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Positive solutions of Schrödinger-Kirchhoff-Poisson system without compact condition

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Abstract

Purpose: The existence of positive solutions for a class of nonlinear Schrödinger-Kirchhoff-Poisson systems.

Methods: Variational method.

Results: Some results on the existence of positive solutions.

MSC: Primary 35A15; 35B38; secondary 35J25

Keywords: Schrödinger-Kirchhoff-Poisson system; positive solution; ground state solution

1 Introduction

In this paper, we are concerned with the existence of positive solutions for the following nonlinear Schrödinger-Kirchhoff-Poisson system:

$$(\mathcal{SK})\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u+u-qK(x)\phi u=f(x)|u|^{p-1}u, & x\in\mathbb{R}^3,\\ -\triangle\phi=qK(x)u^2, & x\in\mathbb{R}^3,\end{cases}$$

by using the variational methods, where $a>0$ and $b\geq 0$ are constants, $1<p<5$, $q>0$, and $K,f:\mathbb{R}^3\to\mathbb{R}$ are two nonnegative functions.

The Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\triangle u+u=f(x,u),\quad x\in\mathbb{R}^3, \tag{1.1}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

arises in many mathematical physics contexts. Equation (1.2) was proposed by Kirchhoff [1] in 1883 as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model considers the changes in the length of the string produced by transverse vibrations.

The Schrödinger-Poisson system

$$\begin{cases} -\triangle u + u + qK(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\triangle \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

describes a charged wave interacting with its own electrostatic field [2]. We also refer the readers to [3, 4] and the references therein for more mathematical and physical background of (1.3). The Schödinger-Kirchhoff-Poisson system $(\mathcal{SK})$ is a more generalized Kirchhoff-type system of (1.3).

In recent years, there have been enormous results on the existence and multiplicity of solutions of problem (1.3) for $q > 0$ (see e.g. [5–8]). To the best of our knowledge, there are a few articles on the existence of solutions to problem (1.3) for $q < 0$. Recently, in [9], the author proved that problem (1.3) has a positive ground state solution for $q < 0$ and $f(x,u) = a(x)|u|^{p-1}u$, $3 < p < 5$. In [10], the author proved that problem (1.3) has a positive solution for $q < 0$. When $q \equiv 1$, the problem $(\mathcal{SK})$ can be reduced to

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u + u - K(x)\phi u = f(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\triangle \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

Li and Ye [11] proved that problem (1.4) has a a positive ground state solution for $K(x) \equiv 0$, $f(x) \equiv 1$, $2 < p < 5$.

Motivated by the works mentioned, we consider system $(\mathcal{SK})$ with the Kirchhoff term $(\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u$, which implies that the equation in $(\mathcal{SK})$ is no longer a pointwise identity and is different from the classical Schrödinger-Poisson system. We must emphasize that the combined effects of the nonlocal term it contains and the negative coefficient at ϕu make problem (1.4) an interesting problem.

To obtain our main results, we first require some conditions on f and K:

(f_1) $f(x) \geq f_\infty = \lim_{|x|\to+\infty} f(x) > 0$, and $\alpha(x) = f(x) - f_\infty \in L^{\frac{6}{5-p}}(\mathbb{R}^3)$;
(f_2) $(\nabla f, x) \in L^{\frac{6}{5-p}}(\mathbb{R}^3) \setminus \{0\}$, and $(\nabla f, x) \geq 0$;
(K_1) $K \in L^2(\mathbb{R}^3)$;
(K_2) $(\nabla K, x) \in L^2(\mathbb{R}^3) \setminus \{0\}$;
(K_3) $K(x) \geq K_\infty = \lim_{|x|\to+\infty} K(x) \geq 0$, and $K - K_\infty \in L^2(\mathbb{R}^3)$;
(K_4) $K(x) + (\nabla K, x) \geq 0$, $x \in \mathbb{R}^3$.

2 Conclusion

In this paper, we get the existence of a positive solution for all $1 < p < 5$. Our main results are as follows.

Theorem 2.1 *Let* (f_1), (f_2), (K_1), *and* (K_2) *hold. Then there exists* $q_0 > 0$ *such that, for any* $0 < q < q_0$, *problem* $(\mathcal{SK})$ *admits a positive solution* $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *for all* $1 < p < 5$.

Theorem 2.2 *Let* (f_1), (f_2), *and* (K_2)-(K_4) *hold. Then problem* (1.4) *admits a positive ground state solution* $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *for all* $1 < p < 5$.

As a consequence of Theorem 2.2, we have that the problem

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\triangle u+u=|u|^{p-1}u, & x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), & x\in\mathbb{R}^3, \end{cases}$$

has a positive ground state solution for all $1<p<5$. This can be viewed as an extension of a recent result of Li and Ye [11] concerning the existence of positive ground state solutions.

The paper is organized as follows. In Section 3, we describe the notation and preliminaries. In Section 4, we give a proof of Theorem 2.1. In Section 5, we give a proof of Theorem 2.2.

3 Notation and preliminaries

- Let $a>0$ is fixed, and let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the scalar product and norm

$$(u,v)=\int_{\mathbb{R}^3}(a\nabla u\nabla v+uv)\qquad\text{and}\qquad\|u\|=\left(\int_{\mathbb{R}^3}\left(a|\nabla u|^2+u^2\right)\right)^{\frac{1}{2}}.$$

- $H_r^1(\mathbb{R}^3):=\{u:u\in H^1(\mathbb{R}^3),u(x)=u(|x|)\}$.
- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}}=\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^{\frac{1}{2}}.$$

- For any $u\in H^1(\mathbb{R}^3)$ and $t>0$, we define $u_t(x)=t^{\frac{1}{2}}u(\frac{x}{t})$.
- For any $z\in\mathbb{R}^3$ and $\rho>0$, $B_\rho(z)$ denotes the ball of radius ρ centered at z.
- For any $s\in[1,+\infty)$, $|\cdot|_s$ denotes the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$.
- S_1 is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is,

$$S_1=\inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\frac{\|u\|}{|u|_6}.$$

- S_2 is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is,

$$S_2=\inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

- $H^{-1}(\mathbb{R}^3)$ denotes the conjugate Sobolev space of $H^1(\mathbb{R}^3)$.
- C and C_i, $i\in\mathbb{N}$, are various positive constants.

It is well known that problem $(\mathcal{SK})$ can be reduced to a nonlinear Schrödinger-Kirchhoff equation with a nonlocal term. For any $u\in H^1(\mathbb{R}^3)$, define the linear functional L_u in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v)=\int_{\mathbb{R}^3}K(x)u^2v.$$

Then (K_1) or (K_3) and the Hölder and Sobolev inequalities imply

$$\left|L_u(v)\right| \le C\|u\|^2 \cdot \|v\|_{D^{1,2}}. \tag{3.1}$$

Hence, by the Lax-Milgram theorem there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v = q\int_{\mathbb{R}^3} K(x)u^2 v, \quad \forall v \in D^{1,2}\left(\mathbb{R}^3\right), \tag{3.2}$$

that is, ϕ_u is a weak solution of $-\triangle\phi = qK(x)u^2$, and

$$\phi_u(x) = \frac{q}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}\,dy. \tag{3.3}$$

Moreover, $\|\phi_u\|^2_{D^{1,2}(\mathbb{R}^3)} = q\int_{\mathbb{R}^3} K(x)\phi_u u^2$ and $\phi_u > 0$ when $u \neq 0$. Then (3.3) inserted into the first equation of $(\mathcal{SK})$ gives

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\triangle u + u - qK(x)\phi_u u = f(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3. \tag{3.4}$$

Problem (3.4) is variational, and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I_q(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{q}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} f(x)|u|^{p+1}. \tag{3.5}$$

It is clear to see that I_q is well defined on $H^1(\mathbb{R}^3)$ and is of class C^1, and for any $u, v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}\left\langle I_q'(u), v\right\rangle = {}& \left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} uv \\ & - q\int_{\mathbb{R}^3} K(x)\phi_u uv - \int_{\mathbb{R}^3} f(x)|u|^{p-1}uv.\end{aligned} \tag{3.6}$$

Thus, if $u \in H^1(\mathbb{R}^3)$ is a critical point of I_q, then the pair (u, ϕ_u) with ϕ_u as in (3.3) is a solution of the problem $(\mathcal{SK})$.

Define the operator $\Phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ as $\Phi[u] = \phi_u$. The following lemma shows that the operator Φ possesses the property.

Lemma 3.1 (See [7]) *If K satisfies (K_1) or (K_3), then*

(i) Φ *is continuous*;

(ii) Φ *maps bounded sets into bounded sets*;

(iii) *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\Phi[u_n] \rightharpoonup \Phi[u]$ in $D^{1,2}(\mathbb{R}^3)$.*

To obtain the boundedness of (PS) sequences, we recall the indirect approach developed by Jeanjean [12].

Proposition 3.1 *Let $(X, \|\cdot\|)$ be a real Banach space, and let $J \subset \mathbb{R}^+$ be an interval. Consider the following family of C^1 functionals on X:*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall\lambda \in J,$$

with $B(u) \geq 0$ *and either* $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to +\infty$ *and such that* $I_\lambda(0) = 0$. *For every* $\lambda \in J$, *set*

$$\Gamma_\lambda := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\} \neq \emptyset$$

and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0. \tag{3.7}$$

Then, for almost every $\lambda \in J$, *there is a bounded* $(\mathrm{PS})_{c_\lambda}$ *sequence* $\{u_n\} \subset X$.

Lemma 3.2 (See [13], Lemma 2.3) *Under the assumptions of Proposition* 3.1, *the map* $\lambda \to c_\lambda$ *is nonincreasing and left-continuous.*

4 A mountain pass solution

In this section, let (f_1), (f_2), (K_1), and (K_2) hold, and let $1 < p < 5$. Using (K_1), (3.2), and the Sobolev inequality, we have

$$\|\phi_u\|_{D^{1,2}} \leq q|K|_2 S_2^{-1} \cdot |u|_6^2, \qquad \|\phi_u\|_{D^{1,2}} \leq q|K|_2 S_2^{-1} S_1^{-2} \cdot \|u\|^2, \tag{4.1}$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_u u^2 \leq q|K|_2^2 S_2^{-2} \cdot |u|_6^4, \qquad \int_{\mathbb{R}^3} K(x)\phi_u u^2 \leq q|K|_2^2 S_2^{-2} S_1^{-4} \cdot \|u\|^4. \tag{4.2}$$

To overcome the difficulty of finding bounded (PS) sequences for the associated function I_q, we use the cut-off function $\chi \in C_0^\infty(\mathbb{R}^+, \mathbb{R})$ satisfying

$$\begin{cases} \chi(s) = 1, & s \in [0,1], \\ 0 \leq \chi(s) \leq 1, & s \in (1,2), \\ \chi(s) = 0, & s \in [2, +\infty), \\ |\chi'|_\infty \leq 2, \end{cases} \tag{4.3}$$

and study the following modified functional $I_q^T : H^1(\mathbb{R}^3) \to \mathbb{R}$:

$$I_q^T(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{q}{4}\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3} K(x)\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} f(x)|u|^{p+1}.$$

Letting $J = [\frac{1}{2}, 1]$, we consider the following family of functionals on $X = H^1(\mathbb{R}^3)$:

$$\begin{aligned} I_{q,\lambda}^T(u) = {} & \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ & - \frac{q\lambda}{4}\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3} K(x)\phi_u u^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} f(x)|u|^{p+1}. \end{aligned} \tag{4.4}$$

Then $I_{q,\lambda}^T(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \to +\infty \quad \text{as } \|u\| \to +\infty$$

and

$$B(u)=\frac{q}{4}\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u u^2+\frac{1}{p+1}\int_{\mathbb{R}^3}f(x)|u|^{p+1}\geq 0.$$

Moreover,

$$\begin{aligned}\left\langle\left(I_{q,\lambda}^T\right)'(u),v\right\rangle &=\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\int_{\mathbb{R}^3}\nabla u\nabla v+\int_{\mathbb{R}^3}uv\\&\quad-\frac{3q\lambda}{2T^6}\chi'\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u u^2\int_{\mathbb{R}^3}|u|^4uv\\&\quad-\lambda q\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u uv-\lambda\int_{\mathbb{R}^3}f(x)|u|^{p-1}uv. \end{aligned}\tag{4.5}$$

The following lemma implies that $I_{q,\lambda}^T$ satisfies the conditions of Proposition 3.1.

Lemma 4.1

(i) *For any* $q_0>0$, *there exists a constant* $\eta>0$ *such that* $c_\lambda\geq\eta>0$ *for all* $\lambda\in J$ *and* $q\in(0,q_0]$;

(ii) $\Gamma_\lambda\neq\emptyset$ *for all* $\lambda\in J$.

Proof (i) For any $\lambda\in J$ and $q_0>0$, from (f_1), (K_1), (4.2)-(4.3), and the Hölder and Sobolev inequalities it follows that

$$\begin{aligned}I_{q,\lambda}^T(u)&=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\\&\quad-\frac{q\lambda}{4}\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u u^2-\frac{\lambda}{p+1}\int_{\mathbb{R}^3}f(x)|u|^{p+1}\\&\geq\frac{1}{2}\|u\|^2-\frac{q}{4}\int_{\mathbb{R}^3}K(x)\phi_u u^2-\frac{1}{p+1}\int_{\mathbb{R}^3}f(x)|u|^{p+1}\\&\geq\frac{1}{2}\|u\|^2-C\left(\|u\|^4+\|u\|^{p+1}\right),\quad\forall q\in(0,q_0].\end{aligned}$$

Since $1<p<5$, there exists $\rho>0$ such that $I_{q,\lambda}^T(u)>0$ for $0<\|u\|\leq\rho$. In particular, there exists $\eta>0$ such that $I_{q,\lambda}^T(u)\geq\eta$ for $\|u\|=\rho$. Now fix $\lambda\in J$ and $\gamma\in\Gamma_\lambda$. Since $I_{q,\lambda}^T(\gamma(1))<0$, $\|\gamma(1)\|>\rho$. By $\gamma(0)=0$ and the continuity of γ we can deduce that there exists $t_\gamma\in(0,1)$ such that $\|\gamma(t_\gamma)\|=\rho$. Hence, for any $\lambda\in J$,

$$c_\lambda\geq\inf_{\gamma\in\Gamma_\lambda}I_{q,\lambda}^T\left(\gamma(t_\gamma)\right)\geq\eta>0.$$

(ii) Set $w\in H^1(\mathbb{R}^3)\backslash\{0\}$ with $|w|_6=1$. Define $\gamma:[0,1]\to H^1(\mathbb{R}^3)$ as

$$\gamma(t)=\begin{cases}0, & t=0,\\ \overline{w}_t, & t\in(0,1],\end{cases}$$

where $\overline{w} = w_\theta$ and $\theta > 2T$. It is clear to see that γ is a continuous path from 0 to $\overline{w}$. Moreover, for all $\lambda \in J$, it follows from (4.3) that

$$\begin{aligned} I_{q,\lambda}^T\big(\gamma(1)\big) \le{}& \frac{a}{2}\theta^2 \int_{\mathbb{R}^3} |\nabla w|^2 + \frac{1}{2}\theta^4 \int_{\mathbb{R}^3} w^2 \\ &+ \frac{b}{4}\theta^4 \left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^2 - \frac{1}{2(p+1)}\theta^{\frac{p+7}{2}} \int_{\mathbb{R}^3} f_\infty |w|^{p+1}. \end{aligned} \tag{4.6}$$

As $1 < p < 5$, we get $I_{q,\lambda}^T(\gamma(1)) < 0$ for θ large enough. □

Lemma 4.2 *For any $\lambda \in J$, each bounded (PS) sequence of the functional $I_{q,\lambda}^T$ admits a convergent subsequence.*

Proof Let $\lambda \in J$, and let $\{u_n\}$ be a bounded (PS) sequence for $I_{q,\lambda}^T$, that is,

$$\sup_{n\in\mathbb{N}} \big|I_{q,\lambda}^T(u_n)\big| < \infty, \qquad \big(I_{q,\lambda}^T\big)'(u_n) \to 0 \quad \text{in } H^{-1}\big(\mathbb{R}^3\big). \tag{4.7}$$

Up to a subsequence, we may suppose that there exists $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} &\text{(a) } u_n \rightharpoonup u \quad \text{in } H^1\big(\mathbb{R}^3\big), \\ &\text{(b) } u_n \to u \quad \text{in } L^s_{\text{loc}}\big(\mathbb{R}^3\big) \text{ for } s \in [2,6), \text{and} \\ &\text{(c) } u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{4.8}$$

Now we prove that

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} K(x)\phi_u u^2 \tag{4.9}$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n w \to \int_{\mathbb{R}^3} K(x)\phi_u u w, \quad \forall w \in H^1\big(\mathbb{R}^3\big). \tag{4.10}$$

First, by $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and Lemma 3.1(iii) we have that

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } D^{1,2}\big(\mathbb{R}^3\big). \tag{4.11}$$

Moreover, in view of the Sobolev embedding theorem and Lemma 3.1(iii), we deduce that

$$\begin{aligned} &\text{(a) } u_n \rightharpoonup u \quad \text{in } L^6\big(\mathbb{R}^3\big), \\ &\text{(b) } u_n^2 \rightharpoonup u^2 \quad \text{in } L^3_{\text{loc}}\big(\mathbb{R}^3\big), \quad \text{and} \\ &\text{(c) } \phi_{u_n} \rightharpoonup \phi_u \quad \text{in } L^6_{\text{loc}}\big(\mathbb{R}^3\big). \end{aligned} \tag{4.12}$$

Thus, given $\varepsilon > 0$, it follows from (4.11) that

$$\left|\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u^2\right| < \varepsilon, \tag{4.13}$$

and for any fixed $w \in H^1(\mathbb{R}^3)$, we have

$$\left|\int_{\mathbb{R}^3} K(x)\phi_u(u_n - u)w\right| < \varepsilon. \tag{4.14}$$

Since $K \in L^2(\mathbb{R}^3)$, there exists $\rho = \rho(\varepsilon) > 0$ such that

$$|K|_{2,\mathbb{R}^3\setminus B_\rho(0)} < \varepsilon, \quad \forall \rho \geq \overline{\rho}, \tag{4.15}$$

and

$$\left|u_n^2 - u^2\right|_{3,B_\rho(0)} < \varepsilon, \tag{4.16}$$

$$|\phi_{u_n} - \phi_u|_{6,B_\rho(0)} < \varepsilon \tag{4.17}$$

for large n. Hence, by (4.12), (4.13), (4.15), and (4.16) we deduce

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 - \int_{\mathbb{R}^3} K(x)\phi_u u^2\right| \\
&\quad \leq \left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}\left(u_n^2 - u^2\right)\right| + \left|\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u^2\right| \\
&\quad \leq |\phi_{u_n}|_6 \cdot |K|_{2,\mathbb{R}^3\setminus B_\rho(0)} \cdot \left|u_n^2 - u^2\right|_3 + |\phi_{u_n}|_6 \cdot |K|_2 \cdot \left|u_n^2 - u^2\right|_{3,B_\rho(0)} + \varepsilon \\
&\quad \leq C\varepsilon
\end{aligned}$$

for large n. Analogously, by (4.12), (4.14), (4.15), and (4.17) we infer

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n w - \int_{\mathbb{R}^3} K(x)\phi_u uw\right| \\
&\quad \leq \left|\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u_n w\right| + \left|\int_{\mathbb{R}^3} K(x)\phi_u(u_n - u)w\right| \\
&\quad \leq |K|_{2,\mathbb{R}^3\setminus B_\rho(0)} \cdot |\phi_{u_n} - \phi_u|_6 \cdot |u_n|_6 \cdot |w|_6 \\
&\qquad + |K|_2| \cdot |\phi_{u_n} - \phi_u|_{6,B_\rho(0)} \cdot |u_n|_6 \cdot |w|_6 + \varepsilon \\
&\quad \leq C\varepsilon
\end{aligned}$$

for large n. This completes the proof of (4.9) and (4.10). Similarly, we can get

$$\int_{\mathbb{R}^3} \alpha(x)|u_n|^{p-1}u_n^2 = \int_{\mathbb{R}^3} \alpha(x)|u|^{p-1}u^2 + o(1), \tag{4.18}$$

$$\int_{\mathbb{R}^3} \alpha(x)|u_n|^{p-1}u_n w = \int_{\mathbb{R}^3} \alpha(x)|u|^{p-1}uw + o(1), \quad \forall w \in H^1\left(\mathbb{R}^3\right), \tag{4.19}$$

and

$$\int_{\mathbb{R}^3} |u_n|^{p+1} = \int_{\mathbb{R}^3} |u_n|^{p-1}u_n u + o(1). \tag{4.20}$$

Thus, it follows from (4.18)-(4.20) that

$$\int_{\mathbb{R}^3} f(x)|u_n|^{p-1}u_n(u_n-u)=o(1). \tag{4.21}$$

Moreover, by (4.7)-(4.10) and (4.3) we have

$$\begin{aligned}
o(1) &= \langle (I^T_{q,\lambda})'(u_n)-(I^T_{q,\lambda})'(u),u_n-u\rangle \\
&= \|u_n-u\|^2+b\int_{\mathbb{R}^3}|\nabla u_n|^2\int_{\mathbb{R}^3}\nabla u_n\nabla(u_n-u) \\
&\quad -b\int_{\mathbb{R}^3}|\nabla u|^2\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u) \\
&\quad -\frac{3q\lambda}{2T^6}\chi'\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2\int_{\mathbb{R}^3}|u_n|^4u_n(u_n-u) \\
&\quad +\frac{3q\lambda}{2T^6}\chi'\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u u^2\int_{\mathbb{R}^3}|u|^4u(u_n-u) \\
&\quad -q\lambda\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n(u_n-u) \\
&\quad +q\lambda\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u u(u_n-u) \\
&\quad -q\lambda\int_{\mathbb{R}^3}f(x)|u_n|^{p-1}u_n(u_n-u) \\
&\quad +q\lambda\int_{\mathbb{R}^3}f(x)|u|^{p-1}u(u_n-u).
\end{aligned} \tag{4.22}$$

By (4.8)(a) and (4.3) we deduce

$$\int_{\mathbb{R}^3}|\nabla u|^2\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)=o(1),\qquad \int_{\mathbb{R}^3}f(x)|u|^{p-1}u(u_n-u)=o(1), \tag{4.23}$$

$$\chi'\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_u u^2\int_{\mathbb{R}^3}|u|^4u(u_n-u)=o(1), \tag{4.24}$$

and

$$\int_{\mathbb{R}^3}|\nabla u_n|^2\int_{\mathbb{R}^3}\nabla u_n\nabla(u_n-u)=\int_{\mathbb{R}^3}|\nabla u_n|^2\int_{\mathbb{R}^3}\left|\nabla(u_n-u)\right|^2+o(1). \tag{4.25}$$

It follows (4.3), (4.9), and (4.10) that

$$\chi'\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2\int_{\mathbb{R}^3}|u_n|^4u_n(u_n-u)=o(1). \tag{4.26}$$

Therefore, it follows from (4.21)-(4.26) that

$$\|u_n-u\|^2+b\int_{\mathbb{R}^3}|\nabla u_n|^2\int_{\mathbb{R}^3}\left|\nabla(u_n-u)\right|^2=o(1).$$

This implies $\|u_n-u\|^2=o(1)$. Thus, $u_n\to u$ in $H^1(\mathbb{R}^3)$. □

Lemma 4.3 *For almost every $\lambda \in J$, there exists $u^\lambda \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that*

$$I^T_{q,\lambda}(u^\lambda) = c_\lambda, \qquad (I^T_{q,\lambda})'(u^\lambda) = 0.$$

Proof By Lemma 3.1, Lemma 3.2, and Proposition 3.1, for almost every $\lambda \in J$, there exists a bounded $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n^\lambda\} \subset H^1(\mathbb{R}^3)$. Up to a subsequence, by Lemma 4.2 we can suppose that there exists $u^\lambda \in H^1(\mathbb{R}^3)$ such that $u_n^\lambda \to u^\lambda$ in $H^1(\mathbb{R}^3)$. Then we have $I^T_{q,\lambda}(u^\lambda) = c_\lambda$ and $(I^T_{q,\lambda})'(u^\lambda) = 0$. □

Lemma 4.4 (Pohožaev identity) *If $u \in H^1(\mathbb{R}^3)$ is a weak solution of*

$$\begin{aligned} &-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\triangle u + u - q\chi\left(\frac{|u|_6^6}{T^6}\right)K(x)\phi_u u \\ &\quad - \frac{3q\lambda}{2T^6}\chi'\left(\frac{|u|_6^6}{T^6}\right)|u|^4 u\int_{\mathbb{R}^3} K(x)\phi_u u^2 \\ &\quad = \lambda f(x)|u|^{p-1}u, \end{aligned}$$

then we have

$$\begin{aligned} 0 &= P(u) \\ &:= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \frac{5q\lambda}{4}\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3} K(x)\phi_u u^2 \\ &\quad - \frac{3q\lambda}{4}\frac{|u|_6^6}{T^6}\chi'\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3} K(x)\phi_u u^2 \\ &\quad - \frac{q\lambda}{4}\chi\left(\frac{|u|_6^6}{T^6}\right)\int_{\mathbb{R}^3} (\nabla K, x)\phi_u u^2 \\ &\quad - \frac{3\lambda}{p+1}\int_{\mathbb{R}^3} f(x)|u|^{p+1} - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} (\nabla f, x)|u|^{p+1}, \end{aligned} \tag{4.27}$$

where ϕ_u is defined by (3.3).

Proof The proof is standard, and we omit it (see [14]). □

Lemma 4.5 *Let m_i $(i = 1,2,3)$ be positive constants, $p > 1$, and $g(t) = m_1 t^2 + m_2 t^4 - m_3 t^{3+\frac{p+1}{2}}$, $t \geq 0$. Then g has a unique positive critical point, which corresponds to its maximum.*

Proof The proof is similar to that of Lemma 3.3 in [6] and is elementary. We omit the proof. □

Lemma 4.6 *Let $\lambda_n \in J$, and let u_n be a critical point of I^T_{q,λ_n} at level c_{λ_n} for every $n \in \mathbb{N}$. Then, for $T > 0$ sufficiently large, there exists $q_0 = q_0(T)$ such that, for any $0 < q < q_0$, up to a subsequence, $|u_n|_6 \leq T$ for all $n \in \mathbb{N}$.*

Proof We argue by contradiction. First of all, it follows from $(I^T_{q,\lambda_n})'(u_n) = 0$ and Lemma 4.4 that u_n satisfies the following Pohožaev identity:

$$\begin{aligned}
&\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{3}{2}\int_{\mathbb{R}^3}u_n^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 - \frac{5q\lambda_n}{4}\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 \\
&\quad - \frac{3q\lambda_n}{4}\frac{|u_n|_6^6}{T^6}\chi'\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 - \frac{q\lambda_n}{4}\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}(\nabla K, x)\phi_{u_n}u_n^2 \\
&\quad - \frac{3\lambda_n}{p+1}\int_{\mathbb{R}^3}f(x)|u_n|^{p+1} - \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3}(\nabla f, x)|u_n|^{p+1} = 0.
\end{aligned} \tag{4.28}$$

Since $I^T_{q,\lambda_n}(u_n) = c_{\lambda_n}$, we have

$$\begin{aligned}
c_{\lambda_n} &= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 \\
&\quad - \frac{q\lambda_n}{4}\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 - \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3}f(x)|u_n|^{p+1}.
\end{aligned} \tag{4.29}$$

Hence, by (4.28) and (4.29) we obtain

$$\begin{aligned}
&a\int_{\mathbb{R}^3}|\nabla u_n|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 + \frac{q\lambda_n}{2}\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 \\
&\quad + \frac{3q\lambda_n}{4}\frac{|u_n|_6^6}{T^6}\chi'\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 + \frac{q\lambda_n}{4}\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}(\nabla K, x)\phi_{u_n}u_n^2 \\
&\quad + \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3}(\nabla f, x)|u_n|^{p+1} = 3c_{\lambda_n}.
\end{aligned}$$

Moreover, combining (f_2), (K_2), (4.1), (4.2), (4.3), (4.30), Lemma 3.1(ii), and the Sobolev embedding theorem, we deduce

$$\begin{aligned}
a\int_{\mathbb{R}^3}|\nabla u_n|^2 &\le 3c_{\lambda_n} - \frac{3q\lambda_n}{4}\frac{|u_n|_6^6}{T^6}\chi'\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 \\
&\quad - \frac{q\lambda_n}{4}\chi\left(\frac{|u_n|_6^6}{T^6}\right)\int_{\mathbb{R}^3}(\nabla K, x)\phi_{u_n}u_n^2 - \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3}(\nabla f, x)|u_n|^{p+1} \\
&\le 3c_{\lambda_n} + C_1\frac{q^2}{T^6}\chi'\left(\frac{|u_n|_6^6}{T^6}\right)|u_n|_6^{10} + C_2q^2\chi\left(\frac{|u_n|_6^6}{T^6}\right)|u_n|_6^4.
\end{aligned}$$

By the definition of c_{λ_n} and by (4.6) we have

$$\begin{aligned}
c_{\lambda_n} &\le \max_{\theta\ge 0} I^T_{q,\lambda_n}(w_\theta) \\
&\le \max_{\theta\ge 0}\left[\frac{a}{2}\theta^2\int_{\mathbb{R}^3}|\nabla w|^2 + \frac{1}{2}\theta^4\int_{\mathbb{R}^3}w^2 + \frac{b}{4}\theta^4\left(\int_{\mathbb{R}^3}|\nabla w|^2\right)^2\right. \\
&\quad \left. - \frac{q\lambda_n}{4}\chi\left(\frac{\theta^6}{T^6}\right)\int_{\mathbb{R}^3}K(x)\phi_{w_\theta}w_\theta^2 - \frac{\lambda_n}{p+1}\theta^{3+\frac{p+1}{2}}\int_{\mathbb{R}^3}|w|^{p+1}\right] \\
&\le \max_{\theta\ge 0}\left[\frac{a}{2}\theta^2\int_{\mathbb{R}^3}|\nabla w|^2 + \frac{1}{2}\theta^4\int_{\mathbb{R}^3}|w|^2 + \frac{b}{4}\theta^4\left(\int_{\mathbb{R}^3}|\nabla w|^2\right)^2\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\lambda_n}{p+1}\theta^{3+\frac{p+1}{2}} \int_{\mathbb{R}^3} |w|^{p+1}\Bigg] + \max_{\theta\geq 0}\left[\frac{q\lambda_n}{4}\chi\left(\frac{\theta^6}{T^6}\right)\int_{\mathbb{R}^3} K(x)\phi_{w_\theta} w_\theta^2\right] \\
&= A_1 + A_2(T),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \max_{\theta\geq 0}\Bigg[\frac{a}{2}\theta^2\int_{\mathbb{R}^3}|\nabla w|^2 + \frac{1}{2}\theta^4\int_{\mathbb{R}^3}|w|^2 + \frac{b}{4}\theta^4\left(\int_{\mathbb{R}^3}|\nabla w|^2\right)^2 \\
&\quad - \frac{\lambda_n}{p+1}\theta^{3+\frac{p+1}{2}}\int_{\mathbb{R}^3}|w|^{p+1}\Bigg], \\
A_2(T) &= \max_{\theta\geq 0}\left[\frac{q\lambda_n}{4}\chi\left(\frac{\theta^6}{T^6}\right)\int_{\mathbb{R}^3} K(x)\phi_{w_\theta} w_\theta^2\right].
\end{aligned}
$$

By Lemma 4.5, A_1 is a finite number. If $\theta^6 \geq 2T^6$, then $A_2(T) = 0$. Otherwise, it follows from (4.2) that

$$
\begin{aligned}
A_2(T) &\leq \frac{q}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_\theta} w_\theta^2 \\
&\leq \frac{q^2}{4}|K|_2^2 S_2^{-2}\cdot |w_\theta|_6^4 \\
&= \frac{q^2}{4}|K|_2^2 S_2^{-2}\theta^4 \leq C_3 q^2 T^4.
\end{aligned}
\tag{4.30}
$$

We also have

$$
\begin{aligned}
&C_1\frac{q^2}{T^6}\chi'\left(\frac{|u_n|_6^6}{T^6}\right)|u_n|_6^{10} \leq C_4 q^2 T^4, \\
&C_2 q^2\chi\left(\frac{|u_n|_6^6}{T^6}\right)|u_n|_6^4 \leq C_5 q^2 T^4.
\end{aligned}
$$

Then we deduce

$$
a\int_{\mathbb{R}^3}|\nabla u_n|^2 \leq 3A_1 + C_6 q^2 T^4. \tag{4.31}
$$

By the Sobolev embedding theorem we have

$$
|u|_6^2 \leq C\int_{\mathbb{R}^3}|\nabla u_n|^2 \leq C_7 + C_8 q^2 T^4. \tag{4.32}
$$

We suppose by contradiction that there exists no subsequence of $\{u_n\}$ that is uniformly bounded by T. Then we can assume that $|u_n|_6 > T$, $n\in\mathbb{N}$. Therefore by (4.32) we conclude that

$$
T^2 < |u_n|_6^2 \leq C\int_{\mathbb{R}^3}|\nabla u_n|^2 \leq C_7 + C_8 q^2 T^4,
$$

which is not true for T large and q small enough. Indeed, we can find $T_0 > 0$ such that $T_0^2 > C_7 + 1$ and $q_0 = q_0(T_0)$ such that $C_8 q^2 T^4 < 1$ for any $0 < q < q_0$. □

Proof of Theorem 2.1 Let T, q_0 be as in Lemma 4.6 and fix $0 < q < q_0$. According to Lemma 4.3, there exist sequences $\{\lambda_n\} \subset J$ and $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$\lambda_n \to 1^-, \qquad I^T_{q,\lambda_n}(u_n) = c_{\lambda_n}, \qquad \left(I^T_{q,\lambda_n}\right)'(u_n) = 0, \tag{4.33}$$

where c_{λ_n} is defined by (3.7). We will prove that $\{u_n\}_n$ is a bounded (PS) sequence for $I_q = I^T_{q,1}$. By Lemma 4.6 we know that

$$|u_n|_6 \le T. \tag{4.34}$$

Using the Hölder inequality, we get

$$\int_{\mathbb{R}^3} |u_n|^{p+1} \le \left(\int_{\mathbb{R}^3} |u_n|^6\right)^{\frac{p-1}{4}} \left(\int_{\mathbb{R}^3} u_n^2\right)^{\frac{5-p}{4}} \le T^{\frac{3(p-1)}{2}} \left(\int_{\mathbb{R}^3} u_n^2\right)^{\frac{5-p}{4}}. \tag{4.35}$$

Since $\langle (I^T_{q,\lambda})'(u_n), u_n\rangle = 0$, it follows from ($f_1$), (4.2), (4.3), (4.5), (4.34), and (4.35) that

$$\begin{aligned}
\int_{\mathbb{R}^3} u_n^2 &= \lambda q \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 + \lambda \int_{\mathbb{R}^3} f(x)|u_n|^{p+1} \\
&\quad - \left(a + b\int_{\mathbb{R}^3} |\nabla u_n|^2\right) \int_{\mathbb{R}^3} |\nabla u_n|^2 \\
&\le q \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 + \int_{\mathbb{R}^3} \alpha(x)|u_n|^{p+1} + \int_{\mathbb{R}^3} f_\infty |u_n|^{p+1} \\
&\le C_9 q^2 T^4 + C_{10} T^{p+1} + f_\infty T^{\frac{3(p-1)}{2}} \left(\int_{\mathbb{R}^3} u_n^2\right)^{\frac{5-p}{4}}.
\end{aligned}$$

As $1 < p < 5$, $\int_{\mathbb{R}^3} u_n^2$ is bounded. Combining (4.31), we deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

On the other hand, by (4.34), (4.3), and (4.4) we have

$$I^T_{q,\lambda_n}(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - \frac{q\lambda_n}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 - \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3} f(x)|u_n|^{p+1}.$$

Since $\lambda_n \to 1^-$, we can prove that $\{u_n\}$ is a (PS) sequence for $I_q = I^T_{q,1}$ by similar arguments as in Theorem 1.1 of [15]. We finish as in Lemma 4.3. □

5 A ground state solution

To apply the global compactness lemma to solve problem (1.4), first of all, we need to consider the existence of ground state solutions of the associated 'limit problem' of (1.4), which is given as

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u + u - K_\infty \phi_u u = f_\infty |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\triangle \phi_u = K_\infty u^2, & x \in \mathbb{R}^3, \end{cases} \tag{5.1}$$

and the corresponding least energy of the associated limited functional

$$I_\infty(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{1}{4}\int_{\mathbb{R}^3} K_\infty \phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} f_\infty |u|^{p+1}. \tag{5.2}$$

Set

$$\mathcal{M}_\infty := \{u \in H^1(\mathbb{R}^3)\setminus\{0\} | G_\infty(u) = 0\},$$

where

$$\begin{aligned} G_\infty(u) &= \frac{1}{2}\langle I'_\infty(u), u\rangle + P_\infty(u) \\ &= a\int_{\mathbb{R}^3} |\nabla u|^2 + 2\int_{\mathbb{R}^3} u^2 + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \frac{7}{4}\int_{\mathbb{R}^3} K_\infty \phi_u u^2 - \frac{p+7}{2(p+1)}\int_{\mathbb{R}^3} f_\infty |u|^{p+1}, \\ P_\infty(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \frac{5}{4}\int_{\mathbb{R}^3} K_\infty \phi_u u^2 - \frac{3}{p+1}\int_{\mathbb{R}^3} f_\infty |u|^{p+1}. \end{aligned}$$

We remark that

$$I_\infty|_{\mathcal{M}_\infty}(u) = \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{16}\int_{\mathbb{R}^3} K_\infty \phi_u u^2 + \frac{p-1}{8(p+1)}\int_{\mathbb{R}^3} f_\infty |u|^{p+1} \tag{5.3}$$

and

$$\begin{aligned} I_\infty|_{\mathcal{M}_\infty}(u) &= \frac{p+3}{2(p+7)} a\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{p-1}{2(p+7)}\int_{\mathbb{R}^3} u^2 \\ &\quad + \frac{p-1}{4(p+7)} b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \frac{7-p}{4(p+7)}\int_{\mathbb{R}^3} K_\infty \phi_u u^2. \end{aligned} \tag{5.4}$$

Lemma 5.1 *I_∞ is not bounded from below.*

Proof For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we have

$$\begin{aligned} I_\infty(u_t) &= \frac{a}{2}t^2\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}t^4\int_{\mathbb{R}^3} u^2 + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad - \frac{1}{4}t^7\int_{\mathbb{R}^3} K_\infty \phi_u u^2 - \frac{1}{p+1}t^{\frac{p+7}{2}}\int_{\mathbb{R}^3} f_\infty |u|^{p+1}. \end{aligned}$$

As $1 < p < 5$, $I_\infty(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 5.2 *Let m_i $(i = 1, 2, 3, 4)$ be positive constants, $1 < p < 5$, and $g(t) = m_1 t^2 + m_2 t^4 - m_3 t^7 - m_4 t^{\frac{p+7}{2}}$ for $t \geq 0$. Then g has a unique positive critical point, which corresponds to its maximum.*

Proof The proof is similar to Lemma 3.3 of [6], and we omit it. □

Lemma 5.3 *For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there is a unique $\bar{t} > 0$ such that $u_{\bar{t}} \in \mathcal{M}_\infty$. Moreover, $I_\infty(u_{\bar{t}}) = \max_{t>0} I_\infty(u_t)$.*

Proof For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $t \geq 0$, we denote

$$\begin{aligned}\gamma(t) &\triangleq I_\infty(u_t)\\ &= \frac{a}{2}t^2\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{2}t^4\int_{\mathbb{R}^3}u^2 + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\\ &\quad - \frac{1}{4}t^7\int_{\mathbb{R}^3}K_\infty\phi_u u^2 - \frac{1}{p+1}t^{\frac{p+7}{2}}\int_{\mathbb{R}^3}f_\infty|u|^{p+1}.\end{aligned}$$

By Lemma 5.2, γ has a unique critical point $\bar{t} > 0$ corresponding to its maximum. Then $\gamma(\bar{t}) = \max_{t>0}\gamma(t)$ and $\gamma'(\bar{t}) = 0$. It follows that $I_\infty(u_{\bar{t}}) = \max_{t>0} I_\infty(u_t)$, $G_\infty(u_{\bar{t}}) = \bar{t}\gamma'(\bar{t}) = 0$ a.e., and $u_{\bar{t}} \in \mathcal{M}_\infty$. □

Set

$$\begin{aligned}&c_1 := \inf\nolimits_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\infty\bigl(\eta(t)\bigr),\\ &c_2 := \inf\nolimits_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\max_{t>0} I_\infty(u_t),\\ &c_3 := \inf\nolimits_{u\in\mathcal{M}_\infty} I_\infty(u),\\ &c_4 := \inf\nolimits_{u\in H_r^1(\mathbb{R}^3)\cap\mathcal{M}_\infty} I_\infty(u),\end{aligned}$$

where

$$\Gamma := \bigl\{\eta \in C\bigl([0,1], H^1\bigl(\mathbb{R}^3\bigr)\bigr) \mid \eta(0) = 0, I_\infty\bigl(\eta(1)\bigr) < 0\bigr\}.$$

The constant c_3 turns out to be a nontrivial number, as we will prove in the next lemma, which contains the statement of the main properties of $\mathcal{M}_\infty$.

Lemma 5.4
(i) *There exists $\sigma > 0$ such that $\|u\| \geq \sigma$ for all $u \in \mathcal{M}_\infty$;*
(ii) *I_∞ is bounded from below on $\mathcal{M}_\infty$ by a positive constant, i.e. $c_3 > 0$;*
(iii) *u is a free critical point of I_∞ if and only if u is a critical point of I_∞ constrained on $\mathcal{M}_\infty$.*

Proof (i) Let $u \in \mathcal{M}_\infty$. By the Hölder inequality we see that

$$\begin{aligned}0 &= a\int_{\mathbb{R}^3}|\nabla u|^2 + 2\int_{\mathbb{R}^3}u^2 + b\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2\\ &\quad - \frac{7}{4}\int_{\mathbb{R}^3}K_\infty\phi_u u^2 - \frac{p+7}{2(p+1)}\int_{\mathbb{R}^3}f_\infty|u|^{p+1}\\ &\geq \|u\|^2 - C\bigl(\|u\|^4 + \|u\|^{p+1}\bigr).\end{aligned}$$

As $1 < p < 5$, there exists $\sigma > 0$ such that

$$\|u\| \geq \sigma > 0, \quad \forall u \in \mathcal{M}_\infty. \tag{5.5}$$

(ii) It follows from (5.4) and (5.5) that

$$\begin{aligned} I_\infty(u) &= \frac{p+3}{2(p+7)} a \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{p-1}{2(p+7)} \int_{\mathbb{R}^3} u^2 \\ &\quad + \frac{p-1}{4(p+7)} b \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{7-p}{4(p+7)} \int_{\mathbb{R}^3} K_\infty \phi_u u^2 \\ &\geq \frac{p-1}{2(p+7)} \sigma^2 > 0. \end{aligned}$$

Thus, $c_3 > 0$.

(iii) The proof consists of two steps.

Step 1. $G'_\infty(u) \neq 0$ for any $u \in \mathcal{M}_\infty$, and hence $\mathcal{M}_\infty$ is a C^1-manifold.

We will prove this by contradiction. Suppose that $G'_\infty(u) = 0$ for some $u \in \mathcal{M}_\infty$. Denote

$$\begin{aligned} &k \triangleq I_\infty(u), \qquad \alpha \triangleq a \int_{\mathbb{R}^3} |\nabla u|^2, \qquad \beta \triangleq \int_{\mathbb{R}^3} u^2, \\ &\mu \triangleq b \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2, \qquad \nu \triangleq \int_{\mathbb{R}^3} K_\infty \phi_u u^2, \qquad \delta \triangleq \int_{\mathbb{R}^3} f_\infty |u|^{p+1}. \end{aligned}$$

In a weak sense, the equation $G'_\infty(u) = 0$ can be written as

$$\begin{aligned} &-2 \left(a + 2b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \triangle u + 4u - 7 K_\infty \phi_u u \\ &\quad = \frac{p+7}{2} f_\infty |u|^{p-1} u, \end{aligned} \tag{5.6}$$

where $\phi_u(x) = \frac{K_\infty}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy$. Combining $G_\infty(u) = 0$, (5.6), and its Pohožave identity, we have

$$\begin{cases} \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{4}\mu - \frac{1}{4}\nu - \frac{1}{p+1}\delta = k > 0, \\ \alpha + 2\beta + \mu - \frac{7}{4}\nu - \frac{p+7}{2(p+1)}\delta = 0, \\ 2\alpha + 4\beta + 4\mu - 7\nu - \frac{p+7}{2}\delta = 0, \\ \alpha + 6\beta + 2\mu - \frac{21}{4}\nu - \frac{3(p+7)}{2(p+1)}\delta = 0. \end{cases}$$

Thus, we can conclude that

$$\nu + \frac{(p+3)(p-1)}{1+p} \delta = -32k < 0,$$

which is impossible since ν and δ are positive. Thus, $G'_\infty(u) \neq 0$ for any $u \in \mathcal{M}_\infty$, and $\mathcal{M}_\infty$ is a C^1-manifold by the implicit function theorem.

Step 2. Every critical point of $I_\infty|_{\mathcal{M}_\infty}$ is the critical point of I_∞ in $H^1(\mathbb{R}^3)$.

If u is a critical point of $I_\infty|_{\mathcal{M}_\infty}$, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$I'_\infty(u) = \lambda G'_\infty(u).$$

We claim that $\lambda = 0$. The equation $I'_\infty(u) - \lambda G'_\infty(u) = 0$ can be written in the weak sense as

$$\begin{aligned}
&-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\triangle u + u - K_\infty \phi_u u - f_\infty |u|^{p-1}u \\
&\quad = \lambda\left[-2\left(a + 2b\int_{\mathbb{R}^3} |\nabla u|^2\right)\triangle u + 4u - 7K_\infty \phi_u u - \frac{p+7}{2} f_\infty |u|^{p-1}u\right],
\end{aligned}$$

that is, u solves the equations $-\triangle\phi = K_\infty u^2$ and

$$\begin{aligned}
&-\left((2\lambda-1)a + (4\lambda-1)b\int_{\mathbb{R}^3} |\nabla u|^2\right)\triangle u + (4\lambda-1)u \\
&\qquad - (7\lambda-1)K_\infty \phi_u u \\
&\quad = \left[\frac{p+7}{2}\lambda - 1\right] f_\infty |u|^{p-1}u
\end{aligned}$$

for $x \in \mathbb{R}^3$. Using the notation in *Step 1*, we have that

$$\begin{cases}
\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{4}\mu - \frac{1}{p+1}\delta = \frac{1}{4}\nu + k > 0, \\
\alpha + 2\beta + \mu - \frac{p+7}{2(p+1)}\delta = \frac{7}{4}\nu, \\
(2\lambda-1)\alpha + (4\lambda-1)\beta + (4\lambda-1)\mu - [\frac{p+7}{2}\lambda - 1]\delta = (7\lambda-1)\nu, \\
\frac{2\lambda-1}{2}\alpha + \frac{3(4\lambda-1)}{2}\beta + \frac{4\lambda-1}{2}\mu - \frac{3(p+7)\lambda-6}{2(p+1)}\delta = \frac{5(7\lambda-1)}{4}\nu.
\end{cases} \tag{5.7}$$

The coefficient matrix of (5.7) is

$$\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{p+1} \\
1 & 2 & 1 & -\frac{p+7}{2(p+1)} \\
2\lambda-1 & 4\lambda-1 & 4\lambda-1 & 1-\frac{p+7}{2}\lambda \\
\frac{2\lambda-1}{2} & \frac{3(4\lambda-1)}{2} & \frac{4\lambda-1}{2} & -\frac{3(p+7)\lambda-6}{2(p+1)}
\end{pmatrix},$$

and its determinant is

$$\det A = \frac{\lambda(1-4\lambda)(p+3)(p-1)}{16(p+1)}.$$

As $1 < p < 5$,

$$\det A = 0 \quad \Longleftrightarrow \quad \lambda = 0, \quad \lambda = \frac{1}{4}.$$

Now we prove that $\lambda = 0$ by excluding the other two possibilities:

(1) If $\lambda = \frac{1}{4}$, then the last equation of (5.7) is

$$\frac{1}{4}\alpha + \frac{15}{16}\nu + \frac{3(p-1)}{8(p+1)}\delta = 0.$$

This is a contradiction since $\alpha > 0$, $\delta > 0$, $\nu \geq 0$, and $1 < p < 5$.

(2) If $\lambda \neq 0$, $\lambda \neq \frac{1}{4}$, then the linear system (5.7) has a unique solution. We obtain

$$\delta = -\frac{(p+1)(32k+15\nu)}{(p+3)(p-1)}.$$

As $1 < p < 5$, $k > 0$ and $\nu \geq 0$, we have $\delta < 0$. This is impossible since $\delta > 0$. □

Lemma 5.5 $c \triangleq c_1 = c_2 = c_3 = c_4$.

Proof (i) $c_1 = c_2 = c_3$. The proof is similar to the argument of Nehari manifold method [16].

(ii) $c_2 = c_4$. It is clear that $c_2 = c_3 \leq c_4$. We next prove that $c_2 \geq c_4$. For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, it follows from (5.3) that there exists a unique $\bar{t} > 0$ such that $u_{\bar{t}} \in \mathcal{M}_\infty$ and $I_\infty(u_{\bar{t}}) = \max_{t>0} I_\infty(u_t)$. Similarly, for u^*, the Schwarz symmetric arrangement of u, there exists a unique $t^* > 0$ such that $u^*_{t^*} \in \mathcal{M}_\infty$, that is, $u^*_{t^*} \in \mathcal{M}_\infty \cap H^1_r(\mathbb{R}^3)$. Note that

$$\int_{\mathbb{R}^3} \left|\nabla u^*\right|^2 \leq \int_{\mathbb{R}^3} |\nabla u|^2, \qquad \int_{\mathbb{R}^3} \left|u^*\right|^2 = \int_{\mathbb{R}^3} |u|^2, \qquad \int_{\mathbb{R}^3} \left|u^*\right|^{p+1} = \int_{\mathbb{R}^3} |u|^{p+1},$$

and (see [17])

$$\int_{\mathbb{R}^3} \phi_u u^2 \leq \int_{\mathbb{R}^3} \phi_{u^*} \left(u^*\right)^2.$$

Then we have that

$$I_\infty(u_{t^*}) \geq I_\infty\left(u^*_{t^*}\right).$$

By the preceding we deduce that

$$c_2 = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_\infty(u_t) \geq I_\infty(u_{t_0}) \geq I_\infty\left(u^*_{t^*}\right) \geq \inf_{u \in H^1_r(\mathbb{R}^3)\cap\mathcal{M}_\infty} I_\infty(u) = c_4.$$ □

Lemma 5.6 (See [6]) *Let $\{u_n\}$ be a sequence satisfying $u_n \rightharpoonup u$ in $H^1_r(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_u u^2.$$

Theorem 5.1 *Problem* (5.1) *admits a positive ground state solution* $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Proof By Lemmas 5.4-5.5 we only need to prove that c is attained at some $u \in H^1_r(\mathbb{R}^3) \cap \mathcal{M}_\infty$. Letting $\{u_n\} \subset H^1_r(\mathbb{R}^3) \cap \mathcal{M}_\infty$ be a minimizing sequence for I_∞, it follows from (5.4) that

$$\begin{aligned} c + 1 > {} & \frac{p+3}{2(p+7)} a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{p-1}{2(p+7)} \int_{\mathbb{R}^3} u_n^2 \\ & + \frac{p-1}{4(p+7)} b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + \frac{7-p}{4(p+7)} \int_{\mathbb{R}^3} K_\infty \phi_{u_n} u_n^2 \end{aligned}$$

for n large enough. Therefore, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Passing, if necessary, to a subsequence, we may assume that there exists a function $\tilde{u} \in H_r^1(\mathbb{R}^3)$ such that

$$\begin{aligned}
&u_n \rightharpoonup \tilde{u} \quad \text{in } H_r^1(\mathbb{R}^3), \\
&u_n \to \tilde{u} \quad \text{in } L^s(\mathbb{R}^3) \text{ for } s \in (2,6), \text{and} \\
&u_n \to \tilde{u} \quad \text{a.e. in } \mathbb{R}^3.
\end{aligned} \tag{5.8}$$

Now we show that $u_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$. Indeed, it follows from (5.8), Fatou's lemma, and Lemma 5.6 that

$$\begin{aligned}
&a\int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \le \liminf_{n\to\infty} a\int_{\mathbb{R}^3} |\nabla u_n|^2, \\
&\int_{\mathbb{R}^3} |\tilde{u}|^2 \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^2, \\
&b\left(\int_{\mathbb{R}^3} |\nabla \tilde{u}|^2\right)^2 \le \liminf_{n\to\infty} b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\tilde{u}|^{p+1} = \liminf_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{p+1}, \\
&\int_{\mathbb{R}^3} \phi_{\tilde{u}} \tilde{u}^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2.
\end{aligned}$$

Clearly, we have $G_\infty(\tilde{u}) \le 0$. Therefore, by Lemma 5.3 there exists a unique $t_0 \in (0,1]$ such that $\tilde{u}_{t_0} \in \mathcal{M}_\infty$. If $t_0 \in (0,1)$, then it follows from (5.3) that

$$I_\infty(\tilde{u}_{t_0}) < \lim_{n\to\infty} I_\infty(u_n) = c,$$

which is impossible. Thus $t_0 = 1$, and c is attained at $\tilde{u} \in \mathcal{M}_\infty$.

By the standard regularity arguments as in the proof of Theorem 1.4 of [11] we see that $\tilde{u}$ is a positive ground state solution for problem (5.1). □

Assume that (f_1)-(f_2) and (K_2)-(K_4) hold. We apply Proposition 3.1 to prove Theorem 2.2. Set $J = [\frac{1}{2}, 1]$. We consider a family of functionals on $H^1(\mathbb{R}^3)$,

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 - \frac{\lambda}{p+1}\int_{\mathbb{R}^3} f(x)|u|^{p+1}. \tag{5.9}$$

Then $I_\lambda(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \to +\infty \quad \text{as } n \to +\infty$$

and

$$B(u) = \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 + \frac{1}{p+1}\int_{\mathbb{R}^3} f(x)|u|^{p+1} \ge 0.$$

Lemma 5.7 *Assume that* (f_1)-(f_2) *and* (K_3) *hold. Then*

(i) *there exists* $v \in H^1(\mathbb{R}^3) \setminus \{0\}$ *such that* $I_\lambda(v) < 0$ *for all* $\lambda \in J$;

(ii) $c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0$ *for all* $\lambda \in J$, *where*

$$\Gamma_\lambda := \bigl\{\gamma \in C\bigl([0,1], H^1\bigl(\mathbb{R}^3\bigr)\bigr) \mid \gamma(0) = 0, \gamma(1) = v\bigr\}.$$

Proof The proof is similar to those of Lemma 3.1 and Lemma 3.2, and we omit it. □

By Proposition 3.1 we see that, for any $\lambda \in [\frac{1}{2}, 1]$, the associated limit problem

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\triangle u + u - \lambda K_\infty \phi u = \lambda f_\infty |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\triangle\phi = K_\infty u^2, & x \in \mathbb{R}^3, \end{cases} \tag{5.10}$$

has a positive ground state solution $(u_\lambda, \phi_{u_\lambda}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, that is, for any $\lambda \in [\frac{1}{2}, 1]$,

$$m_\lambda^\infty := \inf\bigl\{I_\lambda^\infty(u), u \in \mathcal{M}_\lambda^\infty\bigr\} \tag{5.11}$$

is achieved at $u_\lambda \in \mathcal{M}_\lambda^\infty \triangleq \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G_\lambda^\infty(u) = 0\}$, where

$$\begin{aligned} G_\lambda^\infty(u) = {} & a\int_{\mathbb{R}^3} |\nabla u|^2 + 2\int_{\mathbb{R}^3} u^2 + b\biggl(\int_{\mathbb{R}^3} |\nabla u|^2\biggr)^2 \\ & - \frac{7\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_u u^2 - \frac{p+7}{2(p+1)}\lambda\int_{\mathbb{R}^3} f_\infty |u|^{p+1}. \end{aligned} \tag{5.12}$$

Lemma 5.8 *Assume that* (f_1)-(f_2) *and* (K_2)-(K_3) *hold and that* $K \not\equiv K_\infty$ *or* $f \not\equiv f_\infty$. *Then* $c_\lambda < m_\lambda^\infty$ *for any* $\lambda \in J$.

Proof Let u_λ be the minimizer of m_λ^∞. By Lemma 5.3 we see that $I_\lambda^\infty(u_\lambda) = \max_{t>0} I_\lambda^\infty(t^{\frac{1}{2}} u_\lambda(\frac{x}{t}))$. Then, choosing $v(x) = (u_\lambda)_{t_0}$ for t_0 large enough in Lemma 5.7(i), we have that, for any $\lambda \in J$, there exists $\widehat{t} \in (0, t_0)$ such that

$$\begin{aligned} c_\lambda &= \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda\bigl(\gamma(t)\bigr) \\ &\le \max_{0<t<t_0} I_\lambda\bigl((u_\lambda)_{t_0}\bigr) \\ &= I_\lambda\bigl((u_\lambda)_{\widehat{t}}\bigr). \end{aligned}$$

By the assumptions on K and f, since $u_\lambda > 0$, we derive

$$I_\lambda\bigl((u_\lambda)_{\widehat{t}}\bigr) < I_\lambda^\infty\bigl((u_\lambda)_{\widehat{t}}\bigr).$$

Thus, we conclude that

$$c_\lambda < I_\lambda^\infty\bigl((u_\lambda)_{\widehat{t}}\bigr) \le I_\lambda^\infty(u_\lambda) = m_\lambda^\infty. \qquad \square$$

To prove that the functional I_λ satisfies $(\mathrm{PS})_{c_\lambda}$ condition for a.e. $\lambda \in J$, we need the following global compactness lemma, which is suitable for Kirchhoff equations.

Lemma 5.9 *Assume that* (f_1), (f_2), *and* (K_3) *hold. Then for* $c > 0$, $\lambda \in J$, *and a bounded* $(\mathrm{PS})_c$ *sequence* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *of* I_λ, *there exist* $u \in H^1(\mathbb{R}^3)$ *and* $A \in \mathbb{R}$ *such that* $J'_\lambda(u) = 0$, *where*

$$\begin{aligned} J_\lambda(u) = {} & \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 \\ & - \frac{\lambda}{4} \int_{\mathbb{R}^3} K(x)\phi_u u^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} f(x)|u|^{p+1}. \end{aligned} \tag{5.13}$$

Moreover, there exists a finite (possibly, empty) set $\{w^1, w^2, \ldots, w^l\} \subset H^1(\mathbb{R}^3)$ *of nontrivial positive solutions of*

$$-(a + bA^2)\triangle u + u - \lambda K_\infty \phi_u u = \lambda f_\infty |u|^{p-1} u \tag{5.14}$$

and $\{y_n^1, y_n^2, \ldots, y_n^l\} \subset \mathbb{R}^3$ *such that*

$$\begin{aligned} & |y_n^k| \to \infty, \qquad |y_n^k - y_n^{k'}| \to \infty, \quad k \neq k', n \to \infty, \\ & c + \frac{bA^4}{4} = J_\lambda(u) + \sum_{k=1}^{l} J_\lambda^\infty(w^k), \end{aligned}$$

and

$$\begin{aligned} & \left\| u_n - u - \sum_{k=1}^{l} w^k(\cdot - y_n^k) \right\| \to 0, \\ & A^2 = |\nabla u|_2^2 + \sum_{k=1}^{l} |\nabla w^k|_2^2, \end{aligned}$$

where

$$J_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 - \frac{\lambda}{4} \int_{\mathbb{R}^3} K_\infty \phi_u u^2 - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} f_\infty |u|^{p+1}. \tag{5.15}$$

Proof The proof is similar to that of Lemma 3.4 in [11]. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, there exist $u \in H^1(\mathbb{R}^3)$ and $A \in \mathbb{R}$ such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla u_n|^2 \to A^2. \tag{5.16}$$

Moreover, using (f_1)-(f_2) and (K_2)-(K_3), we have

$$K\phi_{u_n} u_n \to K\phi_u u \quad \text{and} \quad f|u_n|^{p-1} u_n \to f|u|^{p-1} u \quad \text{in } H^{-1}(\mathbb{R}^3). \tag{5.17}$$

Therefore, $I'_\lambda(u_n) \to 0$ implies that

$$\begin{aligned} & \int_{\mathbb{R}^3} (a\nabla u \nabla\varphi + u\varphi) + bA^2 \int_{\mathbb{R}^3} \nabla u \nabla \varphi - \lambda \int_{\mathbb{R}^3} K(x)\phi_u u\varphi \\ & \quad - \lambda \int_{\mathbb{R}^3} f(x)|u|^{p-1} u\varphi = 0, \quad \forall \varphi \in H^1(\mathbb{R}^3), \end{aligned}$$

that is, $J_\lambda'(u)=0$. Since

$$\begin{aligned}
J_\lambda(u_n) &= \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2 \\
&\quad -\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2-\frac{\lambda}{p+1}\int_{\mathbb{R}^3}f(x)|u_n|^{p+1} \\
&= \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 \\
&\quad -\frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 \\
&\quad -\frac{\lambda}{p+1}\int_{\mathbb{R}^3}f(x)|u_n|^{p+1}+\frac{bA^2}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2+o(1) \\
&= I_\lambda(u_n)+\frac{bA^4}{4}+o(1)
\end{aligned}$$

and

$$\begin{aligned}
\langle J_\lambda'(u_n),\varphi\rangle &= \left(a+bA^2\right)\int_{\mathbb{R}^3}\nabla u_n\nabla\varphi+\int_{\mathbb{R}^3}u_n\varphi \\
&\quad -\lambda\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n\varphi-\lambda\int_{\mathbb{R}^3}f(x)|u_n|^{p-1}u_n\varphi \\
&= a\int_{\mathbb{R}^3}\nabla u_n\nabla\varphi+\int_{\mathbb{R}^3}u_n\varphi+b\int_{\mathbb{R}^3}|\nabla u_n|^2\int_{\mathbb{R}^3}\nabla u_n\nabla\varphi \\
&\quad -\lambda\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n\varphi-\lambda\int_{\mathbb{R}^3}f(x)|u_n|^{p-1}u_n\varphi+o(1) \\
&= \langle I_\lambda'(u_n),\varphi\rangle+o(1),
\end{aligned}$$

we conclude that

$$J_\lambda(u_n)\to c+\frac{bA^4}{4},\qquad J_\lambda'(u_n)\to 0\quad\text{in } H^{-1}\left(\mathbb{R}^3\right).$$

We next show that

$$\left\|u_n-u-\sum_{k=1}^{l}w^k\left(\cdot-y_n^k\right)\right\|\to 0,$$

where either $l=0$ or $l>0$.

Step 1: Set $u_n^1=u_n-u$. By (5.15) and the Bresiz-Lieb lemma we obtain that

(A.1) $|\nabla u_n^1|_2^2=|\nabla u_n|_2^2-|\nabla u|_2^2+o(1)$,

(B.1) $|u_n^1|_2^2=|u_n|_2^2-|u|_2^2+o(1)$,

(C.1) $J_\lambda^\infty(u_n^1)\to c+\frac{bA^4}{4}-J_\lambda(u)$, and

(D.1) $(J_\lambda^\infty)'(u_n^1)\to 0$ in $H^{-1}(\mathbb{R}^3)$.

Let

$$\sigma^1:=\limsup_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}\left|u_n^1\right|^2.$$

Similarly to (4.10), we have that

$$\int_{\mathbb{R}^3} \phi_{u_n^1} u_n^1 \varphi \to 0. \tag{5.18}$$

Vanishing: If $\sigma^1 = 0$, then it follows from Lemma 1.21 of [12] that $u_n^1 \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 2^*)$. Thus, $\int_{\mathbb{R}^3} |u_n^1|^{p-1} u_n^1 \to 0$ in $H^{-1}(\mathbb{R}^3)$. Since $(J_\lambda^\infty)'(u_n^1) \to 0$, we get that $u_n^1 \to 0$ in $H^1(\mathbb{R}^3)$, and the proof is completed.

Nonvanishing: If $\sigma^1 > 0$, then there exists a sequence $\{y_n^1\} \in \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |u_n^1|^2 > \frac{\sigma^1}{2}.$$

Set $w_n^1 \triangleq u_n^1(\cdot + y_n^1)$. Then $\{w_n^1\}$ is bounded, and we may assume that $w_n^1 \rightharpoonup w^1$ in $H^1(\mathbb{R}^3)$. Hence $(J_\lambda^\infty)'(w^1) = 0$. As

$$\int_{B_1(0)} |w_n^1|^2 > \frac{\sigma^1}{2},$$

we see that $w^1 \not\equiv 0$. Moreover, $u_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $\{y_n^1\}$ is unbounded. Hence, we may assume that $|y_n^1| \to +\infty$.

Step 2: Setting $u_n^2 = u_n - u - w^1(\cdot - y_n^1)$, we can similarly get that

(A.2) $|\nabla u_n^2|_2^2 = |\nabla u_n|_2^2 - |\nabla u|_2^2 - |\nabla w^1|_2^2 + o(1)$,

(B.2) $|u_n^2|_2^2 = |u_n|_2^2 - |u|_2^2 - |w^1|_2^2 + o(1)$,

(C.2) $J_\lambda^\infty(u_n^2) \to c + \frac{bA^4}{4} - J_\lambda(u) - J_\lambda^\infty(w^1)$,

(D.2) $(J_\lambda^\infty)'(u_n^2) \to 0$ in $H^{-1}(\mathbb{R}^3)$.

Similarly to the arguments in Step 1, let

$$\sigma^2 := \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n^2|^2.$$

If vanishing occurs, then $\|u_n^2\| \to 0$, that is, $\|u_n - u - w^1(\cdot - y_n^1)\| \to 0$. Moreover, it follows from (5.16) and from (A.2) and (C.2) that

$$A^2 = |\nabla u|_2^2 + |\nabla w^1|_2^2,$$
$$c + \frac{bA^4}{4} = J_\lambda(u) + J_\lambda^\infty(w^1).$$

If nonvanishing occurs, then there exist a sequence $\{y_n^2\} \in \mathbb{R}^3$ and a nontrivial $w^2 \in H^1(\mathbb{R}^3)$ such that $w_n^2 \triangleq u_n^2(\cdot + y_n^2) \rightharpoonup w^2$ in $H^1(\mathbb{R}^3)$. Then by (D.2) we have that $(J_\lambda^\infty)'(w^2) = 0$. Furthermore, $u_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ implies that $|y_n^2| \to \infty$ and $|y_n^2 - y_n^1| \to \infty$.

We next proceed by iteration. Recall that if w^k is a nontrivial solution of I_λ^∞, then $I_\lambda^\infty(w^k) > 0$. So there exists some finite $l \in \mathbb{N}$ such that only the vanishing case occurs in Step l. Then the lemma is proved. □

Lemma 5.10 *Assume that* (f_1)-(f_2) *and* (K_2)-(K_4) *hold. For* $\lambda \in J$, *let* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *be a bounded* $(\mathrm{PS})_{c_\lambda}$ *sequence of* I_λ. *Then there exists a nontrivial* $u_\lambda \in H^1(\mathbb{R}^3)$ *such that*

$$u_n \to u_\lambda \quad \text{in } H^1(\mathbb{R}^3).$$

Proof By Lemma 5.9 we know that, for $\lambda \in J$, there exist $u_\lambda \in H^1(\mathbb{R}^3)$ and $A_\lambda \in \mathbb{R}$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in } H^1(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} |\nabla u_n|^2 \to A_\lambda^2, \qquad J'_\lambda(u_\lambda) = 0.$$

Now we prove that $u_n \to u_\lambda$ in $H^1(\mathbb{R}^3)$, that is, $\{w_1, w_2, \dots, w_l\} = \emptyset$. By contradiction we assume that there exist a positive integer l and $\{y_n^k\}_{k=1}^l \subset \mathbb{R}^3$ with $|y_n^k| \to \infty$ as $n \to \infty$ for each $1 \le k \le l$, and nontrivial positive solutions $w^1, w^2, \dots, w^l$ of problem (5.14) such that

$$c_\lambda + \frac{bA_\lambda^4}{4} = J_\lambda(u_\lambda) + \sum_{k=1}^{l} J_\lambda^\infty(w^k),$$
$$\left\| u_n - u_\lambda - \sum_{k=1}^{l} w^k(\cdot - y_n^k) \right\| \to 0,$$

and

$$A_\lambda^2 = |\nabla u_\lambda|_2^2 + \sum_{k=1}^{l} |\nabla w^k|_2^2. \tag{5.19}$$

In the weak sense, the equation $J'_\lambda(u_\lambda) = 0$ can be written as

$$-(a + bA_\lambda^2)\Delta u_\lambda + u_\lambda - \lambda K(x)\phi_{u_\lambda} u_\lambda = \lambda f(x)|u_\lambda|^{p-1}u_\lambda, \tag{5.20}$$

where $\phi_{u_\lambda} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u_\lambda^2(y)}{|x-y|}\,dy$. Denote

$$\begin{cases} \alpha \triangleq a\int_{\mathbb{R}^3} |\nabla u_\lambda|^2, \qquad \beta \triangleq \int_{\mathbb{R}^3} |u_\lambda|^2, \qquad \mu \triangleq bA_\lambda^2 \int_{\mathbb{R}^3} |\nabla u_\lambda|^2, \\ \nu \triangleq \int_{\mathbb{R}^3} K(x)\phi_{u_\lambda} u_\lambda^2, \qquad \delta \triangleq \int_{\mathbb{R}^3} f(x)|u_\lambda|^{p+1}, \\ \overline{\nu} \triangleq \int_{\mathbb{R}^3} (\nabla K, x)\phi_{u_\lambda} u_\lambda^2, \qquad \overline{\delta} \triangleq \int_{\mathbb{R}^3} (\nabla f, x)|u_\lambda|^{p+1}. \end{cases}$$

It is clear that α, β, μ are nonnegative. Then (f_1)-(f_2) and (K_3) imply that δ, $\overline{\delta}$, ν are nonnegative too. Then the Pohožaev identity of (5.20) and $\langle J'_\lambda(u_\lambda), u_\lambda\rangle = 0$ can be written as follows:

$$\begin{cases} \frac{1}{2}\alpha + \frac{3}{2}\beta + \frac{1}{2}\mu - \frac{5\lambda}{4}\nu - \frac{3\lambda}{p+1}\delta - \frac{\lambda}{4}\overline{\nu} - \frac{\lambda}{p+1}\overline{\delta} = 0, \\ \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{4}\mu - \frac{\lambda}{4}\nu - \frac{\lambda}{p+1}\delta = J_\lambda(u_\lambda) - \frac{1}{4}\mu. \end{cases}$$

Note that (K_4) implies that $\nu + \overline{\nu}$ is nonnegative. Then we conclude

$$3\left(J_\lambda(u_\lambda) - \frac{1}{4}\mu\right) = \alpha + \frac{1}{4}\mu + \frac{\lambda}{2}(\nu + \overline{\nu}) + \frac{\lambda}{p+1}\overline{\delta} \ge 0.$$

Hence, we derive

$$J_\lambda(u_\lambda) \ge \frac{1}{4}bA_\lambda^2 \int_{\mathbb{R}^3} |\nabla u_\lambda|^2. \tag{5.21}$$

On the other hand, for each nontrivial positive solution w^k $(k = 1, 2, \dots, l)$ of problem (5.14), we have the following Pohožaev identity:

$$\begin{aligned}\widetilde{P}_\lambda^\infty(w^k) \triangleq{}& \frac{a+bA_\lambda^2}{2}\int_{\mathbb{R}^3}|\nabla w^k|^2 + \frac{3}{2}\int_{\mathbb{R}^3}|w^k|^2 \\ &- \frac{5\lambda}{4}\int_{\mathbb{R}^3}K_\infty\phi_{w^k}|w^k|^2 - \frac{3\lambda}{p+1}\int_{\mathbb{R}^3}f_\infty|w^k|^{p+1} = 0.\end{aligned}$$

It follows from (5.19) that

$$\begin{aligned}0 &= \frac{1}{2}\langle (J_\lambda^\infty)'(w^k), w^k\rangle + \widetilde{P}_\lambda^\infty(w^k) \\ &= (a+bA_\lambda^2)\int_{\mathbb{R}^3}|\nabla w^k|^2 + 2\int_{\mathbb{R}^3}|w^k|^2 \\ &\quad - \frac{7\lambda}{4}\int_{\mathbb{R}^3}K_\infty\phi_{w^k}|w^k|^2 - \frac{(p+7)\lambda}{2(p+1)}\int_{\mathbb{R}^3}f_\infty|w^k|^{p+1} \\ &\geq G_\lambda^\infty(w^k).\end{aligned}$$

Then there exists $t_k \in (0,1]$ such that $(w^k)_{t_k} \in \mathcal{M}_\lambda^\infty$, that is, $G_\lambda^\infty((w^k)_{t_k}) = 0$. By direct calculation we obtain

$$\begin{aligned}J_\lambda^\infty(w^k) &= \frac{a(p+3)}{2(p+7)}\int_{\mathbb{R}^3}|\nabla w^k|^2 + \frac{p-1}{2(p+7)}\int_{\mathbb{R}^3}|w^k|^2 + \frac{b(p-1)}{4(p+7)}A_\lambda^2\int_{\mathbb{R}^3}|\nabla w^k|^2 \\ &\quad + \frac{(7-p)\lambda}{4(p+7)}\int_{\mathbb{R}^3}K_\infty\phi_{w^k}|w^k|^2 + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|\nabla w^k|^2 \\ &\geq \frac{a(p+3)}{2(p+7)}t_k^2\int_{\mathbb{R}^3}|\nabla w^k|^2 + \frac{p-1}{2(p+7)}t_k^4\int_{\mathbb{R}^3}|w^k|^2 + \frac{b(p-1)}{4(p+7)}t_k^4\left(\int_{\mathbb{R}^3}|\nabla w^k|^2\right)^2 \\ &\quad + \frac{(7-p)\lambda}{4(p+7)}t_k^7\int_{\mathbb{R}^3}\phi_{w^k}|w^k|^2 + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|\nabla w^k|^2 \\ &= I_\lambda^\infty((w^k)_{t_k}) - \frac{2}{p+7}G_\lambda^\infty((w^k)_{t_k}) + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|\nabla w^k|^2 \\ &\geq m_\lambda^\infty + \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|\nabla w^k|^2.\end{aligned} \tag{5.22}$$

By (5.19)-(5.22) we have

$$\begin{aligned}c_\lambda + \frac{bA_\lambda^4}{4} &= J_\lambda(u_\lambda) + \sum_{k=1}^{l}J_\lambda^\infty(w^k) \\ &\geq \frac{bA_\lambda^2}{4}\int_{\mathbb{R}^3}|\nabla u_\lambda|^2 + lm_\lambda^\infty + \frac{bA_\lambda^2}{4}\sum_{k=1}^{l}\int_{\mathbb{R}^3}|\nabla w^k|^2 \\ &\geq m_\lambda^\infty + \frac{bA_\lambda^4}{4}.\end{aligned}$$

Hence we get that $c_\lambda \geq m_\lambda^\infty$, which contradicts to Lemma 5.8. So $u_n \to u_\lambda$ in $H^1(\mathbb{R}^3)$. As a consequence, we obtain that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$. □

To prove that problem (1.4) has a positive ground state solution, we define

$$m := \inf\{I_1(u), u \in \mathcal{S}\}, \quad \text{where } \mathcal{S} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\}, I_1'(u) = 0\}.$$

Proof of Theorem 2.2 We complete the proof in three steps.

Step 1. $\mathcal{S} \neq \emptyset$.

By Lemma 5.7, Lemma 5.10, and Proposition 3.1, for a.e. $\lambda \in J$, there exists $u_\lambda \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that $I_\lambda'(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$. Choosing a sequence $\{\lambda_n\} \subset J$ satisfying $\lambda_n \to 1^-$, we have a sequence $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)\backslash\{0\}$ such that $I_{\lambda_n}'(u_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}$. For simplicity, denoting $\{u_n\}$ instead of $\{u_{\lambda_n}\}$, we next show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Denote

$$\begin{cases} \alpha_n \triangleq a\int_{\mathbb{R}^3}|\nabla u_n|^2, & \beta_n \triangleq \int_{\mathbb{R}^3}|u_n|^2, \qquad \mu_n \triangleq b(\int_{\mathbb{R}^3}|\nabla u_n|^2)^2, \\ \nu_n \triangleq \int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2, & \delta_n \triangleq \int_{\mathbb{R}^3}f(x)|u_n|^{p+1}, \\ \overline{\nu}_n \triangleq \int_{\mathbb{R}^3}(\nabla K, x)\phi_{u_n}u_n^2, & \overline{\delta}_n \triangleq \int_{\mathbb{R}^3}(\nabla f, x)|u_n|^{p+1}. \end{cases}$$

Then we have

$$\begin{cases} \frac{1}{2}\alpha_n + \frac{3}{2}\beta_n + \frac{1}{2}\mu_n - \frac{5\lambda_n}{4}\nu_n - \frac{3\lambda_n}{p+1}\delta_n - \frac{\lambda_n}{4}\overline{\nu}_n - \frac{\lambda_n}{p+1}\overline{\delta}_n = 0, \\ \frac{1}{2}\alpha_n + \frac{1}{2}\beta_n + \frac{1}{4}\mu_n - \frac{\lambda_n}{4}\nu_n - \frac{\lambda_n}{p+1}\delta_n = c_{\lambda_n}, \\ \alpha_n + \beta_n + \mu_n - \lambda_n\nu_n - \lambda_n\delta_n = 0. \end{cases}$$

Hence, we have

$$\alpha_n + \frac{\lambda_n}{4}\nu_n + \frac{(p-1)\lambda_n}{2(p+1)}\delta_n + \frac{\lambda_n}{2}(\nu_n + \overline{\nu}_n) + \frac{\lambda_n}{p+1}\overline{\delta}_n = 4c_{\lambda_n} \le 4c_{\frac{1}{2}}. \tag{5.23}$$

Note that α_n, β_n, μ_n are nonnegative. Conditions (f_1)-(f_2), (K_3), and (K_4) imply that δ_n, $\overline{\delta}_n$, ν_n, $\nu_n + \overline{\nu}_n$ are nonnegative too. Then we conclude that δ_n and ν_n are bounded. Moreover, by the third equation we have that $\alpha_n + \beta_n$ is bounded, that is, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Therefore, we get

$$\begin{aligned} &\lim_{n\to\infty} I_1(u_n) \\ &\quad = \lim_{n\to\infty}\left(I_{\lambda_n}(u_n) + (\lambda_n - 1)\left(\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2 + \frac{1}{p+1}\int_{\mathbb{R}^3}f(x)|u_n|^{p+1}\right)\right) \\ &\quad = \lim_{n\to\infty} c_{\lambda_n} = c_1 \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty}\langle I_1'(u_n), \varphi\rangle \\ &\quad = \lim_{n\to\infty}\langle I_{\lambda_n}'(u_n), \varphi\rangle + (\lambda_n - 1)\left(\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n\varphi + \int_{\mathbb{R}^3}f(x)|u_n|^{p-1}u_n\varphi\right) \\ &\quad = 0, \end{aligned}$$

that is, $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_1}$ sequence for I_1. By Lemma 5.10 there exists $u_0 \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that $I_1'(u_0) = 0$ and $I_1(u_0) = c_1$.

Step 2. $0 < m < \infty$.

It is clear that $m \le c_1 < \infty$. Next, we prove $m \ge 0$. For all $u \in \mathcal{S}$, we have $\langle I_1'(u), u\rangle = 0$. Then by a standard argument we get that $\|u\| \ge \delta$ for some $\delta > 0$ (see Lemma 5.4(i)). On the other hand, by the Pohožaev identity of problem (1.4),

$$\begin{aligned}
0 &= P_1(u) \\
&\triangleq \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3}u^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 - \frac{5}{4}\int_{\mathbb{R}^3}K(x)\phi_u u^2 \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}(\nabla K, x)\phi_u u^2 - \frac{3}{p+1}\int_{\mathbb{R}^3}f(x)|u|^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^3}(\nabla f, x)|u|^{p+1},
\end{aligned}$$

where $\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)u^2(y)}{|x-y|}\,dy$, we have

$$\begin{aligned}
I_1(u) &= I_1(u) - \frac{1}{8}\left[\left\langle I_1'(u), u\right\rangle + 2P_1(u)\right] \\
&= \frac{a}{4}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{8}\int_{\mathbb{R}^3}K(x)\phi_u u^2 + \frac{p-1}{8(p+1)}\int_{\mathbb{R}^3}f(x)|u|^{p+1} \\
&\quad + \frac{1}{16}\int_{\mathbb{R}^3}\bigl(K(x) + (\nabla K, x)\bigr)\phi_u u^2 + \frac{1}{4(p+1)}\int_{\mathbb{R}^3}(\nabla f, x)|u|^{p+1}.
\end{aligned}$$

From (f_1), (f_2), and (K_2)-(K_4) we infer

$$I_1(u) \ge \frac{a}{4}\int_{\mathbb{R}^3}|\nabla u|^2. \tag{5.24}$$

Therefore, we obtain that $m \ge 0$. In the following, we will prove that $m > 0$. By contradiction, let $\{u_n\}$ be a $(\mathrm{PS})_0$ sequence of I_1. Inequality (5.24) implies that $\lim_{n\to\infty}|\nabla u_n|^2 = 0$. This conclusion, combined with $\langle I_1'(u), u\rangle = 0$, implies that $\lim_{n\to\infty}u_n^2 = 0$. Therefore, we obtain $\lim_{n\to\infty}\|u_n\| = 0$, a contradiction with $\|u_n\| \ge \delta > 0$ for all n.

Step 3. m is is attained at some $u \in \mathcal{S}$.

Let $\{u_n\} \subset \mathcal{S}$ satisfy $I_1(u_n) \to m$. Using the same arguments as in Step 1, we can deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, that is, $\{u_n\}$ is a bounded $(\mathrm{PS})_m$ sequence of I_1. Similarly to the arguments in Lemma 5.10, there exists a nontrivial point $u \in H^1(\mathbb{R}^3)$ such that $I_1(u) = m$ and $I_1'(u) = 0$. By the standard regularity arguments as in the proof of Theorem 1.4 of [11] we see that u is a positive ground state solution for problem (1.4). □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FL participated in the design of the study and drafted the manuscript. SW carried out the theoretical studies and helped to draft the manuscript. Both authors read and approved the final manuscript.

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