Peng et al. *Boundary Value Problems* (2017) 2017:141
DOI 10.1186/s13661-017-0873-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Damped vibration problems with sign-changing nonlinearities: infinitely many periodic solutions

Zhen Peng[1], Haiyan Lv[1] and Guanwei Chen[2*]

[*]Correspondence:
guanweic@163.com
[2]School of Mathematical Sciences, University of Jinan, Jinan, Shandong Province 250022, P.R. China
Full list of author information is available at the end of the article

Abstract
We obtain infinitely many nontrivial periodic solutions for a class of damped vibration problems, where nonlinearities are superlinear at infinity and primitive functions of nonlinearities are allowed to be sign-changing. By using some weaker conditions, our results extend and improve some existing results in the literature. Besides, some examples are given to illuminate our results.

MSC: 34C25; 70H05

Keywords: damped vibration problems; infinitely many periodic solutions; sign-changing; variational method

1 Introduction and main results

In this paper, we shall study the existence of infinitely many nontrivial periodic solutions for the following damped vibration problem:

$$\begin{cases} \ddot{u} + D(t)\dot{u} + V(t)u + H_u(t,u) = 0, & t \in \mathbb{R}, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, & T > 0, \end{cases} \tag{1.1}$$

where

$$D(t) = q(t)I_{N\times N} + B, \qquad V(t) = \frac{1}{2}Bq(t) - A(t),$$

$I_{N\times N}$ is the $N \times N$ identity matrix, $q(t) \in L^1(\mathbb{R};\mathbb{R})$ is T-periodic and satisfies $\int_0^T q(t)\,dt = 0$, $A(t) = [a_{ij}(t)]$ is a T-periodic symmetric $N \times N$ matrix-valued function with $a_{ij} \in L^\infty(\mathbb{R};\mathbb{R})$ ($\forall i,j = 1,2,\ldots,N$), $B = [b_{ij}]$ is an antisymmetric $N \times N$ constant matrix, $u = u(t) \in C^2(\mathbb{R},\mathbb{R}^N)$, $H(t,u) \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ is T-periodic and $H_u(t,u)$ denotes its gradient with respect to the u variable.

In fact, there are only a few results [1–6] of (1.1). In [5], the authors studied a special case ($B = 0$, zero matrix) and obtained the existence and multiplicity of periodic solutions. Recently, Chen [1] obtained infinitely many periodic solutions for (1.1) with H being *asymptotically quadratic* as $|u| \to \infty$. But the authors [2, 4, 6] obtained infinitely many periodic solutions for (1.1) with H being *superquadratic* as $|u| \to \infty$. For related topics, including homoclinic orbits of damped vibration problems, we refer the reader to [4, 7–10].

Inspired by the above papers, we shall study (1.1) with H being *superquadratic* as $|u| \to \infty$. As is shown in Remark 1.1, our results improve and extend the superquadratic results [2, 4, 6] in the positive definite case (*i.e.*, the following (D_0)).

Let $(\cdot,\cdot)$ denote the standard inner product in $\mathbb{R}^N$, and the associated norm is denoted by $|\cdot|$. To state our main result, we assume that:

(D_0) $\int_0^T [(Bu,\dot{u}) + (A(t)u,u)]dt \geq 0$, which implies and the energy functional of (1.1) is positive definite.

(AH_1) There exist $c_1, c_2 > 0$ and $p > 2$ such that

$$\bigl|H_u(t,u)\bigr| \leq c_1|u| + c_2|u|^{p-1}, \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N.$$

(AH_2) $\lim_{|u|\to+\infty} \frac{H(t,u)}{|u|^2} = +\infty$ uniformly in $t \in [0,T]$, and there exists $r_0 \geq 0$ such that

$$H(t,u) \geq 0, \quad \forall t \in [0,T], \forall |u| \geq r_0.$$

(AH_3) $H(t,-u) = H(t,u)$, $\forall (t,u) \in [0,T] \times \mathbb{R}^N$.

(AH_4) $(H_u(t,u),u) - 2H(t,u) \geq 0$, $\forall (t,u) \in [0,T] \times \mathbb{R}^N$, and there exist $c_0 > 0$ and $\varrho > 1$ such that

$$\bigl|H(t,u)\bigr|^{\varrho} \leq c_0|u|^{2\varrho}\bigl[\bigl(H_u(t,u),u\bigr) - 2H(t,u)\bigr], \quad \forall t \in [0,T], \forall |u| \geq r_0.$$

The condition (AH_4) can be replaced by the following condition.

(AH_4') There exist $\mu > 2$ and $\kappa > 0$ such that

$$\mu H(t,u) \leq \bigl(H_u(t,u),u\bigr) + \kappa u^2, \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N.$$

Now, our main results read as follows.

Theorem 1.1 *If* (D_0) *and* (AH_1)-(AH_4) *hold, then* (1.1) *has infinitely many nontrivial periodic solutions.*

Theorem 1.2 *If* (D_0), (AH_1)-(AH_3) *and* (AH_4') *hold, then* (1.1) *has infinitely many nontrivial periodic solutions.*

Example 1.1 Let

(1) $H(t,u) = a(t)(u^4 - 2u^2 \cos u)$, $(t,u) \in [0,T] \times \mathbb{R}$;

(2) $H(t,u) = a(t)[(4u^2 - 1)\ln(\frac{1}{2} + |u|) - 2(\frac{1}{2} + |u|)^2 + 4|u| + 2]$;

where $0 < \inf_{t\in[0,T]} a(t) < \sup_{t\in[0,T]} a(t) < +\infty$. It is easy to verify that the above functions all satisfy our conditions (AH_1)-(AH_4) and (AH_4').

Remark 1.1 Our Theorems 1.1 and 1.2 improve and extend the results [2, 4, 6] in the positive definite case. In all the results of [4, 6], the authors all used the following condition:

$$\limsup_{|u|\to 0} \frac{H(t,u)}{|u|^2} \leq 0 \quad \text{uniformly for a.e. } t \in [0,T], \tag{1.2}$$

besides, some results in the two papers rely on the following condition:

$$H(t,0) = 0, \quad \forall t \in [0,T]. \tag{1.3}$$

In [2], the author used (1.3) and the following conditions:

$$\begin{aligned} &H(t,u) \geq 0, \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N, \\ &\liminf_{|u|\to+\infty} \frac{(H_u(t,u),u) - 2H(t,u)}{|u|^\nu} \geq b \quad \text{for some } b > 0, \nu > 2, \forall t \in [0,T]. \end{aligned} \tag{1.4}$$

It is not hard to check that the functions in our Example 1.1 do not satisfy the conditions used in [2, 4, 6]. For example (the function in Example 1.1(2)),

$$\limsup_{|u|\to 0} \frac{H(t,u)}{|u|^2} = \limsup_{|u|\to 0} \frac{a(t)[(4|u|^2 - 1)\ln(\frac{1}{2} + |u|) - 2(\frac{1}{2} + |u|)^2 + 4|u| + 2]}{|u|^2} = +\infty,$$

that is, the function in Example 1.1 does not satisfy (1.2). We have

$$H(t,0) = a(t)\left[\ln 2 - \frac{1}{2} + 2\right] \neq 0, \quad \forall t \in [0,T].$$

that is, it also does not satisfy (1.3). Besides, the function in Example 1.1(1) does not satisfy (1.4). However, the functions in Example 1.1 all satisfy our conditions (AH_1)-(AH_4) and (AH_4'). Therefore, our results extend and improve the above results.

2 Variational frameworks and the proofs of main results

In this section, we always assume that (AH_1)-(AH_4) ((AH_4')) hold. We shall use $\|\cdot\|_p$ to denote the norm of $L^p([0,T];\mathbb{R}^N)$ for any $p \in [1,\infty]$, and we will use $u^k \rightharpoonup u$ to denote the weak convergence of $\{u^k\}$.

Let $W := H_T^1$ be defined by

$$\begin{aligned} H_T^1 := \big\{ &u = u(t) : [0,T] \to \mathbb{R}^N \mid \\ &u \text{ is absolutely continuous }, u(0) = u(T), \text{ and } \dot{u} \in L^2\big([0,T];\mathbb{R}^N\big)\big\} \end{aligned}$$

with the inner product

$$(u,v)_W := \int_0^T \big[(u,v) + (\dot{u},\dot{v})\big]\,dt, \quad \forall u,v \in W,$$

and the corresponding norm is defined by $\|u\|_W = (u,u)_W^{1/2}$. Obviously, W is a Hilbert space. By the Sobolev embedding theorem, we see that the following embedding is compact:

$$W \hookrightarrow L^q\big([0,T];\mathbb{R}^N\big), \quad \forall q \in [1,+\infty], \tag{2.1}$$

and there exists a $\gamma_q > 0$ such that

$$\|u\|_q \leq \gamma_q \|u\|, \quad \forall u \in W. \tag{2.2}$$

The corresponding functional of (1.1) is defined as follows:

$$\Phi(u) := \frac{1}{2}\int_0^T e^{Q(t)}\bigl[|\dot{u}|^2 + (Bu,\dot{u}) + \bigl(A(t)u,u\bigr)\bigr]\,dt - \int_0^T e^{Q(t)}H(t,u)\,dt, \quad u \in W,$$

where $Q(t) := \int_0^t q(s)\,ds$. By ($D_0$), we can define an equivalent inner product $\langle\cdot,\cdot\rangle$ on W with corresponding norm $\|\cdot\|$ such that

$$\|u\| := \left[\int_0^T e^{Q(t)}\bigl[|\dot{u}|^2 + (Bu,\dot{u}) + \bigl(A(t)u,u\bigr)\bigr]\,dt\right]^{1/2}, \quad \forall u \in W.$$

Then Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_0^T e^{Q(t)}H(t,u)\,dt, \quad u \in W. \tag{2.3}$$

Then by the assumptions of H, we know Φ is continuously differentiable and

$$\Phi'(u)v = \langle u,v\rangle - \int_0^T e^{Q(t)}\bigl(H_u(t,u),v\bigr)\,dt, \tag{2.4}$$

besides, the T-periodic solutions of (1.1) are the critical points of the C^1 functional $\Phi : W \to \mathbb{R}$ ([4]).

We shall use the following theorem to prove our main results.

Lemma 2.1 ([11, 12]) *Let X be an infinite dimensional Banach space,*

$$X = Y \oplus Z, \tag{2.5}$$

where Y is finite dimensional. If $\Phi \in C^1(X,\mathrm{R})$ satisfies $(C)_c$-condition for all $c > 0$ (we say that Φ satisfies $(C)_c$-condition if any sequence $\{u_k\}$ such that

$$\Phi(u_k) \to c, \quad \bigl\|\Phi'(u_k)\bigr\|\bigl(1 + \|u_k\|\bigr) \to 0, \tag{2.6}$$

has a convergent subsequence). Beside,

(1) $\Phi(0) = 0$, $\Phi(-u) = \Phi(u)$, $\forall u \in X$;
(2) $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$ *for some* $\rho, \alpha > 0$;
(3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{X}\backslash B_R$.*

Then we have an unbounded sequence of critical values.

Proofs of Theorems 1.1 *and* 1.2 To apply Lemma 2.1, we set $X = W$, $Y = Y_k$ and $Z = Z_k$, where

$$Y_k := \operatorname{span}\{e_1,\ldots,e_k\}, \qquad Z_k := \overline{\operatorname{span}\{e_{k+1},\ldots\}}, \quad \forall k \in \mathrm{N},$$

and $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of W.

Clearly, the condition (1) of Lemma 2.1 holds. Therefore, if Φ satisfies the $(C)_c$-condition, and conditions (2) and (3) of Lemma 2.1 hold, then we can prove that the problem (1.1) possesses infinitely many nontrivial solutions by Lemma 2.1, *i.e.*, Theorems 1.1 and 1.2 are true. □

Next, we will prove Φ satisfies the $(C)_c$-condition, and conditions (2) and (3) of Lemma 2.1 hold, *i.e.*, the following lemmas. Clearly, the condition (AH_1) implies that

$$\left|H(t,u)\right| \le \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N. \tag{2.7}$$

Lemma 2.2 *If assumptions* (AH_1), (AH_2) *and* (AH_4) (*or* (AH_4')) *hold, then* Φ *satisfies the* $(C)_c$*-condition.*

Proof We assume that, for any sequence $\{u^k\} \subset W$, $\Phi(u^k) \to c$ and $\|\Phi'(u^k)\|(1 + \|u^k\|) \to 0$. Then $\Phi'(u^k) \to 0$, and

$$\left\langle \Phi'\left(u^k\right), u^k \right\rangle \to 0. \tag{2.8}$$

Next, we will divide our proof into two parts by (AH_4) and $(\mathrm{AH}_4'))$.

Part 1. Φ satisfies $(C)_c$-condition under assumptions (AH_1), (AH_2) and (AH_1).

(i) We prove the boundedness of $\{u^k\}$ by contradiction, if $\|u^k\| \to \infty$, we let $v^k = \frac{u^k}{\|u^k\|}$, then $\|v^k\| = 1$. By the definitions of $\Phi(u)$ and $\Phi'(u)$, for k large, we have

$$\int_0^T e^{Q(t)} \left[\frac{1}{2}\left(H_u\left(t,u^k\right), u^k\right) - H\left(t,u^k\right) \right] dt = \Phi\left(u^k\right) - \frac{1}{2}\left\langle \Phi'\left(u^k\right), u^k \right\rangle \le c + 1. \tag{2.9}$$

By (2.3), $\Phi(u^k) \to c$ and $\|u^k\| \to \infty$, we have

$$\limsup_{k\to\infty} \int_0^T e^{Q(t)} \frac{|H(t,u^k)|}{\|u^k\|^2}\, dt \ge \frac{1}{2}. \tag{2.10}$$

Let

$$\Omega_k(a,b) = \left\{ t \in [0,T] : a \le \left|u^k(t)\right| < b \right\}, \quad 0 \le a < b. \tag{2.11}$$

By $\|v^k\| = 1$, we could assume that $v^k \rightharpoonup v = \{v(t)\}_{t\in[0,T]}$ in W passing to a subsequence, which together with (2.1) implies $v^k \to v$ in L^q for $1 \le q < \infty$, and $v^k \to v$ on $[0,T]$.

If $v = 0$, then $v^k \to 0$ in L^q, $1 \le q < \infty$, and $v^k \to 0$ on $[0,T]$. It follows from (2.7) that

$$\begin{aligned} \int_{\Omega_k(0,r_0)} e^{Q(t)} \frac{|H(t,u^k)|}{|u^k|^2} \left|v^k\right|^2 dt &\le \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right) \int_{\Omega_k(0,r_0)} e^{Q(t)} \left|v^k\right|^2 dt \\ &\le \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right) \int_0^T e^{Q(t)} \left|v^k\right|^2 dt \to 0. \end{aligned} \tag{2.12}$$

Let $\varrho' = \varrho/(\varrho - 1)$. Due to $\varrho > 1$ (see (AH$_4$)), we have $2\varrho > 2$. So by (AH$_4$), (2.9), the Hölder inequality and $v^k \to 0$ in L^q for $1 \le q < \infty$, we have

$$\begin{aligned}
&\int_{\Omega_k(r_0,\infty)} e^{Q(t)} \frac{|H(t,u^k)|}{|u^k|^2} |v^k|^2 \,dt \\
&\le \left[\int_{\Omega_k(r_0,\infty)} e^{Q(t)} \left(\frac{|H(t,u^k)|}{|u^k|^2}\right)^{\varrho} dt\right]^{1/\varrho} \left[\int_{\Omega_k(r_0,\infty)} e^{Q(t)} |v^k|^{2\varrho'} \,dt\right]^{1/\varrho'} \\
&\le (2c_0)^{1/\varrho} \left[\int_{\Omega_k(r_0,\infty)} e^{Q(t)} \left(\frac{1}{2}\big(H_u(t,u^k),u^k\big) - H(t,u^k)\right) dt\right]^{1/\varrho} \\
&\quad \times \left[\int_{\Omega_k(r_0,\infty)} e^{Q(t)} |v^k|^{2\varrho'} \,dt\right]^{1/\varrho'} \\
&\le \big[2c_0(c+1)\big]^{1/\varrho} \int_{\Omega_k(r_0,\infty)} e^{Q(t)} \,dt \cdot \|v^k\|_{2\varrho'}^2 \to 0.
\end{aligned} \tag{2.13}$$

By (2.12) and (2.13), we have

$$\begin{aligned}
&\int_0^T e^{Q(t)} \frac{|H(t,u^k)|}{\|u^k\|^2} \,dt \\
&= \int_{\Omega_k(0,r_0)} e^{Q(t)} \frac{|H(t,u^k)|}{|u^k|^2} |v^k|^2 \,dt + \int_{\Omega_k(r_0,\infty)} e^{Q(t)} \frac{|H(t,u^k)|}{|u^k|^2} |v^k|^2 \,dt \to 0,
\end{aligned}$$

which contradicts (2.10).

If $v \neq 0$, we let $A := \{t \in [0,T] : v(t) \neq 0\}$. For all $t \in A$, by $v^k = \frac{u^k}{\|u^k\|}$ and $\|u^k\| \to \infty$, we have $\lim_{k\to\infty} |u^k| = \infty$. We define

$$\chi_{t,\Omega_k(r_0,\infty)} := \begin{cases} 1, & t \in \Omega_k(r_0,\infty), \\ 0, & t \notin \Omega_k(r_0,\infty), \end{cases} \qquad \forall k \in \mathrm{N}. \tag{2.14}$$

For large $k \in \mathrm{N}$, $A \subset \Omega_k(r_0,\infty)$ and $\lim_{k\to\infty} |u^k| = \infty$ for all $t \in A$, since the definition of $Q(t)$ implies that $e^{Q(t)} \ge M$ for some constant $M > 0$ ($\forall t \in [0,T]$), it follows from (2.3), (2.7), (AH$_2$), Fatou's lemma, $\|v^k\| = 1$, $\|u^k\| \to \infty$, $\Phi(u^k) \to c$ and $\|v^k\|_2 \le \gamma_2 \|v^k\|$ (see (2.2)) that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{c + o(1)}{\|u^k\|^2} \\
&= \lim_{k\to\infty} \frac{\Phi(u^k)}{\|u^k\|^2} \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \int_0^T e^{Q(t)} \frac{H(t,u^k)}{(u^k)^2} (v^k)^2 \,dt\right] \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \int_{\Omega_k(0,r_0)} e^{Q(t)} \frac{H(t,u^k)}{(u^k)^2} (v^k)^2 \,dt - \int_{\Omega_k(r_0,\infty)} e^{Q(t)} \frac{H(t,u^k)}{(u^k)^2} (v^k)^2 \,dt\right] \\
&\le \limsup_{k\to\infty} \left[\frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right) \int_0^T e^{Q(t)} |v_n^k|^2 \,dt - \int_{\Omega_k(r_0,\infty)} e^{Q(t)} \frac{H(t,u^k)}{(u^k)^2} (v^k)^2 \,dt\right]
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \liminf_{k\to\infty} \int_{\Omega_k(r_0,\infty)} e^{Q(t)} \frac{H(t,u^k)}{(u^k)^2} \left(v^k\right)^2 dt \\
&= \frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - \liminf_{k\to\infty} \int_0^T e^{Q(t)} \frac{H(t,u^k)}{(u^k)^2} [\chi_{t,\Omega_k(r_0,\infty)}] \left(v^k\right)^2 dt \\
&\le \frac{1}{2} + \left(\frac{c_1}{2} + \frac{c_2}{p} r_0^{p-2}\right)\gamma_2^2 - M \int_0^T \liminf_{k\to\infty} \frac{H(t,u^k)}{(u^k)^2} [\chi_{t,\Omega_k(r_0,\infty)}] \left(v^k\right)^2 dt \\
&= -\infty. \qquad (2.15)
\end{aligned}$$

It is a contradiction. So $\{u^k\}$ is bounded in W.

(ii) The boundedness of $\{u^k\}$ implies that $u^k \rightharpoonup u$ in W passing to a subsequence, where $u = \{u(t)\}_{t\in[0,T]}$. First, we prove

$$\int_0^T e^{Q(t)} \left[H_u\left(t,u^k\right)\left(u^k - u\right)\right] dt \to 0, \quad k \to \infty. \qquad (2.16)$$

Note that (2.1) implies that $u^k \to u$ in L^q for all $1 \le q < \infty$, so we have

$$\left\|u^k - u\right\|_2 \to 0, \qquad \left\|u^k - u\right\|_p \to 0. \qquad (2.17)$$

The boundedness of $\{u^k\}$ and (2.2) imply that $\|u^k\|_q < \infty$ for all $1 \le q < \infty$, since the definition of $Q(t)$ implies that $e^{Q(t)} \le c_1'$ for some constant $c_1' > 0$ ($\forall t \in [0,T]$), it follows from (AH_1), (2.17) and the Hölder inequality that

$$\begin{aligned}
&\left|\int_0^T e^{Q(t)} \left[H_u\left(t,u^k\right)\left(u^k - u\right)\right] dt\right| \\
&\quad \le \int_0^T e^{Q(t)} \left|H_u\left(t,u^k\right)\left(u^k - u\right)\right| dt \\
&\quad \le \int_0^T e^{Q(t)} \left[\left(c_1\left|u^k\right| + c_2\left|u^k\right|^{p-1}\right)\left|u^k - u\right|\right] dt \\
&\quad = c_1 \int_0^T e^{Q(t)} \left[\left|u^k\right|\left|u^k - u\right|\right] dt + c_2 \int_0^T e^{Q(t)} \left[\left(\left|u^k\right|^{p-1}\left|u^k - u\right|\right] dt \\
&\quad \le c_1 c_1' \left\|u^k\right\|_2 \left\|u^k - u\right\|_2 + c_2 c_1' \left\|u^k\right\|_p^{p-1} \left\|u^k - u\right\|_p \to 0. \qquad (2.18)
\end{aligned}$$

So (2.16) holds. Therefore, by (2.16), $\Phi'(u^k) \to 0$, $u^k \rightharpoonup u$ in W and the definition of Φ', we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \left\langle \Phi'\left(u^k\right), u^k - u \right\rangle \\
&= \lim_{k\to\infty} \left(u^k, u^k - u\right) - \lim_{k\to\infty} \int_0^T e^{Q(t)} \left(H_u\left(t,u^k\right)\left(u^k - u\right)\right) dt \\
&= \lim_{k\to\infty} \left\|u^k\right\|^2 - \|u\|^2 - 0. \qquad (2.19)
\end{aligned}$$

That is,

$$\lim_{k\to\infty} \left\|u^k\right\| = \|u\|. \qquad (2.20)$$

It follows from $u^k \rightharpoonup u$ in W that

$$\|u^k - u\|^2 = (u^k - u, u^k - u) \to 0,$$

that is, $\{u^k\}$ has a convergent subsequence in W. Thus Φ satisfies $(C)_c$-condition.

Part 2. Φ satisfies $(C)_c$-condition under assumptions (AH_1), (AH_2) and (AH_4').

Similar to the Part 1, we need prove that $\{u^k\}$ is bounded in W. We prove it by contradiction. If $\|u^k\| \to \infty$, we let $v^k = \frac{u^k}{\|u^k\|}$, then $\|v^k\| = 1$. By (2.8), (AH_4'), $\Phi(u^k) \to c$ and the definitions of Φ and Φ', for large $k \in \mathrm{N}$ we have

$$\begin{aligned} c + 1 &\geq \Phi(u^k) - \frac{1}{\mu}\langle \Phi'(u^k), u^k\rangle \\ &= \frac{\mu - 2}{2\mu}\|u^k\|^2 + \int_0^T e^{Q(t)}\left[\frac{1}{\mu}(H_u(t,u^k), u^k) - H(t,u^k)\right] dt \\ &\geq \frac{\mu - 2}{2\mu}\|u^k\|^2 - \frac{\kappa}{\mu}\int_0^T e^{Q(t)}\, dt \cdot \|u^k\|_2^2. \end{aligned} \tag{2.21}$$

It follows from $\|u^k\| \to \infty$ and $v^k = \frac{u^k}{\|u^k\|}$ that

$$\frac{2\kappa}{\mu - 2}\int_0^T e^{Q(t)}\, dt \limsup_{k\to\infty}\|v^k\|_2^2 \geq 1. \tag{2.22}$$

$\|v^k\| = 1$ implies that $v^k \rightharpoonup v$ in W passing to a subsequence, then it follows from (2.1) and (2.22) that $v \neq 0$. So similar to (2.15), we can conclude a contradiction. Therefore $\{u^k\}$ is bounded in W. The rest of the proof is the same as that in (ii) of Part 1. □

Lemma 2.3 *The condition* (2) *of Lemma* 2.1 *holds, i.e., there exist constants* $\rho, \alpha > 0$ *such that*

$$\Phi|_{\partial B_\rho \cap Z_k} \geq \alpha.$$

Proof Let

$$l_2(k) := \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_2}{\|u\|}, \qquad l_p(k) := \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_p}{\|u\|}. \tag{2.23}$$

It is clear that $0 < l_2(k+1) \leq l_2(k)$, so that $l_2(k) \to l \geq 0$ as $k \to \infty$. For every $k \geq 0$, there exists $u^k \in Z_k$ such that $\|u^k\| = 1$ and $\|u^k\|_2 > l_2(k)/2$. By the definition of Z_k, $u^k \rightharpoonup 0$ in W, then by (2.1), $u^k \to 0$ in L^2. Therefore, we have $l = 0$, that is, $l_2(k) \to 0$. Similarly, $l_p(k) \to 0$.

Note that $e^{Q(t)} \leq c_1'$ ($\forall t \in [0,T]$) for some constant $c_1' > 0$, we can choose a large integer $k > 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2c_1c_1'}\|u\|^2, \qquad \|u\|_p^p \leq \frac{p}{4c_2c_1'}\|u\|^p, \quad \forall u \in Z_k. \tag{2.24}$$

Then by (2.3), (2.7) and (2.24), we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_0^T e^{Q(t)}H(t,u)\,dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1c_1'}{2}\|u\|_2^2 - \frac{c_2c_1'}{p}\|u\|_p^p \\ &\geq \frac{1}{4}\big(\|u\|^2 - \|u\|^p\big) \\ &= \frac{2^{p-2}-1}{2^{p+2}} := \alpha, \quad \forall u \in Z_m, \|u\| = \frac{1}{2} := \rho.\end{aligned}$$

Thus, this lemma is proved. □

Lemma 2.4 *The condition* (3) *of Lemma* 2.1 *holds, i.e., for any finite dimensional subspace* $\tilde{W} \subset W$, *there is* $R = R(\tilde{W}) > 0$ *such that*

$$\Phi(u) \leq 0, \qquad \forall u \in \tilde{W}, \quad \|u\| \geq R. \tag{2.25}$$

Proof In order to prove our conclusion, we only need to prove

$$\Phi(u) \to -\infty, \qquad \|u\| \to \infty, \quad \forall u \in \tilde{W}.$$

By contradiction, if there exists a sequence $\{u^k\} \subset \tilde{W}$ with $\|u^k\| \to \infty$ such that $\Phi(u^k) \geq -M$ for some $M > 0$, $\forall k \in \mathrm{N}$. Let $v^k = \frac{u^k}{\|u^k\|}$, then $\|v^k\| = 1$. Passing to a subsequence, we can assume that $v^k \rightharpoonup v$ in W. Since $\tilde{W}$ is finite dimensional, $v^k \to v$ in W, thus $\|v\| = 1$. Similar to (2.15) we can conclude we have a contradiction. Thus (2.25) holds. Therefore, the proof is finished. □

3 Conclusion

We obtain infinitely many periodic solutions for a class of superlinear damped vibration problems with primitive functions of nonlinearities being allowed to be sign-changing. By using some weaker conditions, our results extend and improve some existing results in the literature.

Acknowledgements
We would like to thank the referees and editors for their valuable comments and suggestions, which have led to an improvement of the presentation of this paper.

Funding
Research supported by National Natural Science Foundation of China (No. 11401011).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan Province 455000, P.R. China. [2]School of Mathematical Sciences, University of Jinan, Jinan, Shandong Province 250022, P.R. China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 May 2017 Accepted: 12 September 2017 Published online: 26 September 2017

References

1. Chen, G: Infinitely many nontrivial periodic solutions for damped vibration problems with asymptotically linear terms. Appl. Math. Comput. **245**, 438-446 (2014)
2. Chen, G: Periodic solutions of superquadratic damped vibration problems. Appl. Math. Comput. **270**, 794-801 (2015)
3. Chen, G: Damped vibration problems with nonlinearities being sublinear at both zero and infinity. Math. Methods Appl. Sci. **39**, 1505-1512 (2015)
4. Li, X, Wu, X, Wu, K: On a class of damped vibration problems with super-quadratic potentials. Nonlinear Anal. **72**, 135-142 (2010)
5. Wu, X, Chen, S, Teng, K: On variational methods for a class of damped vibration problems. Nonlinear Anal. **68**, 1432-1441 (2008)
6. Zhang, XY: Infinitely many solutions for a class of second-order damped vibration systems. Electron. J. Qual. Theory Differ. Equ. **2013**, 15 (2013)
7. Chen, P, Tang, XH: Fast homoclinic solutions for a class of damped vibration problems with subquadratic potentials. Math. Nachr. **286**(1), 4-16 (2013)
8. Chen, P, Tang, XH, Agarwal, RP: Fast homoclinic solutions for a class of damped vibration problems. Appl. Math. Comput. **219**(11), 6053-6065 (2013)
9. Li, J, Li, S, Zhang, Z: Periodic solutions for a singular damped differential equation. Bound. Value Probl. **2015**, 5 (2015). doi:10.1186/s13661-014-0269-1
10. Wu, X, Zhang, W: Existence and multiplicity of homoclinic solutions for a class of damped vibration problems. Nonlinear Anal. **74**, 4392-4398 (2011)
11. Bartolo, P, Benci, V, Fortunato, D: Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity. Nonlinear Anal. **7**, 981-1012 (1983)
12. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Ser. Math., vol. 65. Am. Math. Soc. Providence (1986). 100 pp.