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Stability of the Hartree equation with time-dependent coefficients

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Abstract

In this paper, we investigate the stability for the nonlinear Hartree equation with time-dependent coefficients

$$i\partial_t u + \Delta u + \alpha(t)\frac{1}{|x|}u + \beta(t)(W * |u|^2)u = 0.$$

We first obtain the Lipschitz continuity of the solution $u = u(\alpha, \beta)$ with respect to coefficients α and β, and then prove that this equation is stable under the perturbation of coefficients. Our results improve some recent results.

MSC: 35Q55; 49J20

Keywords: nonlinear Hartree equation; stability; time-dependent coefficients; Lipschitz continuity

1 Introduction

Motivated by the nonlinearity management and dispersion management [1, 2] in the experimental work in Bose-Einstein condensates and optics, nonlinear Schrödinger equations have attracted more and more attention in both the physics and the mathematics fields; see [1–11] and the references therein.

In this paper, we consider the stability for the following nonlinear Schrödinger equation of Hartree type under the perturbation of coefficients:

$$\begin{cases} i\partial_t u + \Delta u + \alpha(t)\frac{1}{|x|}u + \beta(t)(W * |u|^2)u = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function in $(t,x) \in [0,\infty) \times \mathbb{R}^3$, $u_0 \in H^1(\mathbb{R}^3)$, $\alpha(t)$ and $\beta(t)$ are two real-valued functions in $t \in [0,\infty)$.

Equation (1.1) arises as phenomenological models in many different contexts: Hartree-Fock theory, quantum field theory, etc. In particular, when $W = |x|^{-\gamma}$ with $0 < \gamma < N$, this equation describes the mean-field limit of many-body quantum systems and has been extensively studied in [4, 12–22]. An essential feature of the Hartree equations is that the convolution kernel $|x|^{-\gamma}$ still retains the fine structure of microscopic two-body interactions of the quantum system. Therefore, it is interesting to extend mathematical methods developed for nonlinear Schrödinger equations with power nonlinearities $|u|^p u$ to the study of nonlinear Hartree equations.

When $\alpha(t)$ and $\beta(t)$ are two constants, Ma and Zhao in [23] investigated the stability of (1.1) under the perturbation of parameters. More precisely, they proved the following result.

Theorem A ([23], Theorem 1) *Assume that α_j, β_j are positive numbers satisfying $\alpha_j \to 1$, $\beta_j \to 1$. Let u_j be the H^1 solution of the following perturbed Hartree equation*:

$$i\partial_t u_j = \Delta u_j + \frac{\alpha_j}{|x|} u_j + \beta_j \left(\frac{1}{|x|} * |u_j|^2 \right) u_j \quad \text{on } \mathbb{R}^3, \tag{1.2}$$

with initial data

$$u_j|_{t=0} = u(0) \in H^1.$$

Then there exists a subsequence, still denoted by u_j, such that

$$u_j \to u \quad \text{weakly in } L^\infty(H^1), \tag{1.3}$$

where u a weakly continuous solution of (1.1) *with $\alpha(t) \equiv 1$, $\beta(t) \equiv 1$ and the initial data $u(0)$. Moreover, the energy and mass inequalities*

$$E(u(t)) \le E(u(0)), \qquad \|u(t)\|_{L^2} \le \|u(0)\|_{L^2}, \quad \forall t > 0, \tag{1.4}$$

follow for this weak solution u, where

$$E(u(t)) = \int_{\mathbb{R}^3} \left[\frac{1}{2} |\nabla u(t,x)|^2 \, dx - \frac{1}{4} (W * |u|^2)(x) |u(t,x)|^2 \, dx - \frac{1}{2} \frac{|u(t,x)|^2}{|x|} \right] dx. \tag{1.5}$$

In this paper, we will extend and improve this result in several aspects:

1. Our results hold for more general Hartree nonlinearities $(W * |u|^2)u$, where $W \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $p \ge 1$.
2. We extend this result to the time-dependent coefficients $\alpha(t)$ and $\beta(t)$.
3. We prove the locally Lipschitz continuity of the solution $u(\alpha, \beta)$ with respect to the coefficients α and β.
4. We prove that u_j strongly converge to u in $L^\gamma((0,T), W^{1,\rho})$ as $j \to \infty$, for every admissible pair (γ, ρ) and all $0 < T < T^*$.

More precisely, we will prove the following results.

Theorem 1.1 *Let $W : \mathbb{R}^3 \to \mathbb{R}$ be an even, real-valued function and $W \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $p \ge 1$, $\alpha_j \to \alpha$ and $\beta_j \to \beta$ in $H^1(0, T^*)$. Assume that u is the solution of* (1.1) *defined on the maximal interval $[0, T^*)$ with the initial value $u_0 \in H^1$. Suppose that u_j is the solution to the equation*

$$\begin{cases} i\partial_t u + \Delta u + \alpha_j(t) \frac{1}{|x|} u + \beta_j(t)(W * |u|^2) u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1.6}$$

Then:

(i) *Given any* $0 < T < T^*$, *the solution* u_j *exists on* $[0,T]$ *if* j *is sufficiently large.*

(ii) *For every admissible pair* (γ,ρ) *and* $0 < T < T^*$, $u_j \to u$ *in* $L^\gamma((0,T),W^{1,\rho})$ *as* $j \to \infty$. *In particular, convergence holds in* $C([0,T],H^1)$ *for* $0 < T < T^*$. *In addition, for every admissible pair* (γ,ρ)

$$\|u_j - u\|_{L^\gamma((0,T),W^{1,\rho})} \leq C\|\alpha_j - \alpha\|_{H^1(0,T)} + C\|\beta_j - \beta\|_{H^1(0,T)}, \tag{1.7}$$

where C *depends on* u_0, T, γ, ρ.

Remark For physical reasons, in this paper, we only study the three-dimensional case. In fact, we can investigate more general unbounded potentials in $\mathbb{R}^N$. Our results also follow if the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a real-valued function, satisfying:

(V1) $V \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \geq 1$, $q > N/2$;

(V2) $V, \nabla V \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \geq 2$, $q > N/2$.

A typical example satisfying these assumptions on V is $\frac{1}{|x|^\alpha}$ for some $\alpha > 0$.

This paper is organized as follows: in Section 2, we will collect some preliminaries such as local well-posedness, global existence and a Gronwall-type estimate. In Section 3, we firstly obtain the Lipschitz continuity of the solution $u = u(\alpha,\beta)$ with respect to coefficients α and β, and then prove Theorem 1.1.

Notation In this paper, we use the following notation. $C > 0$ denotes various positive constants. Because we only consider $\mathbb{R}^3$, we often use the abbreviations $L^r = L^r(\mathbb{R}^3)$, $H^1 = H^1(\mathbb{R}^3)$. Given any interval $I \subset \mathbb{R}$, the norms of mixed spaces $L^q(I, L^r(\mathbb{R}^3))$ and $L^q(I, H^s(\mathbb{R}^3))$ are denoted by $\|\cdot\|_{L^q_t L^r_x(I)}$ and $\|\cdot\|_{L^q(I,H^s)}$, respectively. We recall that a pair (q,r) is admissible if $\frac{2}{q} = 3(\frac{1}{2} - \frac{1}{r})$ and $2 \leq r \leq 6$. For simplicity, we always denote $V_1 = \frac{1}{|x|}\chi_{B_0}$ and $V_2 = \frac{1}{|x|}(1 - \chi_{B_0})$, where B_0 is the unit ball in $\mathbb{R}^3$ centered at the original point, χ_{B_0} is its characteristic function. It is obvious that $V_1 \in L^{\frac{3}{1+3\varepsilon}}$, $\nabla V_1 \in L^{\frac{3}{2+3\varepsilon}}$ with $\varepsilon > 0$ sufficiently small, $V_2 \in L^\infty$ and $\nabla V_2 \in L^\infty$.

2 Preliminaries

Firstly, we investigate the local well-posedness for (1.1). When α and β are two constants, the local well-posedness of (1.1) has been studied in [4]. In our case, when the terms $\alpha\frac{u}{|x|}$ and $\beta W * |u|^2 u$ have to be estimated in some norms, due to $\alpha, \beta \in H^1(0,\infty) \hookrightarrow L^\infty(0,\infty)$, we only need to take L^∞ norms of α and β. Therefore, by a similar method as that in [4], we can prove the local well-posedness of (1.1).

Lemma 2.1 *Assume that* $W : \mathbb{R}^3 \to \mathbb{R}$ *is an even, real-valued potential and* $W \in L^p + L^\infty$ *for some* $p \geq 1$. *Given* $A, B, M > 0$, *there exists* $\delta = \delta(A,B,M)$ *such that, for all* $\alpha, \beta \in H^1(0,\infty)$ *satisfying* $\|\alpha\|_{L^\infty} \leq A$, $\|\beta\|_{L^\infty} \leq B$ *and all* $u_0 \in H^1$ *satisfying* $\|u_0\|_{H^1} \leq M$, *there exists a unique solution* $u \in C([0,\delta],H^1)$ *of* (1.1) *and* $\|u\|_{L^\infty((0,\delta),H^1)} \leq 2\|u_0\|_{H^1}$. *In addition, the solution* u *of* (1.1) *satisfies*

$$\big\|u(t)\big\|_{L^2} = \|u_0\|_{L^2} \tag{2.1}$$

and

$$\frac{d}{dt}E(t) = -\frac{\alpha'(t)}{2}\int_{\mathbb{R}^3}\frac{|u(t,x)|^2}{|x|}\,dx - \frac{\beta'(t)}{4}\int_{\mathbb{R}^3}\big(W*|u|^2\big)(x)\big|u(t,x)\big|^2\,dx, \tag{2.2}$$

for all $t \in [0,\delta]$, *where*

$$E\big(u(t)\big) = \int_{\mathbb{R}^3}\left[\frac{1}{2}\big|\nabla u(t,x)\big|^2 - \frac{\alpha(t)}{2}\frac{|u(t,x)|^2}{|x|} - \frac{\beta(t)}{4}\big(W*|u|^2\big)(x)\big|u(t,x)\big|^2\right]dx. \tag{2.3}$$

Next, we consider the global existence for (1.1).

Lemma 2.2 *Let* $W:\mathbb{R}^3\to\mathbb{R}$ *be an even, real-valued potential and* $W \in L^\sigma + L^\infty$ *for some* $\sigma > 3/2$. *If* $u_0 \in H^1$, *then the solution* $u(t)$ *of* (1.1) *exists globally.*

Proof We first deduce from Hölder's and Young's inequalities that

$$\int_{\mathbb{R}^3}\frac{|u(x)|^2}{|x|}\,dx \le C\|V_1\|_{L^{\frac{3}{1+3\varepsilon}}}\|u\|^2_{L^{\frac{6}{2-3\varepsilon}}} + C\|V_2\|_{L^\infty}\|u\|^2_{L^2} \tag{2.4}$$

and

$$\int_{\mathbb{R}^3}\big(W*|u|^2\big)|u|^2\,dx \le C\|W_1\|_{L^\sigma}\|u\|^4_{L^{\frac{4\sigma}{2\sigma-1}}} + C\|W_2\|_{L^\infty}\|u\|^4_{L^2}, \tag{2.5}$$

where $W = W_1 + W_2$ with $W_1 \in L^\sigma$ and $W_2 \in L^\infty$.

Thus, by (2.2), (2.4) and (2.5), we derive

$$\begin{aligned}\big\|E'\big\|_{L^2(0,T)} \le{}& C\big\|\alpha'\big\|_{L^2(0,T)}\big(\|V_1\|_{L^{\frac{3}{1+3\varepsilon}}}\|u\|^2_{L^{\frac{6}{2-3\varepsilon}}} + \|V_2\|_{L^\infty}\|u_0\|^2_{L^2}\big)\\ &+ C\big\|\beta'\big\|_{L^2(0,T)}\big(\|W_1\|_{L^\sigma}\|u\|^4_{L^{\frac{4\sigma}{2\sigma-1}}} + \|W_2\|_{L^\infty}\|u_0\|^4_{L^2}\big).\end{aligned}$$

This and (2.3) yield

$$\begin{aligned}E(t) ={}& E(0) + \int_0^t E'(s)\,ds \le E(0) + \left(t\int_0^t \big(E'(s)\big)^2\,ds\right)^{1/2}\\ \le{}& E(0) + Ct^{1/2}\big\|\alpha'\big\|_{L^2(0,T)}\big(\|V_1\|_{L^{\frac{3}{1+3\varepsilon}}}\|u\|^2_{L^{\frac{6}{2-3\varepsilon}}} + \|V_2\|_{L^\infty}\|u_0\|^2_{L^2}\big)\\ &+ Ct^{1/2}\big\|\beta'\big\|_{L^2(0,T)}\big(\|W_1\|_{L^\sigma}\|u\|^4_{L^{\frac{4\sigma}{2\sigma-1}}} + \|W_2\|_{L^\infty}\|u_0\|^4_{L^2}\big).\end{aligned} \tag{2.6}$$

Combining (2.2), (2.4)-(2.6), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\big|\nabla u(t,x)\big|^2\,dx \le{}& E\big(u(t)\big) + C\int_{\mathbb{R}^3}\frac{|u(t,x)|^2}{|x|}\,dx + C\int_{\mathbb{R}^3}\big(W*|u|^2\big)(x)\big|u(t,x)\big|^2\,dx\\ \le{}& E(0) + C\|V_1\|_{L^{\frac{3}{1+3\varepsilon}}}\|u\|^2_{L^{\frac{6}{2-3\varepsilon}}} + C\|V_2\|_{L^\infty}\|u_0\|^2_{L^2}\\ &+ C\|W_1\|_{L^\sigma}\|u\|^4_{L^{\frac{4\sigma}{2\sigma-1}}} + C\|W_2\|_{L^\infty}\|u_0\|^4_{L^2}.\end{aligned} \tag{2.7}$$

On the other hand, we have the following Gagliardo-Nirenberg's inequalities:

$$\|u\|^2_{L^{\frac{6}{2-3\varepsilon}}} \le C\|u\|^{1+3\varepsilon}_{H^1}\|u\|^{1-3\varepsilon}_{L^2},$$

$$\|u\|^4_{L^{\frac{4\sigma}{2\sigma-1}}} \le C\|u\|^{\frac{3}{\sigma}}_{H^1}\|u\|^{\frac{4\sigma-3}{\sigma}}_{L^2}. \tag{2.8}$$

Taking ε such that $1+3\varepsilon<2$, and $\frac{3}{\sigma}<2$, we infer from (2.7) and Young's inequality with ε that

$$\|u(t)\|_{H^1} \le C\big(T, \|u_0\|_{H^1}, \|\alpha\|_{H^1(0,T)}, \|\beta\|_{H^1(0,T)}\big) \quad \text{for every } t\in[0,T], \tag{2.9}$$

for every $0<T<\infty$. This implies that the solution u of (1.1) is global. □

Finally, we recall a Gronwall-type estimate which is vital to obtain the Lipschitz continuity of solution $u(\alpha,\beta)$ with respect to coefficients α and β.

Lemma 2.3 ([24]) *Let $T>0$, $1\le p_2<q_2\le\infty$, $1\le p_1<q_1\le\infty$, and $a,b_1,b_2\ge 0$. If $f_1\in L^{q_1}(0,T)$, $f_2\in L^{q_2}(0,T)$ satisfying*

$$\|f_1\|_{L^{q_1}(0,t)} + \|f_2\|_{L^{q_2}(0,t)} \le a + b_1\|f_1\|_{L^{p_1}(0,t)} + b_2\|f_2\|_{L^{p_2}(0,t)},$$

for all $0<t<T$, then there exists $\Gamma=\Gamma(b_1,b_2,p,q,T)$ such that

$$\|f_1\|_{L^{q_1}(0,T)} + \|f_2\|_{L^{q_2}(0,T)} \le a\Gamma.$$

3 The proof of main results

In this section, we first prove that the solution of (1.1) depends local Lipschitz continuously on the coefficients α and β, and then show our main results.

Lemma 3.1 *Assume that $W:\mathbb{R}^3\to\mathbb{R}$ is an even, real-valued function and $W\in L^p+L^\infty$ for some $p\ge 1$. Given $u_0\in H^1$. Let $u\in L^\infty([0,T^*),H^1)$ be the corresponding solution of* (1.1) *with coefficients $\alpha,\beta\in H^1(0,T)$. There exists $\varepsilon>0$ such that if $\tilde\alpha$ and $\tilde\beta$ satisfy $\|\tilde\alpha-\alpha\|_{H^1(0,T)}<\varepsilon$, $\|\tilde\beta-\beta\|_{H^1(0,T)}<\varepsilon$ and $\tilde u$ is the corresponding solution of* (1.1) *with coefficients $\tilde\alpha$ and $\tilde\beta$. Then*:

(i) *Given any $0<T<T^*$, the solution $\tilde u$ exists on $[0,T]$.*

(ii) *For every admissible pair (γ,ρ)*

$$\|\tilde u-u\|_{L^\gamma((0,T),W^{1,\rho})} \le C\|\tilde\alpha-\alpha\|_{H^1(0,T)} + C\|\tilde\beta-\beta\|_{H^1(0,T)}, \tag{3.1}$$

where C depends on u_0, T, γ, ρ. In particular,

$$\|\tilde u-u\|_{L^\infty((0,T),H^1)} \le C\|\tilde\alpha-\alpha\|_{H^1(0,T)} + C\|\tilde\beta-\beta\|_{H^1(0,T)}.$$

Proof Firstly, we assume that the solution $\tilde u$ exists on $[0,T]$. Note that the following Duhamel's formulation:

$$u(t)=U(t)u_0+i\int_0^t U(t-s)\left(\alpha(s)\frac{1}{|x|}u(s)+\big(\beta(s)W*\big|u(s)\big|^2\big)u(s)\right)ds, \tag{3.2}$$

where $U(t):=e^{it\triangle}$ denotes the free Schrödinger propagator, which is isometric on H^s for every $s\ge 0$; see [4]. This yields

$$\tilde u(t)-u(t)=i\int_0^t U(t-r)\left(\frac{1}{|x|}(\tilde\alpha\tilde u-\alpha u)+\big(\tilde\beta\big(W*|\tilde u|^2\big)\tilde u-\beta\big(W*|u|^2\big)u\big)\right)(r)\,dr. \tag{3.3}$$

In the following, we set $\rho_1 = \frac{4p}{2p-1}$, taking γ_1 such that (γ_1, ρ_1) is an admissible pair. Applying Strichartz's estimate to (3.3), we deduce from Hölder's inequality that for $0 < t \leq T$

$$
\begin{aligned}
&\|\tilde{u} - u\|_{L_t^\gamma L_x^\rho(0,t)} \\
&\quad\leq C\big\|\tilde{\beta}\big(W * |\tilde{u}|^2\big)\tilde{u} - \tilde{\beta}\big(W * |u|^2\big)u\big\|_{L_t^{\gamma_1'} L_x^{\rho_1'}(0,t)} \\
&\qquad + C\big\|(\tilde{\beta} - \beta)\big(W * |u|^2\big)u\big\|_{L_t^{\gamma_1'} L_x^{\rho_1'}(0,t)} \\
&\qquad + C\big\|V_1(\tilde{\alpha}\tilde{u} - \alpha u)\big\|_{L_t^2 L_x^{\frac{6}{5}}(0,t)} + C\big\|V_2(\tilde{\alpha}\tilde{u} - \alpha u)\big\|_{L_t^1 L_x^2(0,t)} \\
&\quad\leq C\|\tilde{\beta}\|_{L^\infty(0,t)}\big(\|\tilde{u}\|^2_{L_t^\infty L_x^{\rho_1}(0,t)} + \|u\|^2_{L_t^\infty L_x^{\rho_1}(0,t)}\big)\|\tilde{u} - u\|_{L_t^{\gamma_1'} L_x^{\rho_1}(0,t)} \\
&\qquad + C\|\tilde{\beta} - \beta\|_{L^\infty(0,t)}\|\tilde{u}\|^3_{L_t^\infty L_x^{\rho_1}(0,t)} + C\|V_1\|_{L^{\frac{3}{1+3\varepsilon}}}\|\tilde{\alpha}\|_{L^\infty(0,t)}\|\tilde{u} - u\|_{L_t^2 L_x^{\frac{2}{1-2\varepsilon}}(0,t)} \\
&\qquad + \|V_1\|_{L^{\frac{3}{1+3\varepsilon}}}\|\tilde{\alpha} - \alpha\|_{L^\infty(0,t)}\|u\|_{L_t^2 L_x^{\frac{2}{1-2\varepsilon}}(0,t)} \\
&\qquad + C\|V_2\|_{L^\infty}\|\tilde{\alpha}\|_{L^\infty(0,t)}\|\tilde{u} - u\|_{L_t^1 L_x^2(0,t)} + \|V_2\|_{L^\infty}\|\tilde{\alpha} - \alpha\|_{L^\infty(0,t)}\|u\|_{L_t^1 L_x^2(0,t)} \\
&\quad\leq C\|\tilde{u} - u\|_{L_t^{\gamma_1'} L_x^{\rho_1}(0,t)} + C\|\tilde{u} - u\|_{L_t^2 L_x^{\frac{2}{1-2\varepsilon}}(0,t)} + C\|\tilde{u} - u\|_{L_t^1 L_x^2(0,t)} \\
&\qquad + C\|\tilde{\alpha} - \alpha\|_{H^1(0,t)} + C\|\tilde{\beta} - \beta\|_{H^1(0,t)},
\end{aligned}
\tag{3.4}
$$

which, together with Lemma 2.3, implies that for every admissible pair (γ, ρ)

$$
\|\tilde{u} - u\|_{L_t^\gamma L_x^\rho(0,T)} \leq C\|\tilde{\alpha} - \alpha\|_{H^1(0,T)} + C\|\tilde{\beta} - \beta\|_{H^1(0,T)},
\tag{3.5}
$$

where C depends on $\|\tilde{u}\|_{L^\infty((0,T),H^1)}$, $\|u\|_{L^\infty((0,T),H^1)}$, T, γ. In addition, by a similar argument to that of (3.4), the embedding $W^{1,\frac{2}{1-2\varepsilon}} \hookrightarrow L^{\frac{6}{1-6\varepsilon}}$, we obtain

$$
\begin{aligned}
&\|\nabla\tilde{u} - \nabla u\|_{L_t^\gamma L_x^\rho(0,t)} \\
&\quad\leq C\big\|\tilde{\beta}\big(W * |\tilde{u}|^2\big)\nabla\tilde{u} - \beta\big(W * |u|^2\big)\nabla u\big\|_{L_t^{\gamma_1'} L_x^{\rho_1'}(0,t)} \\
&\qquad + C\big\|\tilde{\beta}\big(W * \nabla|\tilde{u}|^2\big)\tilde{u} - \beta\big(W * \nabla|u|^2\big)u\big\|_{L_t^{\gamma_1'} L_x^{\rho_1'}(0,t)} \\
&\qquad + C\big\|(\tilde{\alpha} - \alpha)\nabla V_1\tilde{u}\big\|_{L_t^2 L_x^{\frac{6}{5}}(0,t)} + C\big\|\alpha\nabla V_1(\tilde{u} - u)\big\|_{L_t^2 L_x^{\frac{6}{5}}(0,t)} \\
&\qquad + C\big\|(\tilde{\alpha} - \alpha)\nabla V_2\tilde{u}\big\|_{L_t^1 L_x^2(0,t)} + C\big\|\alpha\nabla V_2(\tilde{u} - u)\big\|_{L_t^1 L_x^2(0,t)} \\
&\qquad + C\big\|(\tilde{\alpha} - \alpha)V_1\nabla\tilde{u}\big\|_{L_t^2 L_x^{\frac{6}{5}}(0,t)} + C\big\|\alpha V_1\nabla(\tilde{u} - u)\big\|_{L_t^2 L_x^{\frac{6}{5}}(0,t)} \\
&\qquad + C\big\|(\tilde{\alpha} - \alpha)V_2\nabla\tilde{u}\big\|_{L_t^1 L_x^2(0,t)} + C\big\|\alpha V_2\nabla(\tilde{u} - u)\big\|_{L_t^1 L_x^2(0,t)} \\
&\quad\leq C\|\tilde{u} - u\|_{L_t^{\gamma_1} L_x^{\rho_1}(0,t)} + C\big\|\nabla(\tilde{u} - u)\big\|_{L_t^{\gamma_1'} L_x^{\rho_1}(0,t)} \\
&\qquad + C\|\tilde{\alpha} - \alpha\|_{H^1(0,t)} + C\|\tilde{\beta} - \beta\|_{H^1(0,t)} + C\|\tilde{u} - u\|_{L_t^1 L_x^2(0,t)} \\
&\qquad + C\big\|\nabla(\tilde{u} - u)\big\|_{L_t^2 L_x^{\frac{2}{1-2\varepsilon}}(0,t)} + C\big\|\nabla(\tilde{u} - u)\big\|_{L_t^1 L_x^2(0,t)}.
\end{aligned}
\tag{3.6}
$$

Hence, it follows from Lemma 2.3 and (3.5) that for every admissible pair (γ,ρ)

$$\|\tilde{u}-u\|_{L^\gamma((0,T),W^{1,\rho})} \le C\|\tilde{\alpha}-\alpha\|_{H^1(0,T)} + C\|\tilde{\beta}-\beta\|_{H^1(0,T)}, \tag{3.7}$$

where C depends on $\|u_j\|_{L^\infty((0,T),H^1)}$, $\|u\|_{L^\infty((0,T),H^1)}$, T, γ.

Therefore, in order to prove this lemma, we only need to show that there exist $\varepsilon > 0$ and $M > 0$ such that if $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy $\|\tilde{\alpha}-\alpha\|_{H^1(0,T)} < \varepsilon$ and $\|\tilde{\beta}-\beta\|_{H^1(0,T)} < \varepsilon$, then the corresponding solution $\tilde{u}$ exists on $[0,T]$ and $\|\tilde{u}\|_{L^\infty((0,T),H^1)} \le M$.

To this aim, we set

$$M = 2\sup_{0\le t\le T}\|u(t)\|_{H^1} + 1. \tag{3.8}$$

With the notation of Lemma 2.1, let $\delta = \delta(A,B,M)$, where $A = \|\tilde{\alpha}\|_{L^\infty}$, $B = \|\tilde{\beta}\|_{L^\infty}$ and M is given by (3.8). We infer from Lemma 2.1 that $\tilde{u}$ exists on $[0,\delta]$ and that

$$\|\tilde{u}\|_{L^\infty((0,\delta),H^1)} \le 2\|u_0\|_{H^1}. \tag{3.9}$$

On the other hand, by (3.7), we have

$$\|\tilde{u}(\delta)-u(\delta)\|_{H^1} \le C\|\tilde{\alpha}-\alpha\|_{H^1(0,T)} + C\|\tilde{\beta}-\beta\|_{H^1(0,T)}. \tag{3.10}$$

Taking ε such that $C\varepsilon < M/4$, it follows that $\|\tilde{u}(\delta)\|_{H^1} < M$. Hence, we can repeat the argument to continue the solution also in the time interval $[\delta,2\delta]$, and so on. Since the solution $u(t)$ exists on $[0,T^*)$, and for any $0 < T < T^*$, we consider $[0,T] \subset [0,\delta]\cup\cdots\cup[(N-1)\delta,N\delta]$, $N = [\frac{T}{\delta}]+1$, where $[\cdot]$ denotes the integer part of the number. Thus, the solution $\tilde{u}$ exists on $[0,T]$ and $\|\tilde{u}\|_{L^\infty((0,T),H^1)} \le M$. This completes the proof. □

Proof of Theorem 1.1 Given $T \in (0,T^*)$. Let $A = \|\alpha\|_{L^\infty(0,T)}$, $B = \|\beta\|_{L^\infty(0,T)}$ and $M = 2\sup_{0\le t\le T}\|u(t)\|_{H^1} + 1$. With the notation of Lemma 2.1, setting $\delta = \delta(A,B,M)$. We infer from Lemma 2.1 that u_j exists on $[0,\delta]$ and satisfies

$$\limsup_{j\to\infty}\|u_j\|_{L^\infty((0,\delta),H^1)} < 2\|u_0\|_{H^1}. \tag{3.11}$$

Applying Lemma 3.1, we see that the conclusion holds on the interval $[0,\delta]$.

Let $0 < l \le T$ be such that u_j exists on $[0,l]$ for j sufficiently large and

$$\limsup_{j\to\infty}\|u_j\|_{L^\infty((0,l),H^1)} < \infty. \tag{3.12}$$

Then we infer from Lemma 3.1 that

$$u_j \to u \quad \text{in } L^\gamma\bigl((0,l),W^{1,\rho}\bigr) \text{ as } j\to\infty \tag{3.13}$$

for every admissible pair (γ,ρ). In particular, $u_j(l)\to u(l)$ in H^1 as $j\to\infty$, which, together with the definition of M yields that $\|u_j(l)\|_{H^1} < M$. Applying Lemma 2.1 with the initial value $u_j(l)$, it follows that u_j exists on $[0,l+\delta]$ and

$$\limsup_{j\to\infty}\|u_j\|_{L^\infty((0,l+\delta),H^1)} \le C. \tag{3.14}$$

It follows from Lemma 3.1 that the estimate (3.13) holds with l replaced by $l+\delta$ provided $l+\delta \leq T$. Iterating this argument, we see that

$$\limsup_{j\to\infty} \|u_j\|_{L^\infty((0,T),H^1)} \leq C. \tag{3.15}$$

This estimate with Lemma 3.1 yields the desired results. □

4 Conclusions

In this paper, we consider the stability for the nonlinear Schrödinger equation (1.1) of Hartree type under the perturbation of coefficients. We first obtain the Lipschitz continuity of the solution $u = u(\alpha,\beta)$ with respect to coefficients α and β by using Strichartz's estimates, and then prove that this equation is stable under the perturbation of coefficients by a bootstrap argument. Our results improve some recent results. In particular, the proof of the locally Lipschitz continuity contains a very general method that may be useful for other related problems.

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Competing interests
The authors declare that no competing interests exist.

Authors' contributions
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