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Multiple positive solutions for Kirchhoff-Schrödinger-Poisson system with general singularity

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Abstract

In this paper, we consider the existence of multiple positive solutions for Kirchhoff-Schrödinger-Poisson system with the nonlinear term containing both general singularity and quasicritical nonlinearity. By combining the variational method with the perturbation method, we obtain the existence of two positive solutions with the parameter λ small enough. One of the solutions is the local minimum of the corresponding functional, and the other is the limit of the mountain pass type solution to the perturbation problem.

Keywords: Kirchhoff-Schrödinger-Poisson system; general singularity; multiple solutions; variational method; perturbation

1 Introduction and main results

In this paper, we are interested in discussing the existence and multiple positive solutions to the following general singular Kirchhoff-Schrödinger-Poisson system:

$$\begin{cases} -(a+b\int_\Omega|\nabla u|^2)\Delta u+\phi u=\lambda hf(u)+g(u), & \text{in } \Omega,\\ -\Delta\phi=u^2, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=\phi=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $a>0$, $b\geq 0$, $\lambda>0$ is a parameter, f, g, h satisfy the following assumptions:

(f) $f\in C((0,\infty),\mathbb{R}_+)$ is nonincreasing and $\int_0^1 f(s)\,ds<\infty$. Moreover, there exists $\gamma\in(0,1)$ such that

$$\lim_{s\to 0^+} f(s)s^\gamma=\infty;$$

(g_1) $g\in C(\mathbb{R}_+,\mathbb{R}_+)$, $g(s)=o(s)$ as $s\to 0$ and g has a 'quasicritical growth', namely

$$\lim_{s\to+\infty}\frac{g(s)}{s^5}=0;$$

(g_2) $\lim_{s\to+\infty}\frac{g(s)}{s^3}=\infty$;
(g_3) $g(s)s \geq 4G(s)$, where $G(s)=\int_0^s g(t)\,dt$, $s\in\mathbb{R}_+$;
(h) $h\in L^2(\Omega)$ with $h(x)>0$ a.e. $x\in\Omega$.

Recently, the following singular Kirchhoff type problem has been studied extensively in [1–5]

$$\begin{cases} -(a+b\int_\Omega|\nabla u|^2)\Delta u = \lambda h u^{-\gamma} + \mu g(x,u), & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 3$ is a smooth bounded domain with boundary $\partial\Omega,\gamma\in(0,1),\lambda,\mu\geq 0$ are parameters. In [5], by using the method of Nehari manifold, Liu and Sun discussed the existence of two positive solutions to (1.2) with $g(x,s)=g(x)\frac{s^p}{|x|^t}, t\in[0,1), p\in[3,5-2t)$ for the parameter $\lambda>0$ small enough. Lei, Liao and Tang in [1] combined the variational method with the perturbation argument to discuss (1.2) with $n=3$ and g being critical term: $g(s)=s^5$ and obtained two positive solutions to this problem. In [3, 4], the authors discussed the existence and multiple positive solutions to (1.2) with $n=4$ and g being critical term: $g(s)=s^3$. By using the Nehari manifold method and analyzing the relations between the parameters λ, μ and the first eigenvalue to the Kirchhoff type problem, the authors in [3] obtained multiple positive solutions to this problem. In [4], the authors obtained the existence of two positive solutions to (1.2) with $h(x)=\frac{1}{|x|^\beta}, \beta\in(0,3)$ by using the variational method and the perturbation method. The existence of unique positive solution to (1.2) with $g(x,s)=-s^p, p\in(0,2^*-1)$ was obtained in [2] by using the variational method. Meanwhile, the singular (p,q) Kirchhoff type system was also considered in [6].

In [7], the author of the present paper considered the following singular Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-r}, & \text{in } \Omega,\\ -\Delta\phi = u^2, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=\phi=0, & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega, \eta=\pm1, r\in(0,1)$ is a constant, $\mu>0$ is a parameter. The existence of unique positive solution was obtained for $\eta=1$ and any $\mu>0$ by using the variational method. The multiple positive solutions were also obtained for $\eta=-1$ and $\mu>0$ small enough by combining the variational method with the Nehari manifold method. Recently, the Schrödinger-Poisson system with singular potential was also considered in [8]. The existence of system (1.1) with $h=0, g(s)=-s^p$ was considered in [9].

The Kirchhoff-Schrödinger-Poisson system with general singularity f and $-g$ ($g\geq 0$) in (1.1) was firstly considered in our recent paper [10], where f, g, h satisfy the more weaker assumptions:

(f_0) $f \in C((0,\infty),\mathbb{R}_+)$ satisfies that there exists $\delta > 0$ such that f is nonincreasing on $(0,\delta]$, $\int_0^\delta f(s)\,ds < \infty$, and there exist $\alpha, \gamma \in (0,1)$ such that

$$\lim_{s\to 0^+} f(s)s^\alpha = \infty, \qquad \lim_{s\to\infty} f(s)/s^\gamma = 0;$$

(g_0) $g \in C(\mathbb{R}_+,\mathbb{R}_+)$ and there exists $c_1 > 0$ such that

$$g(s) \le c_1\bigl(s + s^5\bigr), \quad s \in \mathbb{R}_+;$$

(h_0) $h \in L^{6/(5-\gamma)}(\Omega)$ with $h(x) > 0$ a.e. $x \in \Omega$.

Under the weaker assumptions (f_0), (g_0) and (h_0), the corresponding functional is well defined and is coercive. By using the variational method, the negative global minimum is obtained and is the unique positive solution to this problem. Based on our work [7, 10], recently, Mu and Lu in [11] considered the existence and multiplicity of positive solutions for system (1.1) with $f(s) = s^{-\gamma}$, $\gamma \in (0,1)$. A natural question is whether there exist multiple positive solutions to system (1.1) with the nonlinear term containing both general singular nonlinearity and the quasicritical nonlinearity.

Motivated by the above reference, especially by [1, 4, 5], and based on our work [10], in this paper, we would like to continue to study the existence of multiple solutions to the general singular Kirchhoff-Schrödinger-Poisson system (1.1).

Throughout this paper, let $H_0^1(\Omega)$ be the usual Sobolev space with the inner product and the norm

$$(u,v) = \int_\Omega \nabla u \cdot \nabla v, \qquad \|u\| = (u,u)^{1/2}, \quad u,v \in H_0^1(\Omega).$$

We denote the norm of $L^p(\Omega)$ by $|u|_p = (\int_\Omega |u|^p)^{1/p}$. By the Sobolev embedding theorem, $H_0^1(\Omega)$ can be compactly embedded into $L^p(\Omega)$ for all $p \in [1,6)$ and the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is continuous.

For any given $u \in H_0^1(\Omega)$, by using the Lax-Milgram theorem, the Dirichlet boundary problem $-\Delta\phi = u^2$ in Ω has a unique solution $\phi_u \in H_0^1(\Omega)$. Substitute ϕ_u to the first equation of system (1.1), then system (1.1) can be transformed into the following variable equation:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u + \phi_u u = \lambda h f(u) + g(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Some necessary properties of ϕ_u are given in Lemma 2.1.

Since we only consider the positive solution to system (1.1), we can assume that $f(s) = 0$ and $g(s) = 0$ for all $s \in (-\infty, 0)$. By (f), for $s \ge 1/2$,

$$F(s) = F(1/2) + \int_{1/2}^s f(t)\,dt \le F(1/2) + f(1/2)(s - 1/2).$$

Since $F(s) \le F(1/2), s \in [0,1/2]$, then there exist $c_1, c_2 > 0$ such that

$$0 \le F(s) \le c_1 s + c_2, \quad s \in \mathbb{R}. \tag{1.4}$$

It is obvious that F is continuous on $\mathbb{R}$. By (g_1), we easily obtain that for any $\varepsilon > 0$, there exist $C_\varepsilon > 0, p \in (3,5)$ such that

$$g(s) \leq \varepsilon\left(s + s^5\right) + C_\varepsilon s^p, \quad s \in \mathbb{R}, \tag{1.5}$$

and

$$G(s) \leq \varepsilon\left(s^2 + s^6\right) + C_\varepsilon s^{p+1}, \quad s \in \mathbb{R}. \tag{1.6}$$

Thus, by (1.4), (1.6) and (h), the energy functional corresponding to (1.3)

$$J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4}\int_\Omega \phi_u u^2 - \lambda\int_\Omega hF(u) - \int_\Omega G(u), \quad u \in H_0^1(\Omega) \tag{1.7}$$

is well defined and continuous on $H_0^1(\Omega)$.

As we know, under the general singular assumption (f) or (f_0), the functional J fails to be Fréchet differentiable because of the singular term. We then cannot apply the critical point theory to obtain the existence of solution directly. In general, a function $u \in H_0^1(\Omega)$ is called a solution of (1.3), that is, (u, ϕ_u) is a solution of (1.1) and $u(x) > 0$ a.e. in Ω satisfying

$$\left(a + b\|u\|^2\right)(u,v) + \int_\Omega \phi_u uv - \lambda\int_\Omega hf(u)v - \int_\Omega g(u)v = 0, \quad v \in H_0^1(\Omega). \tag{1.8}$$

In fact, under the weaker singular assumption (f_0), from (1.4), (1.6), (1.7), we easily deduce that the functional J has a negative local minimum around the neighborhood of origin with the parameter $\lambda > 0$ small enough. With the two skilled lemmas (Lemmas 2.3, 2.4 in [10]) on the properties of the singular term f, we can show that the negative local minimum point is a solution of problem (1.3). In order to obtain the second solution of system (1.1), here we assume that (f) holds, that is, f is singular at 0 and nonincreasing on $(0,\infty)$. It is obvious that assumption (f) implies that (f_0) holds. Assumption (f) was first introduced in [12] to consider the singular semilinear elliptic equation. To obtain the second solution of problem (1.3), motivated by [1, 4], we also consider the perturbation problem

$$\begin{cases} -(a + b\|u\|^2)\Delta u + \phi_u u = \lambda hf(u+\alpha) + g(u), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $\alpha > 0$. The functional corresponding to problem (1.9) is as follows:

$$J_\alpha(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4}\int_\Omega \phi_u u^2 - \lambda\int_\Omega h\left(F\left(u^+ + \alpha\right) - F(\alpha)\right) - \int_\Omega G(u),$$
$$u \in H_0^1(\Omega).$$

Under the assumptions of (f), (g_1)-(g_3) and (h), we can show $J_\alpha \in C^1(H_0^1(\Omega), \mathbb{R})$ and problem (1.9) has a mountain pass type solution u_α. Finally, we can prove that the limit v_0 of a family of solutions $\{u_\alpha\}$ of problem (1.9) is the second solution of problem (1.3). In the proof, the monotonic property of f and a result from [13] are crucial to showing the uniform boundedness of $\{u_\alpha\}$ and the convergence of $u_\alpha \to v_0$ as $\alpha \to 0$.

Our main result can be described as follows.

Theorem 1.1 *If $a > 0$, $b \geq 0$, and assumptions* (f), (g_1)-(g_3) *and* (h) *hold, then there exists $\lambda^* > 0$ such that system* (1.1) *has at least two solutions for each $\lambda \in (0, \lambda^*)$.*

Remark 1.2 There are a number of functions which satisfy (f), (g_1)-(g_3), (h) respectively. For example,

(i) $f_1(s) = [s^\alpha \arctan(1+s)]^{-1}$ for all $s \in (0,\infty)$, where $0 < \gamma < \alpha < 1$;

(ii) $f_2(s) = \sqrt{1+s^{2\beta}}/s^\alpha$ for all $s \in (0,\infty)$, where $0 < \max\{\gamma, \beta/2\} < \alpha < 1$.

It is easy to verify that the functions f_1, f_2 satisfy condition (f).

Let $g_1(s) = s^{2+\beta}\ln(1+s), s > 0$ with $\beta \in (0,1)$ and $g_2(s) = s^p, s > 0$ with $p \in (3,5)$. Let $\rho \in C^1(\mathbb{R}^+, [0,1])$ be a cut-off function verifying $s\rho'(s) \leq 0$, $|\rho'(s)| \leq 2$,

$$\rho(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \in [2,\infty). \end{cases}$$

Set $G(s) = \rho(s)G_1(s) + (1-\rho(s))G_2(s), g(s) = G'(s)$, where $G_i(s) = \int_0^s g_i(t)\,dt$. Then it is easy to verify that g satisfies conditions (g_1)-(g_3).

Take some $x_0 \in \Omega$ and let $h(x) = |x - x_0|^{-\beta}$ for all $x \in \Omega \setminus \{x_0\}$, where $\beta \in [0, 3/2)$. It is obvious that h satisfies condition (h).

This paper is organized as follows. In Section 2, we give the existence of a negative local minimum of the functional J for $\lambda > 0$ small enough and show that it is a solution of problem (1.3). In Section 3, we firstly discuss the existence of the mountain pass type solution to the perturbation problem (1.9). Furthermore, by approximation, the second solution of problem (1.3) is obtained.

In this paper, c, c_i, C_i denote various positive constants, which may vary from line to line.

2 Existence of the first solution to system (1.1)

Let us first collect some properties of ϕ_u. We refer the readers to [7, 14–16], etc.

Lemma 2.1 *For each $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta\phi = u^2, & in\ \Omega, \\ \phi = 0, & on\ \partial\Omega. \end{cases}$$

The following properties hold for the solution ϕ_u:

(i) $\|\phi_u\|^2 = \int_\Omega \phi_u u^2$;

(ii) $\phi_u \geq 0$. *Moreover, $\phi_u > 0$ in Ω when $u \not\equiv 0$;*

(iii) *for each $t \neq 0$, it holds that $\phi_{tu} = t^2\phi_u$;*

(iv) *if $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then we have*

$$\phi_{u_n} \to \phi_u \quad in\ H_0^1(\Omega),$$

$$\int_\Omega \phi_{u_n} u_n \phi \to \int_\Omega \phi_u u\phi, \quad \phi \in H_0^1(\Omega),$$
$$\int_\Omega \phi_{u_n} u_n(u_n - u) \to 0;$$

(v) $\phi_u \in W^{2,3}_{\mathrm{loc}}(\Omega) \cap C^0(\bar{\Omega})$;
(vi) $\phi_u = \phi_{u^+} + \phi_{u^-}$, *where* $u^\pm = \pm \max\{\pm u, 0\}$.

Under the assumptions of (f), (g_1) and (h), we can show that the functional J defined in (1.7) has a negative local minimum for small $\lambda > 0$. In fact, we have the following lemma.

Lemma 2.2 *Under the assumptions of* (f), (g_1) *and* (h), *there exist* $\lambda^* > 0$ *and* $r, \rho > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *we have*

$$J|_{S_r} \geq \rho \quad \textit{and} \quad m = \inf_{\bar{B}_r} J < 0,$$

where $B_r = \{u \in H_0^1(\Omega) : \|u\| < r\}, S_r = \partial B_r$.

Proof For any $u \in H_0^1(\Omega)$, by (1.4), (1.6) with $0 < \varepsilon < \frac{a}{4}\mu_1$, and (1.7), where $\mu_1 > 0$ is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$, we have

$$\begin{aligned} J(u) &\geq \frac{a}{2}\|u\|^2 - \lambda \int_\Omega h(c_1 u + c_2) - \int_\Omega \varepsilon\left(u^2 + u^6\right) - C_\varepsilon |u|_{p+1}^{p+1} \\ &\geq \frac{a}{4}\|u\|^2 - \lambda c_1 |h|_2 |u|_2 - c_3\|u\|^6 - c_4\|u\|^{p+1} - \lambda c_2 |h|_1 \\ &\geq \|u\|\left(\frac{a}{4}\|u\| - c_3\|u\|^5 - c_4\|u\|^p - \lambda c_0 |h|_2\right) - \lambda c_2 |h|_1. \end{aligned}$$

Let $m(t) = \frac{a}{4}t - c_3 t^5 - c_4 t^p$, since $p > 3$, there exists $r > 0$ such that $m(r) = \max_{t\geq 0} m(t)$. We choose $\lambda_1, \lambda_2 > 0$ respectively such that $\lambda_1 c_0 |h|_2 = \frac{1}{2}m(r), \lambda_2 c_2 |h|_1 = \frac{1}{4}rm(r)$. Thus, when $0 < \lambda < \lambda^* = \min\{\lambda_1, \lambda_2\}$, for any $u \in S_r$, we have

$$\begin{aligned} J(u) &\geq r\left(m(r) - \lambda c_0 |h|_2\right) - \lambda c_2 |h|_1 \\ &\geq \frac{1}{4} rm(r) \\ &=: \rho. \end{aligned} \tag{2.1}$$

Hence, for any $\lambda \in (0, \lambda^*)$, there exist $r, \rho > 0$ such that $J|_{S_r} \geq \rho$.

On the other hand, by assumption (f), there exists $\delta > 0$ such that

$$f(s) \geq s^{-\gamma}, \qquad F(s) \geq \frac{s^{1-\gamma}}{1-\gamma}, \quad s \in (0, \delta]. \tag{2.2}$$

Choose a nonnegative function $\varphi \in C_0^\infty(\Omega)\backslash\{0\}$ with $\max_\Omega \varphi \leq \delta$. Then, for any $t \in (0,1]$, by Lemma 2.1(iii), (2.2), we have

$$J(t\varphi) = \frac{at^2}{2}\|\varphi\|^2 + \frac{bt^4}{4}\|\varphi\|^4 + \frac{t^4}{4}\int_\Omega \phi_\varphi \varphi^2 - \lambda \int_\Omega hF(t\varphi) - \int_\Omega G(t\varphi)$$

$$\leq \frac{at^2}{2}\|\varphi\|^2 + \frac{bt^4}{4}\|\varphi\|^4 + \frac{t^4}{4}\int_\Omega \phi_\varphi \varphi^2 - \frac{t^{1-\gamma}}{1-\gamma}\lambda \int_\Omega h\varphi^{1-\gamma}.$$

Since $1-\gamma \in (0,1)$ and $h(x) > 0$, a.e. $x \in \Omega$, we get that $J(t\varphi) < 0$ for $t > 0$ small enough. Hence, it follows from (2.1) that $m = \inf_{\bar{B}_r} J < 0$. □

In order to prove that the local minimum m can be obtained by some $u_0 \in H_0^1(\Omega)$ and to prove that u_0 is a solution of problem (1.3), we need the following two skilled lemmas which can be found in [10].

Lemma 2.3 *Assume that* (f_0) *holds, for* a_0, $b_0 \geq 0$, *one has that* $\lim_{t\to 0^+} \frac{1}{t}[F(a_0 + tb_0) - F(a_0)] = f(a_0)b_0$, *which equals* ∞ *if* $a_0 = 0$ *and* $b_0 > 0$.

Lemma 2.4 *Assume that* (f_0) *holds. Then, for any* $u \in H_0^1(\Omega)$ *with* $u(x) > 0$, *a.e.* $x \in \Omega$, *we have*

$$\lim_{t\to 0^+} \int_\Omega \frac{1}{t} h\big[F(u+tu) - F(u)\big] = \int_\Omega hf(u)u.$$

Theorem 2.5 *Assume that* (f), (g_1) *and* (h) *hold. Then, for* $\lambda \in (0,\lambda^*)$, *problem* (1.3) *possesses a solution* u_0 *with* $J(u_0) = m$.

Proof According to the definition of m, there exists a sequence $\{u_n\} \subset \bar{B}_r$ such that $\lim_{n\to\infty} J(u_n) = m$. Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u_0 \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega),\\
&u_n \to u_0 \quad \text{in } L^s(\Omega), s \in [1,6),\\
&u_n(x) \to u_0(x), \quad \text{a.e. } x \in \Omega,
\end{aligned}$$

as $n \to \infty$. By (1.4) and the Sobolev embedding theorem, we see that $\{F(u_n)\}$ is bounded in $L^2(\Omega)$. Moreover, it follows from the continuity of F that $F(u_n(x)) \to F(u_0(x))$, a.e. $x \in \Omega$. Thus, we obtain that $F(u_n) \rightharpoonup F(u_0)$ in $L^2(\Omega)$. By $h \in L^2(\Omega)$, it follows that

$$\lim_{n\to\infty} \int_\Omega hF(u_n) = \int_\Omega hF(u_0). \tag{2.3}$$

By (1.6), we easily deduce $\int_\Omega G(u_n) \to \int_\Omega G(u_0)$. Then, by the weak lower semi-continuity of the norm and Lemma 2.1(iv), (2.3), we have

$$\begin{aligned}
m &= \lim_{n\to\infty} J(u_n)\\
&= \liminf_{n\to\infty}\left[\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{1}{4}\int_\Omega \phi_{u_n} u_n^2 - \lambda\int_\Omega hF(u_n) - \int_\Omega G(u_n)\right]\\
&\geq \frac{a}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 + \frac{1}{4}\int_\Omega \phi_{u_0} u_0^2 - \lambda\int_\Omega hF(u_0) - \int_\Omega G(u_0)\\
&= J(u_0).
\end{aligned}$$

On the other hand, $u_n \rightharpoonup u_0$ implies that $\|u_0\| \leq \liminf_{n\to\infty} \|u_n\| \leq r$, then $J(u_0) \geq m$. Hence $J(u_0) = m$.

To show that u_0 is a solution of problem (1.3), we need to show that $u_0(x) > 0$, a.e. $x \in \Omega$ and u_0 satisfies (1.8). The proof is similar to the proof of Theorem 1.1 in [10], for completeness, here we give the details.

By Lemma 2.1(vi), we obtain that $m \leq J(u_0^+) \leq J(u_0) = m$, then $J(u_0^+) = J(u_0) = m$. Thus we may assume $u_0 \geq 0$. While $m < 0$, then $u_0 \not\equiv 0$. Now we divide the proof into two steps. For convenience, we denote $l_0 = a + b\|u_0\|^2$.

Firstly, we prove that $u_0(x) > 0$, a.e. $x \in \Omega$. In fact, for each $v \in H_0^1(\Omega)$ with $v \geq 0$ and $t > 0$ small enough, we have that

$$\begin{aligned}
0 &\leq \frac{J(u_0 + tv) - J(u_0)}{t} \\
&= \frac{a}{2t}\left(\|u_0 + tv\|^2 - \|u_0\|^2\right) + \frac{b}{4t}\left(\|u_0 + tv\|^4 - \|u_0\|^4\right) \\
&\quad + \frac{1}{4t}\int_\Omega \left[\phi_{u_0+tv}(u_0 + tv)^2 - \phi_{u_0}u_0^2\right] - \lambda\int_\Omega \frac{1}{t}h\left[F(u_0 + tv) - F(u_0)\right] \\
&\quad - \int_\Omega \frac{1}{t}\left[G(u_0 + tv) - G(u_0)\right].
\end{aligned}$$

This implies that

$$\liminf_{t\to 0^+} \lambda\int_\Omega \frac{1}{t}h\left[F(u_0 + tv) - F(u_0)\right] \leq l_0(u_0, v) + \int_\Omega \phi_{u_0}u_0 v - \int_\Omega g(u_0)v.$$

Thus, by Fatou's lemma and Lemma 2.3, we have

$$\lambda\int_\Omega hf(u_0)v \leq l_0(u_0, v) + \int_\Omega \phi_{u_0}u_0 v - \int_\Omega g(u_0)v. \tag{2.4}$$

Now let $e_1 \in H_0^1(\Omega)$ be the first eigenfunction of the operator $-\Delta$ in $H_0^1(\Omega)$ and $e_1(x) > 0$ for all $x \in \Omega$. Taking $v = e_1$ in (2.4), one gets that

$$\lambda\int_\Omega hf(u_0)e_1 \leq l_0(u_0, e_1) + \int_\Omega \phi_{u_0}u_0 e_1 - \int_\Omega g(u_0)e_1 < \infty,$$

which implies that $u_0(x) > 0$, a.e. $x \in \Omega$ by assumption (h). If not, there exists $E \subset \Omega$ such that $m(E) > 0$ and $u_0(x) = 0$ for all $x \in E$. Then, by Lemma 2.3,

$$\int_\Omega hf(u_0)e_1 \geq \int_E hf(u_0)e_1 = \infty,$$

it is a contradiction.

Secondly, we shall prove that u_0 is a solution of problem (1.3), namely, u_0 satisfies the following:

$$l_0(u_0, v) + \int_\Omega \phi_{u_0}u_0 v - \lambda\int_\Omega hf(u_0)v - \int_\Omega g(u_0)v = 0, \quad v \in H_0^1(\Omega). \tag{2.5}$$

For this purpose, we define a function $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(t) = J(u_0 + tu_0)$, that is,

$$\Phi(t) = \frac{a(1+t)^2}{2}\|u_0\|^2 + \frac{b(1+t)^4}{4}\|u_0\|^4 + \frac{(1+t)^4}{4}\int_\Omega \phi_{u_0} u_0^2 \\ - \lambda \int_\Omega hF(u_0 + tu_0) - \int_\Omega G(u_0 + tu_0).$$

Then Φ attains its local minimum at $t = 0$. It follows from Lemma 2.4 that Φ is differentiable at $t = 0$ and $\Phi'(0) = 0$, that is,

$$l_0\|u_0\|^2 + \int_\Omega \phi_{u_0} u_0^2 - \lambda \int_\Omega hf(u_0)u_0 - \int_\Omega g(u_0)u_0 = 0. \tag{2.6}$$

For each $v \in H_0^1(\Omega)$ and $\varepsilon > 0$, let us define $v_\varepsilon = u_0 + \varepsilon v$ and

$$\Omega_+ = \{x \in \Omega : u_0(x) + \varepsilon v(x) \geq 0\}, \qquad \Omega_- = \{x \in \Omega : u_0(x) + \varepsilon v(x) < 0\}.$$

Then $v_\varepsilon^-|_{\Omega_+} = 0$ and $v_\varepsilon^-|_{\Omega_-} = u_0 + \varepsilon v$. Inserting v_ε^+ into (2.4) and using (2.6), we obtain that

$$\begin{aligned}
0 &\leq l_0(u_0, v_\varepsilon^+) + \int_\Omega \phi_{u_0} u_0 v_\varepsilon^+ - \lambda \int_\Omega hf(u_0)v_\varepsilon^+ - \int_\Omega g(u_0)v_\varepsilon^+ \\
&= l_0(u_0, v_\varepsilon) + \int_\Omega \phi_{u_0} u_0 v_\varepsilon - \lambda \int_\Omega hf(u_0)v_\varepsilon - \int_\Omega g(u_0)v_\varepsilon \\
&\quad - \left[l_0(u_0, v_\varepsilon^-) + \int_\Omega \phi_{u_0} u_0 v_\varepsilon^- - \lambda \int_\Omega hf(u_0)v_\varepsilon^- - \int_\Omega g(u_0)v_\varepsilon^-\right] \\
&= \varepsilon\left[l_0(u_0, v) + \int_\Omega \phi_{u_0} u_0 v - \lambda \int_\Omega hf(u_0)v - \int_\Omega g(u_0)v\right] - \left[l_0 \int_{\Omega_-} \nabla u_0 \cdot \nabla(u_0 + \varepsilon v)\right. \\
&\quad \left. + \int_{\Omega_-} \phi_{u_0} u_0(u_0 + \varepsilon v) - \lambda \int_{\Omega_-} hf(u_0)(u_0 + \varepsilon v) - \int_{\Omega_-} g(u_0)(u_0 + \varepsilon v)\right] \\
&\leq \varepsilon\left[l_0(u_0, v) + \int_\Omega \phi_{u_0} u_0 v - \lambda \int_\Omega hf(u_0)v - \int_\Omega g(u_0)v\right] \\
&\quad - \varepsilon\left[l_0 \int_{\Omega_-} \nabla u_0 \cdot \nabla v + \int_{\Omega_-} \phi_{u_0} u_0 v\right],
\end{aligned}$$

which implies that

$$l_0 \int_{\Omega_-} \nabla u_0 \cdot \nabla v + \int_{\Omega_-} \phi_{u_0} u_0 v \\ \leq l_0(u_0, v) + \int_\Omega \phi_{u_0} u_0 v - \lambda \int_\Omega hf(u_0)v - \int_\Omega g(u_0)v. \tag{2.7}$$

Now let $E_n = \{x \in \Omega : u_0(x) > 0, v(x) > -\infty, u_0(x) + v(x)/n < 0\}$ for all n. Then $\{E_n\}$ is a nonincreasing sequence of measurable sets and

$$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n = \emptyset.$$

Thus we have

$$\lim_{n\to\infty} m(E_n) = m\Big(\lim_{n\to\infty} E_n\Big) = 0.$$

Select $\varepsilon = 1/n$. Then $\Omega_- \subset \{x \in \Omega : u_0(x) \le 0\} \cup \{x \in \Omega : v(x) = -\infty\} \cup E_n$ and $m(\Omega_-) = m(E_n) \to 0$ as $n \to \infty$. Letting $\varepsilon = 1/n \to 0$ in (2.7), we have

$$0 \le l_0(u_0, v) + \int_\Omega \phi_{u_0} u_0 v - \lambda \int_\Omega hf(u_0)v - \int_\Omega g(u_0)v.$$

According to the arbitrariness of $v \in H_0^1(\Omega)$, this inequality also holds for $-v$. Thus, (2.5) holds. Therefore, u_0 is a solution of system (1.3) with $J(u_0) = m$. □

3 Proof of Theorem 1.1

In order to overcome the difficulty caused by the singular term and to obtain the second solution of problem (1.3) for $\lambda > 0$ small enough, in this section, we firstly consider the following perturbation problem:

$$\begin{cases} -(a + b\|u\|^2)\Delta u + \phi_u u = \lambda hf(u + \alpha) + g(u), & \text{in } \Omega, \\ u \ge 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\alpha > 0$. We define the functional corresponding to problem (3.1)

$$J_\alpha(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4}\int_\Omega \phi_u u^2 - \lambda \int_\Omega h\big(F\big(u^+ + \alpha\big) - F(\alpha)\big) - \int_\Omega G(u),$$
$$u \in H_0^1(\Omega).$$

It is obvious that J_α is a C^1 functional defined on $H_0^1(\Omega)$. The solution of problem (3.1) corresponds to the critical point of the functional J_α. That is, if $u \in H_0^1(\Omega)$ is a solution of problem (3.1), it satisfies

$$\big(a + b\|u\|^2\big)(u, \phi) + \int_\Omega \phi_u u\phi - \lambda \int_\Omega hf\big(u^+ + \alpha\big)\phi - \int_\Omega g(u)\phi = 0,$$
$$\phi \in H_0^1(\Omega). \tag{3.2}$$

For any $s > 0$, since f is nonincreasing, we have

$$F(s + \alpha) - F(\alpha) = \int_\alpha^{s+\alpha} f(t)\,dt = \int_0^s f(\tau + \alpha)\,d\tau \le \int_0^s f(\tau)\,d\tau = F(s), \tag{3.3}$$

by $F(s) = 0$ if $s \le 0$, (3.3) holds for all $s \in \mathbb{R}$. Then, for any $u \in H_0^1(\Omega)$, we have

$$J(u) \le J_\alpha(u) \le I(u), \tag{3.4}$$

where $I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4}\int_\Omega \phi_u u^2 - \int_\Omega G(u)$, $u \in H_0^1(\Omega)$.

In order to show that J_α satisfies the mountain pass geometry and to estimate the mountain pass critical level, we firstly consider the functional I. In fact, under the assumptions of (g_1) and (g_2), we have the following lemma.

Lemma 3.1 *Under the assumptions of* (g_1) *and* (g_2), *there exist* $r_0, \rho_0 > 0$ *such that the functional* I *satisfies*

(i) $I|_{S_{r_0}} \geq \rho_0$;

(ii) *there exists* $u_1 \in H_0^1(\Omega)$ *with* $\|u_1\| > r_0$ *such that* $I(u_1) < 0$.

Proof (i) By (1.6) with $\varepsilon > 0$ small enough, for any $u \in H_0^1(\Omega)$, we have

$$I(u) \geq \left(\frac{a}{2} - c\varepsilon\right)\|u\|^2 - c_1\|u\|^6 - c_2\|u\|^{p+1},$$

it is obvious that the conclusion (i) holds.

(ii) It follows from (g_1) and (g_2), for any given $M > 0$, there exists $R > 0$ such that $g(t) \geq Mt^3$, $t > R$ and

$$\lim_{t\to 0^+} \frac{g(t) - Mt^3}{t} = 0.$$

Then there exists $C > 0$ such that $g(t) - Mt^3 \geq -Ct$, $t \in [0, R]$ and $g(t) \geq Mt^3 - Ct$, $t \geq 0$. For G, we also have

$$G(t) \geq \frac{M}{4}t^4 - \frac{Ct^2}{2}, \quad t \in \mathbb{R}.$$

Thus, for any $u \in H_0^1(\Omega) \setminus \{0\}$, $\int_\Omega G(tu) \geq \frac{M}{4}t^4|u|_4^4 - \frac{C}{2}t^2|u|_2^2$. It follows that

$$\lim_{t\to\infty} \int_\Omega \frac{G(tu)}{t^4} = \infty. \tag{3.5}$$

Thus

$$\begin{aligned} I(tu) &= \frac{at^2}{2}\|u\|^2 + \frac{bt^4}{4}\|u\|^4 + \frac{t^4}{4}\int_\Omega \phi_u u^2 - \int_\Omega G(tu) \\ &= t^4\left(\frac{a}{2t^2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4}\int_\Omega \phi_u u^2 - \int_\Omega \frac{G(tu)}{t^4}\right), \end{aligned}$$

it follows from (3.5) that $\lim_{t\to\infty} I(tu) = -\infty$, hence, there exists $t > 0$ large enough such that $\|u_1\| = \|tu\| > r_0$ and $I(u_1) < 0$. □

Now, we define

$$\Gamma = \left\{\gamma \in C\big([0,1], H_0^1(\Omega)\big) : \gamma(0) = 0, \gamma(1) = u_1\right\}, \qquad c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big).$$

It follows from Lemma 3.1 that $c \geq \rho_0 > 0$. For any given $\alpha > 0$, J_α also has the mountain pass geometry. In fact, we have the following lemma.

Lemma 3.2 *Assume $\lambda \in (0, \lambda^*)$, under the assumptions of* (f), (g_1), (g_2) *and* (h), *for $r, \rho > 0$ (where λ^*, r, ρ are given in Lemma* 2.2), *the functional J_α satisfies the following*:

(i) $J_\alpha|_{S_r} \geq \rho$;

(ii) *there exists $v \in H_0^1(\Omega)$ such that $J_\alpha(v) < 0$.*

Proof (i) By (3.4) and Lemma 2.2, the conclusion holds.

(ii) From (3.4) and (ii) of Lemma 3.1, we choose $v = u_1$ in Lemma 3.1 and the conclusion holds. □

We also can define the mountain pass critical level

$$c_\alpha = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\alpha\big(\gamma(t)\big).$$

From (3.4) and (i) of Lemma 3.2, for $\lambda \in (0, \lambda^*)$,

$$0 < \rho \leq c_\alpha \leq c. \tag{3.6}$$

In the following, we give the existence of the mountain pass type solution to system (3.1).

Lemma 3.3 *Suppose that* (f), (g_1)-(g_3) *and* (h) *hold*, $\lambda \in (0, \lambda^*)$. *Then there exists $u_\alpha \in H_0^1(\Omega)$ such that*

$$J'_\alpha(u_\alpha) = 0, \qquad J_\alpha(u_\alpha) = c_\alpha.$$

Proof By Lemma 3.2 and the mountain pass lemma, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $J_\alpha(u_n) \to c_\alpha$, $J'_\alpha(u_n) \to 0$. By (3.3), (1.4) and (g_3), for n large enough, we have

$$\begin{aligned}
c_\alpha + 1 + \|u_n\| &\geq J_\alpha(u_n) - \frac{1}{4}\big(J'_\alpha(u_n), u_n\big) \\
&= \frac{a}{4}\|u_n\|^2 - \lambda \int_\Omega h\left(F\big(u_n^+ + \alpha\big) - F(\alpha) - \frac{1}{4}f\big(u_n^+ + \alpha\big)u_n\right) \\
&\quad - \int_\Omega \left(G(u_n) - \frac{1}{4}g(u_n)u_n\right) \\
&\geq \frac{a}{4}\|u_n\|^2 - \lambda \int_\Omega hF\big(u_n^+\big) + \frac{\lambda}{4}f(\alpha)\int_\Omega hu_n^- \\
&\geq \frac{a}{4}\|u_n\|^2 - \lambda^*\left(c_0|h|_2\|u_n\| + c_2|h_1| + \frac{1}{4}f(\alpha)c_0|h|_2\|u_n\|\right).
\end{aligned}$$

Then $\{u_n\}$ is bounded. Up to a subsequence, there exists $u_\alpha \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_\alpha$ in $H_0^1(\Omega)$ and

$$u_n \to u_\alpha \quad \text{in } L^s(\Omega), s \in [1, 6),$$

$$u_n(x) \to u_\alpha(x) \quad \text{a.e. } x \in \Omega,$$

there exists $k_1 \in L^2(\Omega)$ such that for all n, $|u_n(x)|, |u_\alpha(x)| \leq k_1(x)$ a.e. in Ω.

It follows from $J'_\alpha(u_n) \to 0$ that

$$\begin{aligned} 0 &\leftarrow \bigl(J'_\alpha(u_n), u_n - u_\alpha\bigr) \\ &= \bigl(a + b\|u_n\|^2\bigr)(u_n, u_n - u_\alpha) + \int_\Omega \phi_{u_n} u_n (u_n - u_\alpha) \\ &\quad - \lambda \int_\Omega hf\bigl(u_n^+ + \alpha\bigr)(u_n - u_\alpha) - \int_\Omega g(u_n)(u_n - u_\alpha). \end{aligned} \tag{3.7}$$

Since $hf(u_n^+ + \alpha)(u_n - u_\alpha) \to 0$ a.e. in Ω and

$$\bigl|hf\bigl(u_n^+ + \alpha\bigr)(u_n - u_\alpha)\bigr| \le 2f(\alpha)hk_1 \in L^1(\Omega),$$

by the dominated convergence theorem, we have

$$\int_\Omega hf\bigl(u_n^+ + \alpha\bigr)(u_n - u_\alpha) \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

By (1.5), we can deduce that

$$\begin{aligned} \int_\Omega g(u_n)(u_n - u_\alpha) \le \varepsilon\bigl(|u_n|_2 |u_n - u_\alpha| + |u_n|_6^5 |u_n - u|_6\bigr) \\ + C_\varepsilon |u_n|_{p+1}^p |u_n - u|_{p+1} \to 0, \end{aligned} \tag{3.9}$$

and

$$\int_\Omega g(u_n)u_n \to \int_\Omega g(u_\alpha)u_\alpha, \qquad \int_\Omega g(u_n)\phi \to \int_\Omega g(u_\alpha)\phi, \quad \forall \phi \in H_0^1(\Omega). \tag{3.10}$$

From (3.7), using (3.8), (3.9), Lemma 2.1(iv) and the boundedness of $\{u_n\}$, we get $\|u_n\| \to \|u_\alpha\|$. This combined with $u_n \rightharpoonup u_\alpha$ implies that $u_n \to u_\alpha$ in $H_0^1(\Omega)$. Consequently, we have $J_\alpha(u_\alpha) = c_\alpha > \rho, J'_\alpha(u_\alpha) = 0$, that is, u_α is a nontrivial solution to problem (3.1). Then u_α satisfies (3.2), taking the test function $\phi = u_\alpha^-$ in (3.2), it follows that $\|u_\alpha^-\| = 0$. Thus, we have $u_\alpha \ge 0, u_\alpha \not\equiv 0$ and $J_\alpha(u_\alpha) = c_\alpha \ge \rho_1$. Hence, by the strong maximum principle, u_α is a positive solution of the perturbation problem (3.1). □

In order to consider the convergence of $\{u_\alpha\}$ as $\alpha \to 0$ and to obtain the second solution of problem (1.3), we need the following result, which can be found in [13].

Lemma 3.4 (Brezis and Nirenberg [13]) *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and assume that, for some $k \ge 0$, u satisfies, in the sense of distributions,*

$$\begin{cases} -\Delta u + ku \ge 0, & \text{in } \Omega, \\ u \ge 0, & \text{in } \Omega. \end{cases}$$

Then either $u \equiv 0$, or there exists $C > 0$ such that

$$u(x) \ge C\,\mathrm{dist}(x, \partial\Omega), \quad x \in \Omega.$$

Remark 3.5 By Lemma 3.3, (3.2) and Lemma 2.1(v), we have

$$-\Delta u_\alpha + Ku_\alpha \geq -\Delta u_\alpha + \frac{\phi_{u_\alpha} u_\alpha}{a+b\|u_\alpha\|^2} = \frac{\lambda hf(u_\alpha+\alpha)+g(u_\alpha)}{a+b\|u_\alpha\|^2} \geq 0, \quad x \in \Omega,$$

where $K > 0$. Then, by Lemma 3.4, there exists $C > 0$ such that $u_\alpha(x) \geq C\operatorname{dist}(x, \partial\Omega), x \in \Omega$.

Finally, let $\alpha \to 0$, we shall prove that the limit of a family of solutions $\{u_\alpha\}$ of the perturbation problem (3.1) is the second solution of problem (1.3) with $\lambda \in (0, \lambda^*)$, where λ^* is defined in Lemma 2.2.

Theorem 3.6 *Suppose that* (f), (g_1)-(g_3) *and* (h) *hold*, $\lambda \in (0, \lambda^*)$. *Then problem* (1.3) *has a solution* v_0 *satisfying* $J(v_0) > 0$.

Proof Let $\alpha \to 0$ and $u_\alpha \geq 0$ is the solution of problem (3.1), that is, $J_\alpha(u_\alpha) = c_\alpha, J'_\alpha(u_\alpha) = 0$. Then, by ($g_3$), (3.3), (3.6) and (1.4), we have

$$\begin{aligned}
c > c_\alpha &= J_\alpha(u_\alpha) - \frac{1}{4}\big(J'_\alpha(u_\alpha), u_\alpha\big) \\
&= \frac{a}{4}\|u_\alpha\|^2 - \lambda \int_\Omega h\left(F(u_\alpha+\alpha) - F(\alpha) - \frac{1}{4}f(u_\alpha+\alpha)u_\alpha\right) \\
&\quad - \int_\Omega \left(G(u_\alpha) - \frac{1}{4}g(u_\alpha)u_\alpha\right) \\
&\geq \frac{a}{4}\|u_\alpha\|^2 - \lambda \int_\Omega h\big(F(u_\alpha)\big) \\
&\geq \frac{a}{4}\|u_\alpha\|^2 - \lambda^*\big(c_0|h|_2\|u_\alpha\| + c_2|h|_1\big),
\end{aligned}$$

then $\{u_\alpha\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, there exists $v_0 \in H_0^1(\Omega)$ such that $u_\alpha \rightharpoonup v_0$ in $H_0^1(\Omega)$ and

$$\begin{aligned}
&u_\alpha \to v_0 \quad \text{in } L^s(\Omega), s \in [1,6), \\
&u_\alpha(x) \to v_0(x) \quad \text{a.e. } x \in \Omega, \\
&\text{there exists } k_2 \in L^2(\Omega) \text{ such that for all } n, \big|u_\alpha(x)\big|, \big|v_0(x)\big| \leq k_2(x) \text{ a.e. in } \Omega.
\end{aligned}$$

Firstly, we show that $v_0(x) > 0$ a.e. in Ω. For that purpose, we denote $w_\alpha = u_\alpha - v_0$ and $l = \lim_{\alpha\to 0}\|w_\alpha\|$. Taking $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in (3.2), we have

$$\lambda \int_\Omega hf(u_\alpha+\alpha)\phi = \big(a+b\|u_\alpha\|^2\big)(u_\alpha, \phi) + \int_\Omega \phi_{u_\alpha} u_\alpha \phi - \int_\Omega g(u_\alpha)\phi.$$

By using Fatou's lemma, Lemma 2.1(iv) and (3.10), we have

$$\lambda \int_\Omega hf(v_0)\phi \leq \big(a+bl^2+b\|v_0\|^2\big)(v_0, \phi) + \int_\Omega \phi_{v_0} v_0 \phi - \int_\Omega g(v_0)\phi.$$

Similar to the proof of $u_0(x) > 0$ in $x \in \Omega$ in Theorem 2.5, we can show that $v_0(x) > 0$ a.e. in Ω.

Next, we show that $u_\alpha \to v_0$ in $H_0^1(\Omega)$ and v_0 is the solution of problem (1.3), that is, we need to show that $l = 0$ and v_0 satisfies (1.8).

We take $\phi \in C_0^\infty(\Omega)$ with $\operatorname{supp}\phi = \Omega_1 \Subset \Omega$ in (3.2). By Remark 3.5, for $x \in \Omega_1$, we have

$$\begin{aligned}\left|hf(u_\alpha + \alpha)\phi\right| &\le \left|hf(u_\alpha)\phi\right| \\ &\le \left|hf\big(\operatorname{dist}(x, \partial\Omega)\big)\phi\right| \\ &\le \left|hf(k_0)\phi\right| \in L^1(\Omega),\end{aligned}$$

where $k_0 = \min_{x\in\Omega_1} \operatorname{dist}(x, \partial\Omega) > 0$. Since $hf(u_\alpha + \alpha)\phi \to hf(v_0)\phi$ a.e. in Ω, then by the dominant convergence theorem, we have

$$\int_\Omega hf(u_\alpha + \alpha)\phi \to \int_\Omega hf(v_0)\phi \quad \text{as } \alpha \to 0.$$

By $(J'_\alpha(u_\alpha), \phi) = 0$, using $\int_\Omega \phi_{u_\alpha} u_\alpha \phi \to \int_\Omega \phi_{v_0} v_0 \phi$ and (3.10), we get that

$$\begin{aligned}&\big(a + bl^2 + b\|v_0\|^2\big)(v_0, \phi) + \int_\Omega \phi_{v_0} v_0 \phi = \lambda \int_\Omega hf(v_0)\phi + \int_\Omega g(v_0)\phi, \\ &\forall \phi \in C_0^\infty(\Omega).\end{aligned} \tag{3.11}$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, then for $\phi \in H_0^1(\Omega)$, there exists a sequence $\{\phi_n\} \subset C_0^\infty(\Omega)$ such that $\phi_n \to \phi$ as $n \to \infty$. For $n, m \in \mathbb{N}$ large enough, replacing ϕ with $\phi_n - \phi_m$ in (3.11), we obtain that

$$\begin{aligned}&\big(a + bl^2 + b\|v_0\|^2\big)(v_0, \phi_n - \phi_m) + \int_\Omega \phi_{v_0} v_0 (\phi_n - \phi_m) \\ &\quad = \lambda \int_\Omega hf(v_0)(\phi_n - \phi_m) + \int_\Omega g(v_0)(\phi_n - \phi_m).\end{aligned} \tag{3.12}$$

Since $\phi_n \to \phi$, from (3.12), we can deduce that $\{hf(v_0)\phi_n\}$ is a Cauchy sequence in $L^1(\Omega)$, hence there exists $v \in L^1(\Omega)$ satisfying $hf(v_0)\phi_n \to v$ in $L^1(\Omega)$, which means that $hf(v_0)\phi_n \to v$ in measure. By Riesz's theorem, $\{hf(v_0)\phi_n\}$ has a subsequence, still denoted by $\{hf(v_0)\phi_n\}$, such that $hf(v_0)\phi_n \to v$ a.e. in Ω. On the other hand, $hf(v_0)\phi_n \to hf(v_0)\phi$ a.e. in Ω. So $v = hf(v_0)\phi$, that is, $\int_\Omega hf(v_0)\phi_n \to \int_\Omega hf(v_0)\phi$ as $n \to \infty$. Then, taking the test function ϕ_n in (3.11) and passing to the limit as $n \to \infty$, we obtain that (3.11) holds for any $\phi \in H_0^1(\Omega)$. We take $\phi = v_0$ in (3.11) and obtain that

$$\big(a + bl^2 + b\|v_0\|^2\big)\|v_0\|^2 + \int_\Omega \phi_{v_0} v_0^2 = \lambda \int_\Omega hf(v_0)v_0 + \int_\Omega g(v_0)v_0. \tag{3.13}$$

On the other hand, by $(J'_\alpha(u_\alpha), u_\alpha) = 0$, we have

$$\big(a + b\|u_\alpha\|^2\big)\|u_\alpha\|^2 + \int_\Omega \phi_{u_\alpha} u_\alpha^2 = \lambda \int_\Omega hf(u_\alpha + \alpha)u_\alpha + \int_\Omega g(u_\alpha)u_\alpha. \tag{3.14}$$

Since f is nonincreasing on $(0, \infty)$, we have

$$sf(s) \le \int_0^s f(t)\,dt = F(s), \quad s \ge 0.$$

Then, by (1.4), we have

$$hf(u_\alpha+\alpha)u_\alpha \le hf(u_\alpha)u_\alpha \le hF(u_\alpha) \le h\big(c_1|u_\alpha|+c_2\big) \le h(c_1k_2+c_2) \in L^1(\Omega).$$

Then, combining with $hf(u_\alpha+\alpha) \to hf(v_0)v_0$ a.e. in Ω and using the dominant convergence theorem, we have

$$\int_\Omega hf(u_\alpha+\alpha)u_\alpha \to \int_\Omega hf(v_0)v_0.$$

Thus, from (3.14), by (3.10) and Lemma 2.1(iv), we obtain that

$$\big(a+bl^2+b\|v_0\|^2\big)\big(l^2+\|v_0\|^2\big)+\int_\Omega \phi_{v_0}v_0^2=\lambda\int_\Omega hf(v_0)v_0+\int_\Omega g(v_0)v_0. \tag{3.15}$$

Combining with (3.13) and (3.15), we get

$$\big(a+bl^2+b\|v_0\|^2\big)l^2=0,$$

from $a>0, b\ge 0$, it implies that $l=0$, that is, $u_\alpha \to v_0$. Hence, from (3.11) with $l=0$ and $\phi \in H_0^1(\Omega)$, we get that v_0 is the solution of problem (1.3) for $\lambda \in (0,\lambda^*)$ and $J(v_0)=\lim_{\alpha\to 0} J_\alpha(u_\alpha) \ge \rho > 0$. □

Proof of Theorem 1.1 By Theorems 2.5 and 3.6, for $\lambda \in (0,\lambda^*)$, there exist two solutions $u_0, v_0 \in H_0^1(\Omega)$ to problem (1.3) with $J(v_0)>0>J(u_0)$, that is, system (1.1) possesses at least two solutions for each $\lambda \in (0,\lambda^*)$. □

4 Conclusion

In this paper, by using the variational method and the perturbation method, we consider the existence and multiple solutions to the singular Kirchhoff-Schrödinger-Poisson system (1.1). The nonlinear terms contain the quasicritical nonlinearity g, which satisfies assumptions (g_1)-(g_3), and the general singularity f, which satisfies (f). The general singular assumption derives from our previous work [10], in which we consider the uniqueness of solution to Kirchhoff-Schrödinger-Poisson system. Therefore, the results in this paper are the continuation of our research in [10]. Our results also improve the results in [11], in which the authors considered the existence of Kirchhoff-Schrödinger-Poisson system with the singular term $f(s)=s^{-r}$, $r \in (0,1)$.

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