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A regularity criterion for the Keller-Segel-Euler system

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Abstract
We consider a Keller-Segel-Euler model and prove a regularity criterion of the local strong solutions in a 3D bounded domain Ω.

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Keywords: Keller-Segel-Euler; chemotaxis; regularity criterion; bounded domain

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and ν be the unit outward normal vector to $\partial\Omega$. We consider the regularity problem for the following Keller-Segel-Euler model:

$$\partial_t u + u\cdot\nabla u + \nabla\pi + n\nabla\phi = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t n + u\cdot\nabla n - \Delta n = -\nabla\cdot\big(nr(p)\nabla p\big), \tag{1.3}$$

$$\partial_t p + u\cdot\nabla p - \Delta p = -nf(p) \quad \text{in } \Omega\times(0,\infty), \tag{1.4}$$

$$u\cdot\nu = 0, \qquad \frac{\partial n}{\partial\nu} = \frac{\partial p}{\partial\nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.5}$$

$$(u,n,p)(\cdot,0) = (u_0,n_0,p_0) \quad \text{in } \Omega\subset\mathbb{R}^3. \tag{1.6}$$

Here u,π,n and p denote the fluid velocity field, scalar pressure, cell concentration, and oxygen concentration, respectively. The functions $f(p)$ and $r(p)$ are two given smooth functions of p denoting the oxygen consumption rate and chemotactic sensitivity, respectively. The function ϕ denotes the potential function.

When $\phi = 0$, system (1.1) and (1.2) reduces to the well-known Euler system, Ferrari [1] showed the regularity criterion

$$\operatorname{rot} u \in L^1\big(0,T;L^\infty(\Omega)\big). \tag{1.7}$$

On the other hand, when $u = 0$, system (1.3) and (1.4) reduces to the classical Keller-Segel chemotaxis model [2–4], which received many studies [5–11] on well-posedness and pattern formation of solutions.

For completeness, we also cite [12–14] which show some regularity criteria for the Keller-Segel-Navier-Stokes model.

The aim of this paper is to prove a regularity criterion of local smooth solutions to problem (1.1)-(1.6). We will prove the following.

Theorem 1.1 *Let $u_0 \in H^3, n_0, p_0 \in H^2, \operatorname{div} u_0 = 0, n_0, p_0 \geq 0$ in Ω and $n_0 \cdot \nu = 0, \frac{\partial n_0}{\partial \nu} = \frac{\partial p_0}{\partial \nu} = 0$ on $\partial\Omega$. Suppose that ϕ is a smooth function. Let (u, n, p) be a local smooth solution to problem* (1.1)-(1.6). *If* (1.7) *and*

$$\nabla p \in L^{\frac{2q}{q-3}}(0, T; L^q), \quad 3 < q \leq \infty, \tag{1.8}$$

hold true with $0 < T < \infty$, then the solution can be extended beyond $T > 0$.

Remark 1.1 We observe that (1.1)-(1.4) is invariant under the scaling transform $(u, \pi, n, p, \phi) \to (u_\lambda, \pi_\lambda, n_\lambda, p_\lambda, \phi_\lambda)$, where

$$\begin{aligned} &u_\lambda := \lambda u(\lambda^2 t, \lambda x), \qquad \pi_\lambda := \lambda^2 \pi(\lambda^2 t, \lambda x), \\ &n_\lambda := \lambda^2 n(\lambda^2 t, \lambda x), \qquad p_\lambda := p(\lambda^2 t, \lambda x), \qquad \phi_\lambda := \phi(\lambda^2 t, \lambda x). \end{aligned}$$

This implies that the regularity criteria (1.7) and (1.8) are optimal in the sense of scaling.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since local existence results can be proved by using standard arguments, say, Galerkin method, we only deal with the a priori estimates.

First of all, from the equations of n, p and the maximum principle, we easily see that

$$n \geq 0, 0 \leq p \leq C, \qquad \int n\,dx = \int n_0\,dx, \tag{2.1}$$

where the constant depends only on the initial data.

For any $m \geq 2$, testing (1.3) by n^{m-1}, using the boundary and incompressibility conditions, and denoting $w := n^{\frac{m}{2}}$, we calculate

$$\frac{1}{m}\frac{d}{dt}\int w^2\,dx + \frac{4(m-1)}{m^2}\int |\nabla w|^2\,dx = (m-1)\int wr(p)(\nabla p \cdot \nabla w)\,dx.$$

Using the smoothness of $r(p)$ and (2.1), we infer that

$$\begin{aligned} &\frac{1}{m}\frac{d}{dt}\int w^2\,dx + \frac{4(m-1)}{m^2}\int |\nabla w|^2\,dx \\ &\quad \leq C\int |\nabla p| w |\nabla w|\,dx \\ &\quad \leq C\|\nabla p\|_{L^p}\|w\|_{L^{\frac{2q}{q-2}}}\|\nabla w\|_{L^2} \\ &\quad \leq C\|\nabla p\|_{L^p}\big(\|w\|_{L^2}^{1-\frac{3}{q}}\|\nabla w\|_{L^2}^{1+\frac{3}{q}} + \|w\|_{L^2}\|\nabla w\|_{L^2}\big) \\ &\quad \leq \frac{m-1}{m^2}\|\nabla w\|_{L^2}^2 + C\big(\|\nabla p\|_{L^p}^{\frac{2q}{q-3}} + 1\big)\|w\|_{L^2}^2, \end{aligned}$$

which gives

$$\|n\|_{L^2(0,T;H^1)} + \|n\|_{L^\infty(0,T;L^m)} \le C, \quad \forall m \ge 2. \tag{2.2}$$

Here we have used Young's inequality and the Gagliardo-Nirenberg inequality for functions on a bounded domain:

$$\|f\|_{L^{\frac{2q}{q-2}}} \le C\big(\|f\|_{L^2}^{1-\frac{3}{q}}\|\nabla f\|_{L^2}^{\frac{3}{q}} + \|f\|_{L^2}\big). \tag{2.3}$$

Testing (1.1) by u, using (1.2) and (2.2), we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |u|^2\,dx &= -\int n\nabla\phi\cdot u\,dx\\ &\le \|n\|_{L^3}\|\nabla\phi\|_{L^6}\|u\|_{L^2} \le C\|\nabla\phi\|_{L^6}\|u\|_{L^2},\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;L^2)} \le C. \tag{2.4}$$

Taking curl to (1.1), using (1.2), we infer that

$$\partial_t\omega + u\cdot\nabla\omega = \omega\cdot\nabla u - \nabla n\times\nabla\phi, \tag{2.5}$$

where $\omega := \operatorname{curl} u$. Testing (2.5) by ω, using (1.2) and (2.2), we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\omega|^2\,dx &= \int(\omega\cdot\nabla u - \nabla n\times\nabla\phi)\cdot\omega\,dx\\ &\le \|\omega\|_{L^\infty}\|\nabla u\|_{L^2}\|\omega\|_{L^2} + \|\nabla n\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\omega\|_{L^2}\\ &\le C\|\omega\|_{L^\infty}\|\omega\|_{L^2}^2 + \|\nabla n\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\omega\|_{L^2},\end{aligned}$$

which implies

$$\|\omega\|_{L^\infty(0,T;L^2)} \le C, \tag{2.6}$$

$$\|u\|_{L^\infty(0,T;L^6)} \le C. \tag{2.7}$$

By using the regularity theory of parabolic equations [15], it follows from (1.3), (1.5), (1.6), (2.1), (2.2), and (2.7) that

$$\begin{aligned}&\|\nabla n\|_{L^2(0,T;L^{\tilde r})}\\ &\quad\le C\big(1 + \|un\|_{L^2(0,T;L^{\tilde r})} + \big\|nr(p)\nabla p\big\|_{L^2(0,T;L^{\tilde r})}\big)\\ &\quad\le C\big(1 + \|u\|_{L^\infty(0,T;L^6)}\|n\|_{L^\infty(0,T;L^{\frac{6\tilde r}{6-\tilde r}})} + \big\|r(p)\big\|_{L^\infty}\|n\|_{L^\infty(0,T;L^{\frac{q\tilde r}{q-\tilde r}})}\|\nabla p\|_{L^2(0,T;L^{\tilde q})}\big)\\ &\quad\le C\end{aligned} \tag{2.8}$$

for some $3 < \tilde r < 6$ and $\tilde r < q$.

Now we turn to the higher order regularity of the velocity field. Testing (2.5) by $|\omega|^{\tilde{r}-2}\omega$, using (1.2) and (2.8), we obtain

$$\frac{d}{dt}\|\omega\|_{L^{\tilde{r}}}^{\tilde{r}} \le C\|\omega\|_{L^\infty}\|\omega\|_{L^{\tilde{r}}}^{\tilde{r}} + C\|\nabla\phi\|_{L^\infty}\|\nabla n\|_{L^{\tilde{r}}}\|\omega\|_{L^{\tilde{r}}}^{\tilde{r}-1},$$

which gives

$$\|\omega\|_{L^\infty(0,T;L^{\tilde{r}})} \le C, \tag{2.9}$$

$$\|u\|_{L^\infty(0,T;L^\infty)} \le C. \tag{2.10}$$

Testing (1.1) by u_t, using (1.2), (2.2), (2.9), and (2.10), we get

$$\|u_t\|_{L^2} \le \|u\cdot\nabla u + n\nabla\phi\|_{L^2} \le \|u\|_{L^\infty}\|\nabla u\|_{L^2} + \|n\|_{L^3}\|\nabla\phi\|_{L^6} \le C,$$

whence

$$\|u_t\|_{L^\infty(0,T;L^2)} \le C. \tag{2.11}$$

Testing (1.4) by $-\Delta p$, using (2.1) and (2.2), we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla p|^2\,dx + \int|\Delta p|^2\,dx &= \int\big(u\cdot\nabla p - nf(p)\big)\Delta p\,dx\\ &\le \big(\|u\|_{L^\infty}\|\nabla p\|_{L^2} + \|f(p)\|_{L^\infty}\|n\|_{L^2}\big)\|\Delta p\|_{L^2}\\ &\le C\big(\|\nabla p\|_{L^2}+1\big)\|\Delta p\|_{L^2}\\ &\le \frac{1}{2}\|\Delta p\|_{L^2}^2 + C\|\nabla p\|_{L^2}^2,\end{aligned}$$

which implies

$$\|p\|_{L^\infty(0,T;H^1)} + \|p\|_{L^2(0,T;H^2)} \le C. \tag{2.12}$$

To achieve higher order regularity of p, we decompose p as

$$p := p_1 + p_2,$$

where p_1 and p_2 satisfy

$$\begin{cases}\partial_t p_1 - \Delta p_1 = -\operatorname{div}(up) & \text{in } \Omega\times(0,T),\\ \frac{\partial p_1}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ p_1(x,0) = 0 & \text{in } \Omega\end{cases}$$

and

$$\begin{cases}\partial_t p_2 - \Delta p_2 = -nf(p) & \text{in } \Omega\times(0,T),\\ \frac{\partial p_2}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ p_2(x,0) = p_0(x) & \text{in } \Omega,\end{cases}$$

respectively.

By using the regularity theory of general parabolic equations (cf. [15]), (2.2), (2.5), and (2.7), we have

$$\|\nabla p_1\|_{L^m(0,T;L^m)} \le C, \quad \forall m > 2, \tag{2.13}$$

$$\|p_2\|_{W_m^{2,1}(\overline{\Omega}\times[0,T])} \le C, \quad \forall m > 5, \tag{2.14}$$

whence

$$\|\nabla p\|_{L^m(0,T;L^m)} \le C. \tag{2.15}$$

Similarly, by the regularity theory of heat equations [15], we have

$$\|\nabla n\|_{L^m(0,T;L^m)} \le C, \quad \forall m > 3. \tag{2.16}$$

By the well-known L^∞-estimate of the heat equation, we discover that

$$\|n\|_{L^\infty(\Omega\times[0,T])} \le C. \tag{2.17}$$

Applying ∂_t to (1.3), testing by n_t, using (1.2), (2.11), and (2.17), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int n_t^2\,dx + \int |\nabla n_t|^2\,dx \\
&\quad = \int u_t n \nabla n_t\,dx + \int \big(n_t r(p)\nabla p + n r'(p) p_t \nabla p + n r(p)\nabla p_t\big)\nabla n_t\,dx \\
&\quad \le \|u_t\|_{L^2}\|n\|_{L^\infty}\|\nabla n_t\|_{L^2} + C\|n_t\|_{L^3}\|\nabla p\|_{L^6}\|\nabla n_t\|_{L^2} \\
&\qquad + C\|n\|_{L^\infty}\|p_t\|_{L^3}\|\nabla p\|_{L^6}\|\nabla n_t\|_{L^2} + C\|n\|_{L^\infty}\|\nabla p_t\|_{L^2}\|\nabla n_t\|_{L^2} \\
&\quad \le C\|\nabla n_t\|_{L^2} + C\|n_t\|_{L^2}^{\frac{1}{2}}\|\nabla p\|_{L^6}\|\nabla n_t\|_{L^2}^{\frac{3}{2}} \\
&\qquad + C\|p_t\|_{L^3}\|\nabla p\|_{L^6}\|\nabla n_t\|_{L^2} + C\|\nabla p_t\|_{L^2}\|\nabla n_t\|_{L^2}.
\end{aligned} \tag{2.18}$$

Here we used the fact $\int n_t\,dx = 0$ and the Gagliardo-Nirenberg inequality

$$\|n_t\|_{L^3}^2 \le C\|n_t\|_{L^2}\|\nabla n_t\|_{L^2}. \tag{2.19}$$

Applying ∂_t to (1.4), testing by p_t, using (1.2), (2.11), (2.1), and (2.17), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int p_t^2\,dx + \int |\nabla p_t|^2\,dx \\
&\quad = \int u_t p \nabla p_t\,dx - \int \big(n_t f(p) + n f'(p) p_t\big) p_t\,dx \\
&\quad \le \|u_t\|_{L^2}\|p\|_{L^\infty}\|\nabla p_t\|_{L^2} + C\|n_t\|_{L^2}\|p_t\|_{L^2} + C\|n\|_{L^\infty}\|p_t\|_{L^2}^2 \\
&\quad \le C\|\nabla p_t\|_{L^2} + C\|n_t\|_{L^2}\|p_t\|_{L^2} + C\|p_t\|_{L^2}^2.
\end{aligned} \tag{2.20}$$

Combining (2.18) and (2.20) and using the Gronwall inequality, we conclude that

$$\|n_t\|_{L^\infty(0,T;L^2)} + \|n_t\|_{L^2(0,T;H^1)} \le C, \tag{2.21}$$

$$\|p_t\|_{L^\infty(0,T;L^2)} + \|p_t\|_{L^2(0,T;H^1)} \le C. \tag{2.22}$$

Now using the H^2-theory of Poisson's equation, we have

$$\|p\|_{L^\infty(0,T;H^2)} + \|p\|_{L^2(0,T;H^3)} \le C, \tag{2.23}$$

$$\|n\|_{L^\infty(0,T;H^2)} + \|n\|_{L^\infty(0,T;H^3)} \le C. \tag{2.24}$$

To further improve the regularity of u, we recall some technical lemmas in [1, 16, 17].

Lemma 2.1 ([1]) *If $f, g \in H^s(\Omega) \cap C(\Omega)$, then*

$$\|fg\|_{H^s} \le C\big(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\big). \tag{2.25}$$

If $f \in H^s(\Omega) \cap C^1(\Omega)$ and $g \in H^{s-1}(\Omega) \cap C(\Omega)$, then for $|\alpha| \le s$,

$$\big\|D^\alpha(fg) - fD^\alpha g\big\|_{L^2} \le C\big(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{W^{1,\infty}}\|g\|_{H^{s-1}}\big). \tag{2.26}$$

Lemma 2.2 ([1, 17]) *For any $u \in H^3(\Omega)$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$, there holds*

$$\|\nabla u\|_{L^\infty} \le \big(1 + \|\operatorname{curl} u\|_{L^\infty} \log\big(e + \|u\|_{H^3}\big)\big). \tag{2.27}$$

Lemma 2.3 ([16]) *For any $u \in W^{s,p}$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$, there holds*

$$\|u\|_{W^{s,p}} \le C\big(\|u\|_{L^p} + \|\operatorname{curl} u\|_{W^{s-1,p}}\big) \tag{2.28}$$

for any $s > 1$ and $p \in (1, \infty)$.

Now, applying Δ to (2.5), testing by $\Delta\omega$, using (1.2), (2.25), (2.26), (2.10), (2.28), (2.27), and (2.24), we conclude that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\Delta\omega|^2\,dx &= -\sum_i \int \big[\partial_i\Delta(u_i\omega) - u_i\partial_i\Delta\omega\big]\cdot\Delta\omega\,dx \\
&\quad + \int \Delta(\omega\cdot\nabla u)\cdot\Delta\omega\,dx - \int \Delta(\nabla n \times \nabla\phi)\cdot\Delta\omega\,dx \\
&\le C\big(\|\nabla u\|_{L^\infty}\|\Delta\omega\|_{L^2} + \|\omega\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\big)\|\Delta\omega\|_{L^2} \\
&\quad + C\big(\|\nabla\phi\|_{L^\infty}\|\nabla\Delta n\|_{L^2} + \|\nabla n\|_{L^\infty}\|\nabla\Delta\phi\|_{L^2}\big)\|\Delta\omega\|_{L^2},
\end{aligned}$$

which gives

$$\|\Delta\omega\|_{L^\infty(0,T;L^2)} \le C,$$

$$\|u\|_{L^\infty(0,T;H^3)} \le C.$$

This completes the proof of Theorem 1.1.

3 Conclusion

We consider the 3D Keller-Segel-Euler system in a bounded domain. It is a challenging open problem whether the local solution exists globally. Here, a regularity criterion in terms of the vorticity and oxygen concentration is established to guarantee smoothness up to time T. It will help people to gain understanding of the model. We hope to find more inside structures and establish refined regularity criteria.

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Competing interests
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Authors' contributions
All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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