x) &= \left(\int_0^{\infty}\left|\int_{|x-y|\le t}K_j^L(x,y)\big(b(x)-b(y)\big)f(y)\,dy\right|^2\frac{dt}{t^3}\right)^{1/2} \\ &\le |b(x)-b_{2B}|\mu_j^L(f_2)(x)+\mu_j^L\big((b-b_{2B})f_2\big)(x). \end{aligned}$$

By (2.2), Lemma 2.2 and (3.2) we have

$$\begin{aligned}\sup_{x\in B(x_0,r)}\mu_j^L(f_2)(x) &\lesssim \int_{(2B)^c}\frac{1}{(1+\frac{|x-y|}{\rho(x)})^N}\frac{|f(y)|}{|x_0-y|^{n-1}}\left(\int_{|x_0-y|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy \\ &\lesssim \frac{1}{(1+\frac{2r}{\rho(x)})^N}\sum_{k=1}^{\infty}(2^{k+1}r)^{-n}\int_{2^{k+1}B}|f(y)|\,dy \\ &\lesssim \frac{1}{(1+\frac{2r}{\rho(x_0)})^{N/(k_0+1)}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}. \end{aligned}$$

Then by Lemma 2.3, and taking $N \geq (k_0+1)\theta$ we get

$$\begin{aligned}\big\|\big(b(x)-b_{2B}\big)\mu_j^L(f_2)\big\|_{L_p(B(x_0,r))} &\lesssim [b]_\theta r^{\frac{n}{p}}\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta-N/(k_0+1)}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}\\ &\lesssim [b]_\theta r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}. \end{aligned}\tag{4.3}$$

Finally, let us estimate $\|\mu_j^L((b-b_{2B})f_2)\|_{L_p(B(x_0,r))}$. By (2.2), Lemma 2.2 and (3.2) we have

$$\begin{aligned}\mu_j^L\big((b-b_{2B})f_2\big)(x) &\lesssim \int_{(2B)^c}\frac{1}{(1+\frac{|x-y|}{\rho(x)})^N}\frac{|b(y)-b_{2B}||f(y)|}{|x_0-y|^{n-1}}\left(\int_{|x_0-y|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy\\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(2^kr)^n(1+\frac{2^kr}{\rho(x)})^N}\int_{2^{k+1}B}\big|b(y)-b_{2B}\big|\big|f(y)\big|\,dy\\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(2^kr)^n(1+\frac{2^kr}{\rho(x_0)})^{N/(k_0+1)}}\int_{2^{k+1}B}\big|b(y)-b_{2B}\big|\big|f(y)\big|\,dy.\end{aligned}$$

Note that

$$\begin{aligned}\int_{2^{k+1}B}\big|b(y)-b_{2B}\big|\big|f(y)\big|\,dy &\lesssim \left(\int_{2^{k+1}B}\big|b(y)-b_{2B}\big|^{p'}\right)^{1/p'}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta k\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\theta'}\big(2^kr\big)^{\frac{n}{p'}}\|f\|_{L_p(B(x_0,2^{k+1}r))}.\end{aligned}$$

Then

$$\begin{aligned}\sup_{x\in B(x_0,r)}\mu_j^L\big((b-b_B)f_2\big)(x) &\lesssim [b]_\theta\sum_{k=1}^{\infty}\frac{k}{(1+\frac{2^kr}{\rho(x_0)})^{N/(k_0+1)-\theta'}}\big(2^kr\big)^{-\frac{n}{p}}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta\sum_{k=1}^{\infty}k\big(2^kr\big)^{-\frac{n}{p}}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta\sum_{k=1}^{\infty}k\int_{2^kr}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}.\end{aligned}$$

Since $2^kr\leq t\leq 2^{k+1}r$, $k\approx\ln\frac{t}{r}$. Thus

$$\begin{aligned}\sup_{x\in B(x_0,r)}\mu_j^L\big((b-b_B)f_2\big)(x) &\lesssim [b]_\theta\sum_{k=1}^{\infty}k\int_{2^kr}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}\\ &\lesssim [b]_\theta\sum_{k=1}^{\infty}\int_{2^kr}^{2^{k+1}r}\ln\frac{t}{r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}\\ &\lesssim [b]_\theta\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}.\end{aligned}$$

Then

$$\big\|\mu_j^L\big((b-b_{2B})f_2\big)\big\|_{L_p(B(x_0,r))}\lesssim [b]_\theta r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\,\frac{dt}{t}.\tag{4.4}$$

Combining (4.2), (4.3) and (4.4), the proof of Theorem 4.1 is completed. □

Remark 4.1 Note that, in the case $b \in BMO$, Theorem 4.1 was proved in [18].

Proof of Theorem 1.2 Since $f \in M^{\alpha,V}_{p,\varphi_1}$ and (φ_1, φ_2) satisfies the condition (1.4), by (3.5) we have

$$\begin{aligned}
&\int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&= \int_{2r}^{\infty} \frac{(1 + \frac{t}{\rho(x_0)})^{\alpha} \|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{(1 + \frac{t}{\rho(x_0)})^{\alpha} t^{\frac{n}{p}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{(1 + \frac{t}{\rho(x_0)})^{\alpha} t^{\frac{n}{p}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r).
\end{aligned}$$

Then from Theorem 4.1 we get

$$\begin{aligned}
&\big\| \big[b, \mu_j^L\big](f) \big\|_{M^{\alpha,V}_{p,\varphi_2}} \\
&\lesssim \sup_{x_0 \in \mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x_0, r)^{-1} r^{-n/p} \big\| \big[b, \mu_j^L\big](f) \big\|_{L_p(B(x_0,r))} \\
&\lesssim [b]_\theta \sup_{x_0 \in \mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x_0, r)^{-1} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\lesssim [b]_\theta \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}$$
□

5 Proof of Theorem 1.3

The statement is derived from the estimate (3.4). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from Theorem 1.1. So we only have to prove that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f; x, r) = 0 \quad \Rightarrow \quad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_2}\big(\mu_j^L(f); x, r\big) = 0 \tag{5.1}$$

and

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f; x, r) = 0 \quad \Rightarrow \quad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{1,\varphi_2}\big(\mu_j^L(f); x, r\big) = 0. \tag{5.2}$$

To show that $\sup_{x\in\mathbb{R}^n} (1 + \frac{r}{\rho(x)})^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mu_j^L(f)\|_{L_p(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (3.4):

$$\left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \big\| \mu_j^L(f) \big\|_{L_p(B(x,r))} \le C\big[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\big], \tag{5.3}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{(1+\frac{r}{\rho(x)})^\alpha}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))}\,dt$$

and

$$J_{\delta_0}(x,r) := \frac{(1+\frac{r}{\rho(x)})^\alpha}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))}\,dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_1(x,t)^{-1} t^{-n/p} \|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (1.6) and (5.3). This allows us to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now can be made by the choice of sufficiently small $r > 0$. Indeed, thanks to the condition (1.5) we have

$$J_{\delta_0}(x,r) \le c_{\sigma_0} \frac{(1+\frac{r}{\rho(x)})^\alpha}{\varphi_1(x,r)} \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1.2). Then, by (1.5) it suffices to choose r small enough so that

$$\sup_{x\in\mathbb{R}^n} \frac{(1+\frac{r}{\rho(x)})^\alpha}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (5.1).

The proof of (5.2) is similar to the proof of (5.1).

6 Proof of Theorem 1.4

The norm inequality is provided by Theorem 1.2, therefore, we only have to prove the implication

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_1(x,t)^{-1} t^{-n/p} \|f\|_{L_p(B(x,t))} = 0$$
$$\implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_2(x,t)^{-1} t^{-n/p} \left\|\left[b,\mu_j^L(f)\right]\right\|_{L_p(B(x,t))} = 0.$$

To check that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^\alpha \varphi_2(x,t)^{-1} t^{-n/p} \left\|\left[b,\mu_j^L(f)\right]\right\|_{L_p(B(x,t))} < \varepsilon \quad \text{for small } r,$$

we use the estimate (4.1):

$$\varphi_2(x,t)^{-1}t^{-n/p}\big\|\big[b,\mu_j^L(f)\big]\big\|_{L_p(B(x,t))}\lesssim\frac{[b]_\theta}{\varphi_2(x,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}.$$

We take $r<\delta_0$ where δ_0 will be chosen small enough and split the integration:

$$\left(1+\frac{t}{\rho(x)}\right)^\alpha\varphi_2(x,t)^{-1}t^{-n/p}\big\|\big[b,\mu_j^L(f)\big]\big\|_{L_p(B(x,t))}\le C\big[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)\big], \tag{6.1}$$

where

$$I_{\delta_0}(x,r):=\frac{(1+\frac{t}{\rho(x)})^\alpha}{\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r):=\frac{(1+\frac{t}{\rho(x)})^\alpha}{\varphi_2(x,r)}\int_{\delta_0}^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^\alpha\varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))}<\frac{\varepsilon}{2CC_0},\quad t\le\delta_0,$$

where C and C_0 are constants from (6.1) and (1.7), which yields the estimate of the first term uniform in $r\in(0,\delta_0)$: $\sup_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, $0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\le1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r)\le\frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ln\frac{1}{r}}{\varphi_2(x,r)}\|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (1.9) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.8) we can choose small r such that $\sup_{x\in\mathbb{R}^n}J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof.

7 Conclusions

In this paper, we study the boundedness of the Marcinkiewicz integral operators μ_j^L and their commutators $[b,\mu_j^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1,φ_2) which ensure the boundedness of the operators μ_j^L from one vanishing generalized Morrey space $VM^{\alpha,V}_{p,\varphi_1}$ to another $VM^{\alpha,V}_{p,\varphi_2}$, $1<p<\infty$ and from the space $VM^{\alpha,V}_{1,\varphi_1}$ to the weak space $VWM^{\alpha,V}_{1,\varphi_2}$. When b belongs to $BMO_\theta(\rho)$ and (φ_1,φ_2) satisfies some conditions, we also show that $[b,\mu_j^L]$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{p,\varphi_2}$ and from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{p,\varphi_2}$, $1<p<\infty$.

Our results about the boundedness of μ_j^L and $[b,\mu_j^L]$ from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{p,\varphi_2}$ (Theorems 1.1 and 1.2) are based on the local estimates for the Marcinkiewicz integrals (Theorem 3.1) and their commutators (Theorem 4.1).

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. VSG raised these interesting problems in the research. VSG, AA and MNO proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, and they read and approved the manuscript.

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