Zhao and Tang *Boundary Value Problems* (2017) 2017:123
DOI 10.1186/s13661-017-0855-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Multiplicity results for impulsive fractional differential equations with p-Laplacian via variational methods

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Abstract
In this paper, we apply critical point theory and variational methods to study the multiple solutions of boundary value problems for an impulsive fractional differential equation with p-Laplacian. Some new criteria guaranteeing the existence of multiple solutions are established for the considered problem.

Keywords: fractional p-Laplacian; variational methods; multiple solutions; impulsive effects

1 Introduction and main results

Considering the following impulsive fractional differential equations:

$$\begin{cases} {}_tD_T^\alpha \Phi_p({}_0^cD_t^\alpha u(t)) + |u(t)|^{p-2}u(t) = f(t,u(t)), \quad 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}\Phi_p({}_0^cD_t^\alpha u))(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, $\Phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, $\Phi_p(0) = 0$, $p > 1$ and ${}_tD_T^\alpha$ denotes the right Riemann-Liouville fractional derivative of order α; $0 = t_0 < t_1 < \cdots < t_{m+1} = T$ and

$$\Delta\big({}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)\big)(t_j) = {}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)\big(t_j^+\big) - {}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)\big(t_j^-\big),$$

where

$${}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)\big(t_j^+\big) = \lim_{t\to t_j^+} {}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)(t),$$

$${}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)\big(t_j^-\big) = \lim_{t\to t_j^-} {}_tD_T^{\alpha-1}\Phi_p\big({}_0^cD_t^\alpha u\big)(t).$$

${}_0^cD_t^\alpha$ is the left Caputo fractional derivative of order α, $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ and $I_j \in C(\mathbb{R},\mathbb{R})$.

Fractional calculus is a generalization of classical derivatives and integrals to an arbitrary (non-integer) order. It represents a powerful tool in applied mathematics to deal with a myriad of problems from different fields such as physics, mechanics, electricity, control theory, rheology, signal and image processing, aerodynamics, electricity, *etc.* (see [1–3] and the references therein). Recently the theory and application of fractional differential equations have been rapidly developed. The existence and multiplicity of solutions to such problems have been extensively studied by many mathematicians; see the monographs of Podlubny [1], Kilbas [4], Diethelm [5], and also [6–12] and the references therein. The main classical techniques to study fractional differential equations are degree theory, the method of upper and lower solutions, and fixed point theorems. For some related work on the theory and application of fractional differential equations, we refer the interested reader to [11–17] and the references therein.

In recent years, variational methods and critical point theory have already been applied successfully to investigate the existence of solutions for nonlinear fractional boundary value problems [18–27]. By establishing a corresponding variational structure and using the Mountain Pass theorem, the authors [18] first dealt with the existence of solutions for a class of fractional boundary value problems. Since then the variational methods are applied to discuss the existence of solutions for fractional differential equations. The literature on this approach has been extended by many authors as [20–26]. Moreover, the p-Laplacian introduced by Leibenson (see [28]) often occurs in non-Newtonian fluid theory, nonlinear elastic mechanics and so on. Note that, when $p = 2$, the nonlinear and nonlocal differential operator ${}_tD_T^{\alpha}\Phi_p({}_0^cD_t^{\alpha})$ reduces to the linear differential operator ${}_tD_T^{\alpha}{}_0^cD_t^{\alpha}$, and further reduces to the local second-order differential operator $-d^2/dt^2$ when $\alpha = 1$.

Further, some authors have started to discuss the existence of solutions for impulsive fractional boundary value problems by using variational methods [29–37]. Taking a fractional Dirichlet problem with impulses as a model, Bonanno *et al.* [29] and Rodríguez-López and Tersian [30] investigated the following fractional differential equations with impulsive effects:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u), & 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), & j = 1,2,\dots,m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where $\lambda, \mu \in (0,+\infty)$ are two parameters. By applying the critical point theorem and variational methods, they obtained the existence results of at least one and three solutions for problem (1.2). In [32], the authors considered a class of nonlinear impulsive fractional differential systems including Lipschitz continuous nonlinear terms. Under suitable hypotheses and by applying variational methods, they obtained some new criteria guaranteeing that the studied systems have at least two nontrivial and nonnegative solutions. Furthermore, under appropriate hypotheses and by applying Morse theory coupled with local linking arguments, Zhao *et al.* [36] obtained the existence of at least one nontrivial solution for problem (1.2), in the case $\lambda = \mu = 1$.

Motivated by the described work, our goal is to apply variational methods to problem (1.1) and prove the existence of weak solutions under some suitable assumptions. With the impulsive effects and p-Laplacian operator taken into consideration, the corresponding variational functional φ will be more complicated. To the best of our knowledge, with exception of [38], little work is done on the existence and multiplicity of solutions for impulsive fractional differential problems with p-Laplacian by using variational methods. The main results of this paper are different from the aforementioned results, and extend the recent results studied in [22, 25, 29–35] in the sense that we deal with the case $p \neq 2$. The effectiveness of our results is illustrated by some examples.

In this paper, we need the following assumptions on the nonlinearity f and the impulsive terms I_j:

(H1) There exists a constant $\mu > p$ such that $I_j(u)u \leq \mu \int_0^u I_j(s)\,ds < 0$ for any $u \in E^{\alpha,p} \setminus \{0\}$, $j = 1, 2, \ldots, m$, where $E^{\alpha,p}$ will be introduced in Definition 2.3.

(H2) There exists a constant $\vartheta \in (p, \mu]$ such that $\vartheta F(t,u) \leq f(t,u)u$ for all $u \in E^{\alpha,p}$, $t \in [0,T]$, where $F(t,u) = \int_0^u f(t,s)\,ds$.

(H3) There exist constants $\delta, \gamma > 0$ such that $F^0 \leq \delta$ and $F_\infty \geq \gamma$, where

$$F^0 = \lim_{|u|\to 0} \sup \frac{F(t,u)}{|u|^\vartheta}, \qquad F_\infty = \lim_{|u|\to\infty} \inf \frac{F(t,u)}{|u|^\vartheta}.$$

(H4) There exist constants $\delta_j > 0$ such that $\int_0^u I_j(s)\,ds \geq -\delta_j |u|^\mu$, for all $u \in E^{\alpha,p} \setminus \{0\}$, where $j = 1, \ldots, m$.

Here are our main results.

Theorem 1.1 *Suppose that* (H1)-(H4) *hold. Then problem* (1.1) *admits at least two weak solutions.*

Theorem 1.2 *Suppose that* (H1)-(H4) *hold. Moreover, $f(t,u)$ and $I_j(u)$ are odd about u, where $j = 1, \ldots, m$. Then problem* (1.1) *admits infinitely many weak solutions.*

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas and a variational setting. In Section 3, we give the proofs of our main results.

2 Variational setting and preliminaries

To apply critical point theory to discuss the existence of solutions for problem (1.1), we present some basic notations and lemmas and construct a variational framework, which will be used in the proof of our main results.

Suppose that X is a real Banach space and the functional $\phi : X \to \mathbb{R}$ is differentiable. The functional ϕ satisfies the Palais-Smale condition if each sequence $\{u_n\}$ in the space X such that $\{\phi(u_n)\}$ is bounded and $\lim_{n\to\infty} \phi'(u_n) = 0$ admits a convergent subsequence.

Lemma 2.1 (Mountain pass theorem; see [39]) *Let $\phi \in C^1(X, \mathbb{R})$, and ϕ satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 such that u_1 is not in Ω and*

$$\inf_{v\in\partial\Omega} \phi(v) > \max\bigl\{\phi(u_0), \phi(u_1)\bigr\}.$$

Then there exists a critical point u of ϕ, i.e., $\phi'(u) = 0$, with

$$\phi(u) > \max\bigl\{\phi(u_0), \phi(u_1)\bigr\}.$$

Obviously, if either u_0 or u_1 is a critical point of ϕ then one obtains the existence of at least two critical points for ϕ.

Lemma 2.2 (Theorem 38.A in [40]) *For the functional $\phi : B \subseteq X \to \mathbb{R}$ with B not empty, $\min_{u\in B} \phi(u) = c$ admits a solution in the case that the following hold*:

(i) *X is a real reflexive Banach space*;

(ii) *B is bounded and weak sequentially closed*;

(iii) *ϕ is weakly sequentially lower semi-continuous in B, i.e., by definition, for every sequence $\{u_n\}$ in B such that $u_n \rightharpoonup u$ as $n \to \infty$, one has $\phi(u) \le \underline{\lim}_{n\to\infty}\phi(u_n)$.*

Lemma 2.3 (Theorem 9.12 in [41]) *Let X be an infinite dimensional Banach space and let $\phi \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition, and $\phi(0) = 0$. If $X = Y \oplus Z$, where Y is finite dimensional, and ϕ satisfies*

(i) *there exist constants $\rho, \eta > 0$ such that $\phi|_{\partial B_\rho \cap Z} \ge \eta$, and*

(ii) *for each finite dimensional subspace $W \subset E$, there is an $r = r(W)$ such that $\phi \le 0$ on $W \setminus B_{r(W)}$,*

then ϕ has an unbounded sequence of critical values.

Now we present some definitions and notations of the fractional calculus as follows (for details, see [1, 4, 5, 13, 18]):

Denote by $\mathrm{AC}([a, b])$ the space of absolutely continuous functions on $[a, b]$.

Definition 2.1 For $\alpha > 0$, the left and right Riemann-Liouville fractional derivatives of order α of a function $f \in \mathrm{AC}([a, b])$ are defined by

$$
\begin{aligned}
{}_aD_t^\alpha f(t) &\equiv \frac{d}{dt}{}_aD_t^{\alpha-1}f(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\alpha} f(s)\,ds\right), \quad t > a, \\
{}_tD_b^\alpha f(t) &\equiv -\frac{d}{dt}{}_tD_b^{\alpha-1}f(t) := -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\alpha} f(s)\,ds\right), \quad t < b.
\end{aligned}
$$

Definition 2.2 For $\alpha > 0$, the left and right Caputo fractional derivatives of order α of a function $f \in \mathrm{AC}([a, b])$ are defined by

$$
\begin{aligned}
{}_a^cD_t^\alpha f(t) &\equiv {}^cD_{a^+}^\alpha f(t) := {}_aD_t^{\alpha-1}f'(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha} f'(s)\,ds\right), \\
{}_t^cD_b^\alpha f(t) &\equiv {}^cD_{b^-}^\alpha f(t) := -{}_tD_b^{\alpha-1}f'(t) = -\frac{1}{\Gamma(1-\alpha)}\left(\int_t^b (s-t)^{-\alpha} f'(s)\,ds\right).
\end{aligned}
$$

When $\alpha = 1$, we obtain from Definitions 2.1 and 2.2 ${}_a^cD_t^1 f(t) = f'(t)$, ${}_t^cD_b^1 f(t) = -f'(t)$.

Let $C_0^\infty([0, T], \mathbb{R})$ be the set of all functions $x \in C^\infty([0, T], \mathbb{R})$ with $x(0) = x(T) = 0$ and the norm $\|x\|_\infty = \max_{[0,T]} |x(t)|$. Denote the norm of the space $L^p([0, T], \mathbb{R})$ for $1 \le p < \infty$ by $\|x\|_{L^p} = (\int_0^T |x(s)|^p\,ds)^{1/p}$.

Definition 2.3 Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}(0, T)$ (denoted by $E^{\alpha,p}$ for short) is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$, with respect to the following norm:

$$
\|u\|_{E^{\alpha,p}}^p = \int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^p dt + \int_0^T \left|u(t)\right|^p dt, \quad \forall u \in E^{\alpha,p}(0, T). \tag{2.1}
$$

According to [18], Proposition 3.1, it is well known that the space $E^{\alpha,p}$ is a reflexive and separable Banach space. For $u \in E^{\alpha,p}$, we have $u, {}_0^cD_t^\alpha u \in L^p([0,T],R)$, $u(0) = u(T) = 0$.

Proposition 2.1 ([13]) *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For any $u \in E^{\alpha,p}$, we have*

$$\|u\|_{L^p} \le A_p \left\| {}_0^cD_t^\alpha u \right\|_{L^p}, \tag{2.2}$$

where $A_p := \frac{T^\alpha}{\Gamma(\alpha+1)}$ is a positive constant, and if $\alpha > \frac{1}{p}$, then

$$\|u\|_\infty \le A_\infty \left\| {}_0^cD_t^\alpha u \right\|_{L^p}, \tag{2.3}$$

where $A_\infty := \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}$, $q = p(p-1)^{-1} > 1$ are two positive constants.

Proposition 2.2 ([13]) *Let $\frac{1}{p} < \alpha \le 1$ and $1 < p < \infty$. For any $u \in E^{\alpha,p}$, the imbedding of $E^{\alpha,p}$ in $C([0,T],\mathbb{R})$ is compact.*

Proposition 2.3 ([18], Proposition 3.3) *Assume that $\frac{1}{p} < \alpha \le 1$ and $1 < p < \infty$, and the sequence $\{u_k\}$ converges weakly to u in $E^{\alpha,p}$, i.e., $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R})$, i.e., $\|u_k - u\|_\infty \to 0$, as $k \to \infty$.*

Definition 2.4 Let $\mathrm{AC}([0,T])$ be the space of absolutely continuous functions on $[0,T]$. A function

$$u \in \left\{ u \in \mathrm{AC}\big([0,T]\big) : \int_{t_j}^{t_{j+1}} \left(\left| {}_0^cD_t^\alpha u(t) \right|^p + \left| u(t) \right|^p \right) dt < \infty, j = 0,1,\dots,m \right\}$$

is called a classical solution of (1.1), if u satisfies the first equation of (1.1) a.e. on $[0,T] \setminus \{t_1,\dots,t_m\}$, the limits ${}_tD_T^{\alpha-1}\Phi_p({}_0^cD_t^\alpha u)(t_j^+)$ and ${}_tD_T^{\alpha-1}\Phi_p({}_0^cD_t^\alpha u)(t_j^-)$ exist and satisfy the impulsive conditions of (1.1), and boundary condition $u(0) = u(T) = 0$ holds.

Definition 2.5 We say that a function $u \in E^{\alpha,p}$ is a weak solution of problem (1.1), if the following identity:

$$\begin{aligned} &\int_0^T \left| {}_0^cD_t^\alpha u(t) \right|^{p-2} {}_0^cD_t^\alpha u(t) {}_0^cD_t^\alpha v(t)\, dt + \int_0^T \left| u(t) \right|^{p-2} u(t) v(t)\, dt \\ &\quad + \sum_{j=1}^m I_j\big(u(t_j)\big) v(t_j) = \int_0^T f\big(t,u(t)\big) v(t)\, dt \end{aligned} \tag{2.4}$$

holds for any $v \in E^{\alpha,p}$.

In order to study problem (1.1), we define the functional $\varphi : E^{\alpha,p} \to \mathbb{R}$ by putting

$$\begin{aligned} \varphi(u) :&= \int_0^T \frac{1}{p} \left(\left| {}_0^cD_t^\alpha u(t) \right|^p + \left| u(t) \right|^p \right) dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\, ds - \int_0^T F\big(t,u(t)\big)\, dt \\ &= \frac{1}{p} \|u\|_{E^{\alpha,p}}^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\, ds - \int_0^T F\big(t,u(t)\big)\, dt. \end{aligned} \tag{2.5}$$

It is clear that φ is continuous and differentiable at any $u \in E^{\alpha,p}$ and

$$\begin{aligned}\varphi'(u)(v) = {} & \int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t)\,dt + \int_0^T \left|u(t)\right|^{p-2}u(t)v(t)\,dt \\ & + \sum_{j=1}^m I_j\big(u(t_j)\big)v(t_j) - \int_0^T f\big(t,u(t)\big)v(t)\,dt\end{aligned}$$

for any $v \in E^{\alpha,p}$. Moreover, the critical point of φ is a weak solution of (1.1).

Proposition 2.4 (see [4, 13])

(i) *Let* $\alpha > 0$, $p \geq 1$, $p \geq 1$, *and* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$, $q \neq 1$, *in the case when* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). *If* $u \in L^p(a,b)$ *and* $v \in L^q(a,b)$, *then*

$$\int_a^b \big({}_aD_t^{-\alpha}u(t)\big)v(t)\,dt = \int_a^b u(t)\big({}_tD_b^{-\alpha}v(t)\big)\,dt. \tag{2.6}$$

(ii) *Let* $0 < \alpha < 1$, $u \in \mathrm{AC}([a,b])$ *and* $v \in L^p(a,b)$ $(1 \leq p < \infty)$. *Then*

$$\int_a^b u(t)\big({}_a^cD_t^\alpha v(t)\big)\,dt = {}_tD_b^{\alpha-1}u(t)v(t)\big|_{t=a}^{t=b} + \int_a^b {}_tD_b^\alpha u(t)v(t)\,dt. \tag{2.7}$$

Proposition 2.5 *For any* $u, v \in E^{\alpha,p}$, *the following identity holds*:

$$\begin{aligned}&\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t)\,dt \\ &\quad = \int_0^T {}_tD_T^\alpha\big(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\big)v(t)\,dt \\ &\qquad - \sum_{j=1}^m \Delta\big({}_tD_T^{\alpha-1}\big(\left|{}_0^cD_t^\alpha u(t_j)\right|^{p-2}{}_0^cD_t^\alpha u(t_j)\big)\big)v(t_j).\end{aligned} \tag{2.8}$$

Proof Since ${}_0^cD_t^\alpha g(t) = {}_0D_t^{\alpha-1}g'(t)$ and ${}_tD_T^\alpha g(t) = -({}_tD_T^{\alpha-1}g(t))'$, it follows from (2.6) and (2.7) that

$$\begin{aligned}\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t)\,dt &= \sum_{j=0}^m \int_{t_j}^{t_{j+1}} {}_tD_T^{\alpha-1}\big(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\big)v'(t)\,dt \\ &= \sum_{j=0}^m {}_tD_T^{\alpha-1}\big(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\big)v(t)\big|_{t_j}^{t_{j+1}} \\ &\quad + \sum_{j=0}^m \int_{t_j}^{t_{j+1}} {}_tD_T^\alpha\big(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\big)v(t)\,dt \\ &= \int_0^T {}_tD_T^\alpha\big(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\big)v(t)\,dt \\ &\quad - \sum_{j=1}^m \Delta\big({}_tD_T^{\alpha-1}\big(\left|{}_0^cD_t^\alpha u(t_j)\right|^{p-2}{}_0^cD_t^\alpha u(t_j)\big)\big)v(t_j).\end{aligned}$$

This completes the proof. □

Proposition 2.6 *If the function $u \in E^{\alpha,p}$ is a weak solution of* (1.1), *then u is a classical solution of* (1.1).

Proof By standard arguments, if u is a classical solution of (1.1), then u is a weak solution. Conversely, if $u \in E^{\alpha,p}$ is a weak solution of (1.1), then by the definition of a weak solution, (2.4) holds for any $v \in E^{\alpha,p}$. For $j \in \{1,\dots,m\}$ we choose a function $v \in E^{\alpha,p}$ with $v(t) = 0$ for every $t \in [0,t_j] \cup [t_{j+1},T]$. Then

$$\int_{t_j}^{t_{j+1}} \left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t)\,dt + \int_0^T \left|u(t)\right|^{p-2}u(t)v(t)\,dt = \int_{t_j}^{t_{j+1}} f\bigl(t,u(t)\bigr)v(t)\,dt$$

and

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha v(t)\,dt &= \int_{t_j}^{t_{j+1}} {}_tD_T^\alpha\bigl(\left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\bigr)v(t)\,dt \\ &< \infty, \end{aligned} \tag{2.9}$$

which implies

$${}_tD_T^\alpha\Phi_p\bigl({}_0^cD_t^\alpha u(t)\bigr) + \left|u(t)\right|^{p-2}u(t) = f\bigl(t,u(t)\bigr) \tag{2.10}$$

for almost every $t \in (t_j,t_{j+1})$. Since $u \in E^{\alpha,p} \subseteq C([0,T])$, one has $\int_{t_j}^{t_{j+1}}(|{}_0^cD_t^\alpha u(t)|^p + |u(t)|^p)\,dt < \infty$, $j = 0,1,\dots,m$ and u satisfies the first equation of (1.1) for almost every $t \in (0,T)$. By (2.9), since $v \in L^p(t_j,t_{j+1})$, ${}_tD_T^\alpha\Phi_p({}_0^cD_t^\alpha u(t)) = ({}_tD_T^{\alpha-1}\Phi_p({}_0^cD_t^\alpha u(t)))' \in L^p(t_j,t_{j+1})$ and then ${}_tD_T^{\alpha-1}\Phi_p({}_0^cD_t^\alpha u(t)) \in \mathrm{AC}([t_j,t_{j+1}])$. Hence the following limits:

$$\begin{aligned}{}_tD_T^{\alpha-1}\Phi_p\bigl({}_0^cD_t^\alpha\bigr)u\bigl(t_j^+\bigr) &= \lim_{t\to t_j^+}\bigl({}_tD_T^{\alpha-1}\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)(t)\bigr), \\ {}_tD_T^{\alpha-1}\Phi_p\bigl({}_0^cD_t^\alpha\bigr)u\bigl(t_j^-\bigr) &= \lim_{t\to t_j^-}\bigl({}_tD_T^{\alpha-1}\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)(t)\bigr)\end{aligned}$$

exist.

Now multiplying (2.10) by $v \in E^{\alpha,p}$, $v(T) = 0$ and integrating between 0 and T, we have

$$\sum_{j=1}^m \Delta\bigl[{}_tD_T^{\alpha-1}\bigl(\left|{}_0^cD_t^\alpha u(t_j)\right|^{p-2}{}_0^cD_t^\alpha u(t_j)\bigr)\bigr]v(t_j) = \sum_{j=1}^m I_j\bigl(u(t_j)\bigr)v(t_j).$$

Hence, $\Delta[{}_tD_T^{\alpha-1}(|{}_0^cD_t^\alpha u(t_j)|^{p-2}{}_0^cD_t^\alpha u(t_j))] = I_j(u(t_j))$ for every $j = 1,\dots,m$. So u satisfies the impulsive conditions of problem (1.1). Similarly, u satisfies the boundary conditions. Therefore, u is a classical solution of (1.1). □

3 Proof of main results

In this section, we will study the existence and multiplicities of problem (1.1). First, we give a Lemma.

Lemma 3.1 *Assume that* (H1) *and* (H2) *hold. Then the function $\varphi : E^{\alpha,p} \to \mathbb{R}$ defined by* (2.5) *is continuous differentiable and weakly sequentially lower semi-continuous. Moreover, it satisfies the Palais-Smale condition.*

Proof From the continuity of f and I_j, we know that φ and φ' are continuous and differentiable. Let $u_k \rightharpoonup u$ in $E^{\alpha,p}$. Then $\|u\|_{E^{\alpha,p}} \leq \underline{\lim}_{k\to\infty} \inf \|u_k\|_{E^{\alpha,p}}$, and $\{u_k\}$ converges uniformly to u in $C([0,T])$. So

$$\begin{aligned}\lim_{k\to\infty} \inf \varphi(u_k) &= \lim_{k\to\infty} \left\{ \frac{1}{p}\|u_k\|^p_{E^{\alpha,p}} + \sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)\,ds - \int_0^T F\big(t,u_k(t)\big)\,dt \right\} \\ &\geq \frac{1}{p}\|u\|^p_{E^{\alpha,p}} + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds - \int_0^T F\big(t,u(t)\big)\,dt = \varphi(u), \end{aligned} \tag{3.1}$$

which implies that φ is weakly sequentially lower semi-continuous.

We will verify that φ satisfies the Palais-Smale condition. Assume that $\{u_k\}_{k\in N} \subset E^{\alpha,p}$ is a sequence such that $\{\varphi(u_k)\}_{k\in N}$ is bounded and $\lim_{k\to\infty} \varphi'(u_k) = 0$. We firstly prove that $\{u_k\}_{k\in N}$ is bounded in $E^{\alpha,p}$. Obviously, there exists a constant $c > 0$ such that

$$\big|\varphi(u_k)\big| \leq c, \qquad \varphi'(u_k) \to 0 \quad \text{as } k \to \infty. \tag{3.2}$$

Then we have

$$\int_0^T f\big(t,u_k(t)\big)u_k(t)\,dt = \|u_k\|^p_{E^{\alpha,p}} + \sum_{j=1}^m I_j\big(u_k(t_j)\big) - \varphi'(u_k)u_k(t). \tag{3.3}$$

From condition (H1), we have

$$\vartheta \int_0^{u_k(t_j)} I_j(s)\,ds - I_j\big(u_k(t_j)\big)u_k(t_j) \geq (\vartheta - \mu) \int_0^{u_k(t_j)} I_j(s)\,ds \geq 0 \quad (\mu > \vartheta),$$

which together with (3.2), (3.3) and the condition (H2) makes

$$\begin{aligned}\vartheta\varphi(u_k) - \varphi'(u_k)u_k(t) &\geq \left(\frac{\vartheta}{p} - 1\right)\|u_k\|^p_{E^{\alpha,p}} + \sum_{j=1}^m \left(\vartheta \int_0^{u_k(t_j)} I_j(s)\,ds - I_j\big(u_k(t_j)\big)u_k(t_j)\right) \\ &\quad + \int_0^T \big(f\big(t,u_k(t)\big)u_k(t) - \vartheta F\big(t,u_k(t)\big)\big)\,dt \\ &\geq \left(\frac{\vartheta}{p} - 1\right)\|u_k\|^p_{E^{\alpha,p}} \quad (\text{since } \vartheta > p),\end{aligned}$$

which implies $\{u_k\}$ is bounded in $E^{\alpha,p}$.

Since $E^{\alpha,p}$ is a reflexive Banach space, going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u$ in $E^{\alpha,p}$, $u_k \to u$ in $L^p([0,T])$ and $u_k \to u$ uniformly in $C([0,T])$. Hence

$$\begin{cases} \int_0^T (f(t,u_k(t)) - f(t,u(t)))(u_k(t) - u(t))\,dt \to 0, \\ \sum_{j=1}^m (I_j(u_k(t_j)) - I_j(t,u(t_j)))(u_k(t_j) - u(t_j)) \to 0, \end{cases} \tag{3.4}$$

as $k \to \infty$. Moreover, by $\varphi'(u_k) \to 0$ as $k \to \infty$, we have

$$\big\langle \varphi'(u_k) - \varphi'(u), u_k - u\big\rangle \leq \big\|\varphi'(u_k)\big\|_{(E^{\alpha,p})^*} \cdot \|u_k - u\|_{E^{\alpha,p}} - \big\langle \varphi'(u), u_k - u\big\rangle \to 0 \tag{3.5}$$

as $k \to \infty$.

Let

$$\phi(p,\alpha,k) := \int_0^T \left(\left|{}_0^cD_t^\alpha u_k(t)\right|^{p-2}{}_0^cD_t^\alpha u_k(t) - \left|{}_0^cD_t^\alpha u(t)\right|^{p-2}{}_0^cD_t^\alpha u(t)\right)\left({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right) dt$$

and

$$\phi(p,k) := \int_0^T \left(\left|u_k(t)\right|^{p-2}u_k(t) - \left|u(t)\right|^{p-2}u(t)\right)\left(u_k(t) - u(t)\right) dt.$$

Notice that

$$\begin{aligned}\left\langle \varphi'(u_k) - \varphi'(u), u_k - u\right\rangle &= \phi(p,\alpha,k) + \phi(p,k) + \sum_{j=1}^m \left(I_j\left(u_k(t_j)\right) - I_j\left(t, u(t_j)\right)\right)\left(u_k(t_j) - u(t_j)\right) \\ &\quad - \int_0^T \left(f\left(t, u_k(t)\right) - f\left(t, u(t)\right)\right)\left(u_k(t) - u(t)\right) dt,\end{aligned}$$

which together with (3.4) and (3.5) yields

$$\phi(p,\alpha,k) + \phi(p,k) \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

From the well-known inequality (see [9], Lemma 4.2)

$$|x-y|^p \le \begin{cases} (|x|^{p-2}x - |y|^{p-2}y)(x-y), & \text{if } p \ge 2, \\ ((|x|^{p-2}x - |y|^{p-2}y)(x-y))^{\frac{p}{2}}(|x|^p + |y|^p)^{\frac{2-p}{2}}, & \text{if } 1 < p < 2, \end{cases}$$

for all $x, y \in \mathbb{R}$. Then there exist constants $c_i > 0$ ($i = 1,2,3,4$) such that

$$\phi(p,\alpha,k) \ge \begin{cases} c_1 \int_0^T |{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^p\, dt, & \text{if } p \ge 2, \\ c_2 \int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\, dt, & \text{if } 1 < p < 2, \end{cases} \tag{3.7}$$

and

$$\phi(p,k) \ge \begin{cases} c_3 \int_0^T |u_k(t) - u(t)|^p\, dt, & \text{if } p \ge 2, \\ c_4 \int_0^T \frac{|u_k(t) - u(t)|^2}{(|u_k(t)| + |u(t)|)^{2-p}}\, dt, & \text{if } 1 < p < 2. \end{cases} \tag{3.8}$$

When $1 < p < 2$, by the Hölder inequality, we have

$$\begin{aligned}\int_0^T \left(\left|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right|^p\right) dt &\le \left(\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\, dt\right)^{\frac{p}{2}} \\ &\quad \cdot \left(\int_0^T \left(\left|{}_0^cD_t^\alpha u_k(t)\right| + \left|{}_0^cD_t^\alpha u(t)\right|\right)^p dt\right)^{\frac{2-p}{2}} \\ &\le M\left(\left\|{}_0^cD_t^\alpha u_k\right\|_{L^p}^p + \left\|{}_0^cD_t^\alpha u\right\|_{L^p}^p\right)^{\frac{2-p}{2}} \\ &\quad \cdot \left(\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\, dt\right)^{\frac{p}{2}},\end{aligned} \tag{3.9}$$

where $M = 2^{(p-1)(2-p)/2}$ is a positive constant. Similarly, we have

$$\int_0^T \left(\left|u_k(t) - u(t)\right|^p\right) dt \le M\left(\|u_k\|_{L^p}^p + \|u\|_{L^p}^p\right)^{\frac{2-p}{2}} \cdot \left(\int_0^T \frac{|u_k(t) - u(t)|^2}{(|u_k(t)| + |u(t)|)^{2-p}}\, dt\right)^{\frac{p}{2}}. \tag{3.10}$$

From (3.7) and (3.9), we have

$$\begin{aligned} \phi(p,\alpha,k) &\ge c_2 M^{-\frac{2}{p}}\left(\|u_k\|_{L^p}^p + \|u\|_{L^p}^p\right)^{\frac{p-2}{p}} \cdot \left(\int_0^T \left|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right|^p dt\right)^{\frac{2}{p}} \\ &= c_2 M^{-\frac{2}{p}}\left(\|u_k\|_{L^p}^p + \|u\|_{L^p}^p\right)^{\frac{p-2}{p}} \cdot \left\|{}_0^cD_t^\alpha (u_k - u)\right\|_{L^p}^2. \end{aligned} \tag{3.11}$$

It follows from (3.8) and (3.10) that

$$\phi(p,k) \ge c_4 M^{-\frac{2}{p}}\left(\|u_k\|_{L^p}^p + \|u\|_{L^p}^p\right)^{\frac{p-2}{p}} \cdot \|u_k - u\|_{L^p}^2. \tag{3.12}$$

When $1 < p < 2$, by (3.11) and (3.12), we get

$$\phi(p,\alpha,k) + \phi(p,k) \ge M_1\left(\left\|{}_0^cD_t^\alpha (u_k - u)\right\|_{L^p}^2 + \|u_k - u\|_{L^p}^2\right) = M_1\|u_k - u\|_{E^{\alpha,p}}^2, \tag{3.13}$$

where

$$M_1 := M^{-\frac{2}{p}} \min\left\{c_2\left(\|u_k\|_{L^p}^p + \|u\|_{L^p}^p\right)^{\frac{p-2}{p}}, c_4\left(\|u_k\|_{L^p}^p + \|u\|_{L^p}^p\right)^{\frac{p-2}{p}}\right\}.$$

When $p \ge 2$, in view of (3.7) and (3.8), we have

$$\phi(p,\alpha,k) + \phi(p,k) \ge M_2\left(\left\|{}_0^cD_t^\alpha (u_k - u)\right\|_{L^p}^p + \|u_k - u\|_{L^p}^p\right) = M_2\|u_k - u\|_{E^{\alpha,p}}^p, \tag{3.14}$$

where $M_2 = \min\{c_1, c_3\}$. Therefore, it follows from (3.6), (3.13) and (3.14) that $\|u_k - u\|_{E^{\alpha,p}} \to 0$ as $k \to +\infty$. That is, $u_k \to u$ in $E^{\alpha,p}$. Hence, φ satisfies the Palais-Smale condition. □

Now we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 *Step I*: Obviously, $\varphi(0) = 0$, and Lemma 3.1 has shown that φ satisfies the Palais-Smale condition. For any $r > 0$, take $\Omega_r = \{u \in E^{\alpha,p} : \|u\|_{E^{\alpha,p}} < r\}$. It is easy to show that $\overline{\Omega}_r$ is bounded and weakly sequentially closed. Indeed, if we let $\{u_n\} \subseteq \overline{\Omega}_r$ and $u_n \rightharpoonup u$ as $n \to \infty$, by the Mazur Theorem [10], there is a sequence of convex combinations $v_n = \sum_{i=1}^n \beta_{n_i} u_i$ with $\sum_{i=1}^n \beta_{n_i} = 1$, $\beta_{n_i} \ge 0$, $i \in \mathbf{N}$ such that $v_n \to u$ in $E^{\alpha,p}$. Since $\overline{\Omega}_r$ is a closed convex set, we have $\{v_n\} \subseteq \overline{\Omega}_r$ and $u \in \overline{\Omega}_r$.

From Lemma 3.1 we know that φ is weakly sequentially lower semi-continuous on $\overline{\Omega}_r$. Besides, $E^{\alpha,p}$ is a reflexive Banach space, so by Lemma 2.2 we see that φ has a local minimum $u_0 \in \overline{\Omega}_r$. Without loss of generality, we assume that $\varphi(u_0) = \min\{\varphi(u) : u \in \overline{\Omega}_r\}$. Now we will prove that $\varphi(u_0) \leq \inf\{\varphi(u) : u \in \overline{\Omega}_r\}$ for some $r = r_0$. Indeed, from (H3) we may choose $r_0, \varepsilon > 0$ satisfying

$$F(t,u) \leq \delta|u|^{\vartheta}, \quad \text{for } \|u\|_{E^{\alpha,p}} \leq r_0, \qquad 0 < \varepsilon < \frac{r_0^p}{p} - r_0^{\mu}\sum_{j=1}^{m}\delta_j A_{\infty}^{\mu} - r_0^{\vartheta}\delta A_p^{\vartheta}. \tag{3.15}$$

For every $u \in \partial\Omega_{r_0}$ with $\|u\|_{E^{\alpha,p}} = r_0$, from (2.2), (2.3), (2.5), (H4) and (3.15) we have

$$\begin{aligned}\varphi(u) &\geq \frac{1}{p}\|u\|_{E^{\alpha,p}}^p - \sum_{j=1}^{m}\delta_j\big|u(t_j)\big|^{\mu} - \delta\int_0^T \big|u(t)\big|^{\vartheta}\,dt \\ &\geq \frac{r_0^p}{p} - r_0^{\mu}\sum_{j=1}^{m}\delta_j A_{\infty}^{\mu} - r_0^{\vartheta}\delta A_p^{\vartheta} > \varepsilon,\end{aligned}$$

which implies that $\varphi(u) > \varepsilon$ for every $u \in \partial\Omega_{r_0}$. Moreover, $\varphi(u_0) \leq \varphi(0) = 0$. Then $\varphi(u_0) \leq \varphi(0) < \varepsilon < \varphi(u)$ for every $u \in \partial\Omega_{r_0}$. Hence $\varphi(u_0) < \inf\{\varphi(u) : u \in \partial\Omega_{r_0}\}$. So φ has a local minimum $u_0 \in \partial\Omega_{r_0}$.

Step II: We will verify that there exists u_1 with $\|u_1\|_{E^{\alpha,p}} > r_0$ such that $\varphi(u_1) < \inf\{\varphi(u) : u \in \partial\Omega_{r_0}\}$, where r_0 is given above.

In view of (H3), we choose a sufficiently large r_1 such that for all $\|u\|_{E^{\alpha,p}} \geq r_1 > r_0$

$$F(t,u) \geq \gamma|u|^{\vartheta}. \tag{3.16}$$

From (H1), we have the following:

$$\frac{\mu}{u} \leq \frac{I_j(u)}{\int_0^u I_j(s)\,ds}, \quad \text{for } u > 0, \tag{3.17}$$

$$\frac{\mu}{u} \geq \frac{I_j(u)}{\int_0^u I_j(s)\,ds}, \quad \text{for } u < 0. \tag{3.18}$$

Integrating (3.17) and (3.18) from T to u and u to $-T$, respectively, we get

$$\begin{aligned}&\int_0^u I_j(s)\,ds \leq \frac{u^{\mu}}{T^{\mu}}\int_0^T I_j(s)\,ds, \quad \text{for } u > T, \\ &\int_0^u I_j(s)\,ds \leq \frac{(-u)^{\mu}}{T^{\mu}}\int_0^{-T} I_j(s)\,ds, \quad \text{for } u < -T.\end{aligned}$$

Note that $\int_0^T I_j(s)\,ds < 0$ and $\int_0^{-T} I_j(s)\,ds < 0$. We take

$$\gamma_j = \quad T^{-\mu}\cdot\min\left\{\left|\int_0^T I_j(s)\,ds\right|, \left|\int_0^{-T} I_j(s)\,ds\right|\right\} > 0,$$

and we get

$$\int_0^u I_j(s)\,ds \leq -\gamma_j|u|^{\mu}, \quad \forall u \in (-\infty,-T] \cup [T,+\infty). \tag{3.19}$$

Clearly, $\int_0^u I_j(s)\,ds$ is continuous in $[-T,T]$, so there is a constant $K>0$ such that

$$\int_0^u I_j(s)\,ds \le K, \quad \forall u\in[-T,T]. \tag{3.20}$$

Combining (3.19) and (3.20), we get

$$\int_0^u I_j(s)\,ds \le -\gamma_j|u|^\mu + K, \quad \forall u\in R. \tag{3.21}$$

For any $u\in E^{\alpha,p}$ with $u\neq 0$, $\tau>0$, it follows from (3.16) and (3.21) that

$$\varphi(\tau u)\le \frac{\tau^p}{p}\|u\|^p_{E^{\alpha,p}} - \tau^\mu\sum_{j=1}^m \gamma_j\bigl|u(t_j)\bigr|^\mu - \tau^\vartheta\gamma\|u\|^\vartheta_{L^\vartheta} + K_* \to -\infty \quad (\mu>\vartheta>p),$$

as $\tau\to\infty$, where K_* is a positive constant. So there exists a sufficiently large τ_0 such that $\varphi(\tau_0 u)\le 0$. That is to say, we can choose u_1 with $\|u_1\|_{E^{\alpha,p}}\ge r_1$ sufficiently large such that $\varphi(u_1)<0$. Therefore, from *Step I* and *Step II* we get $\max\{\varphi(u_0),\varphi(u_1)\}<\inf\{\varphi(u): u\in\partial\Omega_{r_0}\}$. Then Lemma 2.1 admits the critical point u_*. So, u_0 and u_* are two different critical points of φ, and they are weak solutions of (1.1). □

Example 3.1 Let $\alpha=0.6$, $T>0$, $p=\frac{5}{2}$, $t_1\in(0,T)$, $a_0>0$, $a(t)\in C([0,T])$ with $a(t)>0$. Consider the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^{0.6}\Phi_{\frac{5}{2}}({}_0^cD_t^{0.6}u(t)) + |u(t)|^{\frac{1}{2}}u(t) = (2+\sin t)a(t)u^4(t), \quad 0<t<T, t\neq t_1,\\ \Delta({}_tD_T^{-0.4}\Phi_{\frac{5}{2}}({}_0^cD_t^{0.6}u))(t_1) = -a_0u^5(t_1),\\ u(0)=u(T)=0.\end{cases} \tag{3.22}$$

Obviously, $f(t,u)=(2+\sin t)a(t)u^4(t)$, $I_1(u)=-a_0u^5(t_1)$, $\frac{1}{p}=0.4<\alpha=0.6$. Let $\mu=6$, $\vartheta=5$, $\delta_1=\frac{a_0}{6}$, and $\delta=\frac{1}{5}\max\{(2+\sin t)a(t): t\in[0,T]\}$, $\gamma=\frac{1}{5}\min\{(2+\sin t)a(t): t\in[0,T]\}$. By simple computation, the conditions (H1)-(H4) are satisfied. From Theorem 1.1, problem (3.22) has at least two weak solutions.

Proof of Theorem 1.2 We will apply Lemma 2.3 to finish the proof. Obviously, $\varphi\in C^1(E^{\alpha,p},\mathbb{R})$ is even and $\varphi(0)=0$. Moreover, Lemma 3.1 shows that φ satisfies the Palais-Smale condition.

As $E^{\alpha,p}$ is a reflexive and separable Banach space, then there are $e_i\in E^{\alpha,p}$ such that $E^{\alpha,p}=\overline{\operatorname{span}\{e_i: i=1,2,\dots\}}$. For $k=1,2,\dots$, denote

$$X_i := \operatorname{span}\{e_i\},\qquad Y_k:=\bigoplus_{i=1}^k X_i,\qquad Z_k:=\overline{\bigoplus_{i=k}^\infty X_i}.$$

Then $E^{\alpha,p}=Y_k\oplus Z_k$.

For any $u\in Z_k$ with $\|u\|_{E^{\alpha,p}}\le r_0$, combining (2.5) and the conditions (H1)-(H4), we have

$$\varphi(u)\ge\frac{1}{p}\|u\|^p_{E^{\alpha,p}} - \sum_{j=1}^m\delta_jA_\infty^\mu\|u\|^\mu_{E^{\alpha,p}} - \delta A_p^\vartheta\|u\|^\vartheta_{E^{\alpha,p}}, \quad \|u\|_{E^{\alpha,p}}\le r_0,$$

which implies that there exists $\rho > 0$ small enough such that $\varphi(u) \geq \eta > 0$ with $\|u\|_{E^{\alpha,p}} = \rho$.

For any $u \in Y_k$, let

$$\|u\|_* := \left(\int_0^T \left|u(t)\right|^{\vartheta} dt\right)^{\frac{1}{\vartheta}}, \tag{3.23}$$

and it is easy to show that $\|\cdot\|_*$ defined by (3.23) is a norm of Y_k. Since all the norms of a finite dimensional normed space are equivalent, there is a positive constant A_1 such that

$$A_1\|u\|_{E^{\alpha,p}} \leq \|u\|_*, \quad \text{for } u \in Y_k. \tag{3.24}$$

From (H1), we have

$$\sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)\, ds \leq 0.$$

The above inequality, (3.17) and (3.24) imply that, for any finite dimensional space $W \subset E^{\alpha,p}$,

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\|u\|_{E^{\alpha,p}}^p + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)\, ds - \int_0^T F\big(t, u(t)\big)\, dt \\
&\leq \frac{1}{p}\|u\|_{E^{\alpha,p}}^p - \int_{\Omega_1} F\big(t, u(t)\big)\, dt - \int_{\Omega_2} F\big(t, u(t)\big)\, dt \\
&\leq \frac{1}{p}\|u\|_{E^{\alpha,p}}^p - \gamma \int_{\Omega_1} \left|u(t)\right|^{\vartheta} dt - \int_{\Omega_2} F\big(t, u(t)\big)\, dt \\
&= \frac{1}{p}\|u\|_{E^{\alpha,p}}^p - \gamma \int_0^T \left|u(t)\right|^{\vartheta} dt + \gamma \int_{\Omega_1} \left|u(t)\right|^{\vartheta} dt - \int_{\Omega_2} F\big(t, u(t)\big)\, dt \\
&\leq \frac{1}{p}\|u\|_{E^{\alpha,p}}^p - A_1^{\vartheta} \gamma \|u\|_{E^{\alpha,p}}^{\vartheta} + M^*, \quad \forall u \in W,
\end{aligned}$$

where $\Omega_1 := \{t \in [0,T] : |u(t)| \geq A_\infty r_1\}$ (r_1 is given in (3.17)), $\Omega_2 := [0,T] \setminus \Omega_1$ and M^* is a positive constant. Since $\vartheta > p$, the above inequality implies that $\varphi(u) \to -\infty$ as $\|u\|_{E^{\alpha,p}} \to +\infty$. That is, there exists $r > 0$ such that $\varphi(u) \leq 0$ for $u \in W \setminus B_{r(W)}$. By Lemma 2.3, the functional $\varphi(u)$ possesses infinitely many critical points, *i.e.*, the fractional impulsive problem (1.1) admits infinitely many weak solutions. The proof is complete. □

Example 3.2 Let $\alpha = 0.75$, $T > 0$, $p = \frac{3}{2}$, $t_1 \in (0,T)$, $a_0 > 0$, $a(t) \in C([0,T])$ with $a(t) > 0$. Consider the following fractional boundary value problem:

$$\begin{cases}
{}_tD_T^{0.75}\Phi_{\frac{3}{2}}({}_0^cD_t^{0.75}u(t)) + |u(t)|^{-\frac{1}{2}}u(t) = (1+t^2)a(t)u^{\frac{5}{3}}(t), \quad 0 < t < T, t \neq t_1, \\
\Delta({}_tD_T^{-0.25}\Phi_{\frac{3}{2}}({}_0^cD_t^{0.75}u))(t_1) = -a_0u^9(t_1), \\
u(0) = u(T) = 0.
\end{cases} \tag{3.25}$$

Obviously, $\frac{1}{p} = \frac{2}{3} < \alpha = 0.75$ and $f(t,u) = (1+t^2)a(t)u^{\frac{5}{3}}(t)$, $I_1(u) = -a_0u^9(t_1)$ are odd about u. Let $\mu = 10$, $\vartheta = \frac{8}{3}$, $\delta = \frac{3}{8}\max\{(1+t^2)a(t) : t \in [0,T]\}$, $\gamma = \frac{3}{8}\min\{(1+t^2)a(t) : t \in [0,T]\}$, and

$\delta_1 = \frac{a_0}{10}$. Then by simple computation, the conditions in Theorem 1.2 are satisfied. Hence, problem (3.25) has infinitely many weak solutions.

4 Conclusion

In this paper, we have proved the existence and multiplicity of the solutions for an impulsive fractional differential equation with p-Laplacian operator. Our approach is based on the well-known mountain pass theorem and minimax methods in critical point. With the impulsive effects and p-Laplacian operator taken into consideration, the corresponding variational functional is more complicated. Therefore, the existence of solutions for impulsive fractional differential problems with p-Laplacian is interesting. As applications, two examples are presented to illustrate the main results. In the future, we will consider the existence of solutions for the impulsive fractional differential equation with p-Laplacian via Morse theory.

Acknowledgements
The authors are highly grateful for the referees' careful reading and comments on this paper. The research is supported by a Project of the Department of Education in Hunan Province (12C0088), and the Aid program for Science and Technology Innovative Research Team in Higher Educational Instituions of Hunan Province.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 18 January 2017 Accepted: 10 August 2017 Published online: 25 August 2017

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