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REVIEW

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Fractional-order differential equations with anti-periodic boundary conditions: a survey

Ravi P Agarwal[1], Bashir Ahmad[2*] and Ahmed Alsaedi[2]

*Correspondence:
bashirahmad_qau@yahoo.com
[2]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

We will present an up-to-date review on anti-periodic boundary value problems of fractional-order differential equations and inclusions. Some recent and new results on nonlinear coupled fractional differential equations supplemented with coupled anti-periodic boundary conditions will also be highlighted.

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1 Introduction

Non-integer (arbitrary) order calculus has evolved into an interesting area of research for mathematicians and modelers during the last few decades. It has been mainly due to the varied applications of fractional differential equations in applied and technical sciences. Fractional derivatives can take into account memory and hereditary properties of various materials and processes in contrast to classical ones. In [1], Mehaute found the fractional derivatives a nearly perfect tool for describing the turbulent flow in a porous medium. Nowadays, fractional-order differential and integral operators, which are nonlocal in nature, appear in mathematical models of many real world phenomena such as synchronization of chaotic systems [2, 3], anomalous diffusion [4], disease models [5–7], ecological models [8], etc.

Anomalous diffusion phenomena exhibit features different from the classical ones, for instance, the deviation of observed data in the saturated zone of an actual aquifer from simulated results for the classical advection-diffusion equation was noticed by Adams and Gelhar [9]. Some anomalous diffusion can be interpreted as slow diffusion, and it is characterized by the long-tailed profile in spatial distribution of densities with the passage of time. For more details, we refer the reader to the work presented in [10]. For anomalous diffusion, a microscopic model was proposed by the continuous-time random walk with the mean square displacement $\langle u^2(t)\rangle$ growing as t^σ, where $u(t)$, $t > 0$ is the probability density function and σ is a positive constant. The anomalous diffusion subject to this condition can be described by a macroscopic model which is known as fractional diffusion equation [11]. The case $\sigma = 1$ corresponds to the classical diffusion, and the transport phenomenon exhibits sub-diffusion for $\sigma < 1$ while super-diffusion is associated with $\sigma > 1$. Hatano and Hatano [12] used many column experiments on reactive flow in heterogeneous media to determine the value of σ for suitable simulation of the anomalous diffusion. For more details, see [13].

Wide-spread application of fractional calculus has motivated many researchers to develop the theoretical aspects of this branch of mathematical analysis. In particular, there has been shown a great interest in the study of fractional-order boundary value problems (FBVPs). The literature on FBVPs is now much enriched and contains a variety of interesting results involving different kinds of boundary conditions. For theoretical development of the topic, we refer the reader to the works [14–26] and the references cited therein.

In this survey, we will review some recent works on fractional-order anti-periodic boundary value problems and discuss some new results.

2 Some definitions and examples

Let us now recall some basic definitions of fractional derivative [27] and see how such derivatives appear in the mathematical modeling of real world problems.

Definition 2.1 The fractional integral of order r with the lower limit zero for a function $f:[0,\infty)\to R$ is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)}\int_0^t \frac{f(s)}{(t-s)^{1-r}}\,ds, \quad t>0, r>0,$$

provided the right-hand side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r)=\int_0^\infty t^{r-1}e^{-t}\,dt$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $r>0$, $n-1<r<n$, $n\in N$, is defined as

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1} f(s)\,ds,$$

where the function $f(t)$ has an absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 The Caputo derivative of order r for a function $f:[0,\infty)\to R$ with $f(t)\in C^n[0,\infty)$ is defined by

$${}^cD^r f(t) = \frac{1}{\Gamma(n-r)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}}\,ds = I^{n-r}f^{(n)}(t), \quad t>0, n-1<r<n.$$

In particular, for $n=1$, we have

$${}^cD^r f(t) = \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r} f'(s)\,ds, \quad t>0, 0<r<1,$$

which can be interpreted as the distribution of f' scaled by the factor $\frac{(t-s)^{-r}}{\Gamma(1-r)}$ over the interval $[0,t]$.

Next we present some examples of fractional differential equations.

(a) *Population model*. Consider a population with a density function $u(x,t)$ at a position $x \in R^d$ and time $t > 0$. We take account of births, deaths and migration with rate of growth $f(x,u)$, and let $K(x,y)$ denote the fraction of population that will migrate from position x to y within the time interval $(t, t + \Delta t)$. Expressing it in the mathematical form, we have

$$u_t = \int_{R^d} K(x,y)\big[u(y,t) - u(x,t)\big]\,dy + f(x,u). \tag{2.1}$$

Assume that $K \geq 0$ and the range of migration is not too large. Choose

$$K(x,y) = \frac{1}{\epsilon^d} k\left(\frac{y-x}{\epsilon}\right)$$

with $k \in L^1(R^d)$. There may be several cases for k, for instance, let k be compactly supported or at least have bounded third moment. Then, for $(y-x)/\epsilon = z$, the integral on the right-hand side of (2.1) takes the form

$$\frac{1}{\epsilon^{d-1}} \int_{R^d} k\big(|z|\big)\big[u(x+\epsilon z) - u(x)\big]\,dz.$$

In case we select

$$k(x,y) = \left\| \frac{y-x}{\epsilon} \right\|^{-d-2\alpha}, \qquad \|y-x\| \geq \epsilon,$$

then the right-hand side of (2.1) takes the following form:

$$\epsilon^{2\alpha} \int_{\|y-x\|\geq\epsilon} \frac{u(y)-u(x)}{\|y-x\|^{d+2\alpha}}\,dy,$$

which, on taking the limit $\epsilon \to 0$, yields the following form of (2.1):

$$\partial_t u = (-\Delta)^\alpha u + f(x,u).$$

(b) *Local fractional versions of the Korteweg-de Vries equation.* Based on local fractional conservation laws of mass, energy and momentum in fractal media, linear and nonlinear local fractional versions of the Korteweg-de Vries equation describing fractal waves on shallow water surfaces, derived in [28], are respectively given by

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} + \frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} = 0, \qquad \frac{\partial^\alpha \eta}{\partial t^\alpha} - \eta\frac{\partial^\alpha \eta}{\partial x^\alpha} + \frac{\partial^{3\alpha} \eta}{\partial x^{3\alpha}} = 0.$$

(c) *Polarographic equation* [29]. Polarographic equation, examined by Wiener in [30, 31] under the assumption that the derivative of non-integer order appearing in the equation is in the sense of Hadamard, is

$$y^{1/2}(x) - \nu x^\beta y(x) = x^{-1/2}, \quad x > 0, -1/2 < \beta \leq 0, \nu \in \mathbb{R}^+. \tag{2.2}$$

For some recent works on Hadamard fractional differential equations, see the text [32].

3 Classical anti-periodic boundary conditions

In [33], Ahmad and Nieto initiated the study of fractional-order boundary value problems by considering the following problem:

$$\begin{cases} {}^cD^q u(t) = f(t,u(t)), & t \in [0,T], 1 < q \le 2, \\ u(0) = -u(T), & u'(0) = -u'(T), \end{cases} \tag{3.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and T is a fixed positive constant.

Some existence results for problem (3.1) were obtained by transforming the problem into an equivalent fixed point problem

$$u = \mathcal{F}(u),$$

where $\mathcal{F}$ is given by

$$\begin{aligned}(\mathcal{F}u)(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,u(s)\bigr)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f\bigl(s,u(s)\bigr)\,ds \\ & + \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f\bigl(s,u(s)\bigr)\,ds, \quad t \in [0,T].\end{aligned}$$

Here we mention two results from [33] which were proved by applying fixed point theorems due to Altman and Schauder.

Theorem 3.1 *Assume that there exist constants* $0 \le \kappa < \frac{4\Gamma(q+1)}{(6+q)}$ *and* $M > 0$ *such that* $|f(t,u)| \le \frac{\kappa}{T^q}|u| + M$ *for all* $t \in [0,T]$, $u \in C[0,T]$. *Then the anti-periodic boundary value problem* (3.1) *has at least one solution on* $[0,T]$.

Theorem 3.2 *Suppose that* f *is of class* C^1 *in the second variable and there exists a constant* $0 \le M_2 < \frac{4\Gamma(q+1)}{T^q(6+q)}$ *such that* $|f_u(t,u)| \le M_2$ *for all* $t \in [0,T]$, $u \in C[0,T]$. *Then problem* (3.1) *has at least one solution on* $[0,T]$.

Ahmad and Otero-Espinar [34] considered the inclusions (multivalued) case of problem (3.1) by replacing $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ with $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ and proved the following result by means of Bohnenblust-Karlin's fixed point theorem for the resulting problem:

$$\begin{cases} {}^cD^q u(t) \in F(t,x(t)), & t \in [0,T], 1 < q \le 2, \\ u(0) = -u(T), & u'(0) = -u'(T). \end{cases} \tag{3.2}$$

Theorem 3.3 *Suppose that the following assumptions hold*:

(A_1) *Let* $F : [0,T] \times \mathbb{R} \to \mathrm{BCC}(\mathbb{R})$; $(t,x) \to f(t,x)$ *be measurable with respect to* t *for each* $x \in \mathbb{R}$, *u.s.c. with respect to* x *for a.e.* $t \in [0,T]$, *and for each fixed* $x \in \mathbb{R}$, *the set* $S_{F,y} := \{f \in L^1([0,T],\mathbb{R}) : f(t) \in F(t,x)$ *for a.e.* $t \in [0,T]\}$ *is nonempty* ($\mathrm{BCC}(\mathbb{R})$ *denotes the set of all nonempty bounded, closed and convex subsets of* $\mathbb{R}$);

(A_2) *For each* $r > 0$, *there exists a function* $m_r \in L^1([0,T],\mathbb{R}_+)$ *such that* $\|F(t,x)\| = \sup\{|v| : v(t) \in F(t,x)\} \leq m_r(t)$ *for each* $(t,x) \in [0,T] \times \mathbb{R}$ *with* $|x| \leq r$, *and*

$$\liminf_{r\to+\infty}\left(\frac{\int_0^T m_r(t)\,dt}{r}\right) = \gamma < \infty.$$

Then the anti-periodic inclusion problem (3.2) *has at least one solution on* $[0,T]$ *provided that* $\gamma < 4\Gamma(q)/(5+q)T^{q-1}$.

Alsaedi [35] studied the following anti-periodic boundary value problem of integro-differential equations of the form:

$$\begin{cases} {}^cD^q u(t) = f(t,u(t),\int_0^t \gamma(t,s)u(s)\,ds), & t\in[0,T], 1<q\leq 2,\\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{cases} \tag{3.3}$$

where $\gamma : [0,t] \times [0,T] \to [0,\infty)$ is a given function.

In [36], Benchohra et al. studied (3.1) with the nonlinearity of the form $f(t,u(t),{}^cD^{q-1}u(t))$ and investigated the existence of solutions of the resulting problem by means of Banach's fixed point theorem and Schauder's fixed point theorem.

Ahmad and Nieto [37] studied an anti-periodic boundary value problem for impulsive fractional differential equations:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \quad 1<q\leq 2, t\in J_1 = [0,T]\setminus\{t_1,t_2,\ldots,t_p\},\\ \Delta x(t_k) = \mathcal{I}_k(x(t_k^-)), \quad \Delta x'(t_k) = \mathcal{J}_k(x(t_k^-)), \quad t_k\in(0,T), k=1,2,\ldots,p,\\ x(0) = -x(T), \quad x'(0) = -x'(T), \end{cases} \tag{3.4}$$

where ${}^cD$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $J = [0,T]$, $\mathcal{I}_k, \mathcal{J}_k : \mathbb{R} \to \mathbb{R}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h\to 0^+} x(t_k+h)$, $x(t_k^-) = \lim_{h\to 0^-} x(t_k+h)$, $k = 1,2,\ldots,p$ for $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$.

The following results were obtained for problem (3.4).

Theorem 3.4 *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a jointly continuous function and* $I_k, J_k : \mathbb{R} \to \mathbb{R}$ *be continuous functions. Assume that there exist positive constants* L_1, L_2, L_3, M_2, M_3 *such that*

(A_1) $\|f(t,x) - f(t,y)\| \leq L_1\|x-y\|$, $\forall t \in [0,T]$, $x,y \in \mathbb{R}$;

(A_2) $\|\mathcal{I}_k(x) - \mathcal{I}_k(y)\| \leq L_2\|x-y\|$, $\|\mathcal{J}_k(x) - \mathcal{J}_k(y)\| \leq L_3\|x-y\|$ *with* $\|\mathcal{I}_k(x)\| \leq M_2$, $\|\mathcal{J}_k(x)\| \leq M_3$, $\forall x,y \in \mathbb{R}$, $k = 1,2,\ldots,p$.

Further $L_1T^q(\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)}) + \frac{p}{4}(6L_2 + 7TL_3) < 1$ *with* $L_1 \leq \frac{1}{2}[T^q\{\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)}\}]^{-1}$. *Then the impulsive anti-periodic boundary value problem* (3.4) *has a unique solution on* J.

Theorem 3.5 *Let* (A_1) *and* (A_2) *assumed in Theorem* 3.4 *hold with* $\frac{p}{4}(6L_2 + 7TL_3) < 1$ *and* $\|f(t,x)\| \leq \mu(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, *where* $\mu \in C([0,T],\mathbb{R}^+)$. *Then the boundary value problem* (3.4) *has at least one solution on* $[0,T]$.

The inclusions case of problem (3.4) was discussed in [38].

In [39], by applying the contraction mapping principle and Krasnoselskii's fixed point theorem, the author proved the existence and uniqueness results for the following anti-periodic fractional boundary value problem:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), & t \in [0,T], T > 0, 2 < q \le 3, \\ x(0) = -x(T), \quad x'(0) = -x'(T), & x''(0) = -x''(T), \end{cases} \tag{3.5}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q.

Later, Cernea [40] considered the inclusion case of problem (3.5) and obtained several results for convex and non-convex values of the multivalued map by applying nonlinear alternative of Leray-Schauder type, the Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and the Covitz and Nadler set-valued contraction principle.

Wang et al. [41] obtained some existence and uniqueness results for problem (3.5) with impulsive conditions given by

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t)), \quad 2 < \alpha \le 3, t \in J', \\ \Delta u(t_k) = Q_k(u(t_k)), \qquad \Delta u'(t_k) = I_k(u(t_k)), \\ \Delta u''(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \dots, p, \\ u(0) = -u(1), \qquad u'(0) = -u'(1), \qquad u''(0) = -u''(1), \end{cases}$$

where ${}^cD$ is the Caputo fractional derivative, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $Q_k, I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $J = [0,1]$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $J' = J \setminus \{t_1, t_2, \dots, t_p\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+)$ and $u(t_k^-)$ denoting the right and the left limit of $u(t)$ at $t = t_k$, $k = 1, 2, \dots, p$, respectively, and $\Delta u'(t_k)$ and $\Delta u''(t_k)$ have a similar meaning for $u'(t)$ and $u''(t)$, respectively.

In [42], Agarwal and Ahmad developed the existence theory for the following anti-periodic boundary value problems of fractional differential equations and inclusions of order $\alpha \in (3, 4]$:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0,T], T > 0, 3 < q \le 4, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \\ x''(0) = -x''(T), \qquad x'''(0) = -x'''(T), \end{cases} \tag{3.6}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function and

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0,T], T > 0, 3 < q \le 4, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \\ x''(0) = -x''(T), \qquad x'''(0) = -x'''(T). \end{cases} \tag{3.7}$$

In (3.7), $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

In order to transform problem (3.6) into an equivalent fixed point problem, an operator $\mathcal{G} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ was defined as

$$\begin{aligned}(\mathcal{G}x)(t) = {}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f\bigl(s,x(s)\bigr)\,ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f\bigl(s,x(s)\bigr)\,ds \\ &+ \frac{(6t^2T-4t^3-T^3)}{48}\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f\bigl(s,x(s)\bigr)\,ds, \quad t\in[0,T]. \end{aligned} \tag{3.8}$$

The following results were obtained for problems (3.6) and (3.7).

Theorem 3.6 *Assume that there exists a positive constant* L_1 *such that* $|f(t,x)| \le L_1$ *for* $t\in[0,T]$, $x\in\mathbb{R}$. *Then problem* (3.6) *has at least one solution.*

Theorem 3.7 *Let* $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ *and* $\lim_{x\to 0}\frac{f(t,x)}{x} = 0$. *Then problem* (3.6) *has at least one solution.*

Theorem 3.8 *Let* $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function satisfying the following assumptions*:

(B$_1$) $|f(t,x)-f(t,y)| \le L|x-y|$, $\forall t\in[0,T]$, $x,y\in\mathbb{R}$;
(B$_2$) $\|f(t,x)\| \le \mu(t)$, $\forall (t,x)\in[0,1]\times\mathbb{R}$, *and* $\mu\in C([0,T],\mathbb{R}^+)$.

Then the anti-periodic boundary value problem (3.6) *has at least one solution on* $[0,T]$ *if*

$$\frac{LT^q}{\Gamma(q+1)}\left(1+\frac{q(q^2+11)}{24}\right) < 1.$$

Theorem 3.9 *Assume that* $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function satisfying the assumption* (B$_1$) *with*

$$L \le \frac{\Gamma(q+1)}{T^q(3+\frac{q(q^2+11)}{24})}.$$

Then the anti-periodic boundary value problem (3.6) *has a unique solution.*

Theorem 3.10 *Let* $f : [0,T]\times\mathbb{R}\to\mathbb{R}$. *Assume that there exist constants* $0\le\kappa<\frac{1}{\delta}$, *where*

$$\delta = \frac{T^q}{\Gamma(q+1)}\left(\frac{3}{2}+\frac{q(q^2+11)}{48}\right),$$

and $M>0$ *such that* $|f(t,x)| \le \kappa|x|+M$ *for all* $t\in[0,T]$, $x\in\mathbb{R}$. *Then the boundary value problem* (3.6) *has at least one solution.*

Theorem 3.11 *Assume that*

(H$_1$) $F : [0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is* L^1-*Carathéodory and has convex values*;

(H_2) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty) \quad \textit{for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_3) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\gamma_1 \psi(M)\|p\|} > 1, \tag{3.9}$$

where

$$\gamma_1 = \frac{T^q(3 + \frac{q(q^2+11)}{24})}{\Gamma(q+1)}.$$

Then the boundary value problem (3.7) *has at least one solution on* $[0,T]$.

Theorem 3.12 *Assume that the following conditions hold*:

(H_4) $F : [0,T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x) : [0,T] \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

(H_5) $H_d(F(t,x),F(t,\bar{x})) \le m(t)|x - \bar{x}|$ *for almost all* $t \in [0,T]$ *and* $x,\bar{x} \in \mathbb{R}$ *with* $m \in C([0,T],\mathbb{R}^+)$ *and* $d(0,F(t,0)) \le m(t)$ *for almost all* $t \in [0,T]$.

Then the boundary value problem (3.7) *has at least one solution on* $[0,T]$ *if*

$$\frac{T^q(3 + \frac{q(q^2+11)}{24})}{\Gamma(q+1)} < 1.$$

Alsaedi et al. [43] found further insight into the characteristics of fractional-order anti-periodic boundary value problems by extending problem (3.6) to the order $\alpha \in (4,5]$.

In [44], the authors studied a boundary value problem of fractional differential inclusions with anti-periodic type integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t)x(t) \in F(t,x(t)), & 0 < t < T, 2 < q \le 3, \\ x^{(j)}(0) - \lambda_j x^{(j)}(T) = \mu_j \int_0^T g_j(s,x(s))\,ds, & j = 0,1,2, \end{cases} \tag{3.10}$$

where ${}^cD^q$ denotes the Caputo derivative of fractional order q, $x^{(j)}(\cdot)$ denotes jth derivative of $x(\cdot)$ with $x^{(0)}(\cdot) = x(\cdot)$, $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $g_j : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_j, \mu_j \in \mathbb{R}$ ($\lambda_j \neq 1$). The existence of solutions for problem (3.10) was investigated for convex as well as nonconvex valued maps by using nonlinear alternative of Leray-Schauder type and a fixed point theorem for contraction multivalued maps due to Covitz and Nadler, respectively.

In [45], Agarwal et al. introduced nonlocal (parametric type) anti-periodic conditions involving a nonlocal intermediate point $0 < a < T$ and the right end point ($t = T$). This consideration led to a new kind of anti-periodic conditions: $x(a) = -x(T)$, $x'(a) = -x'(T)$. With these conditions, the following anti-periodic boundary value problem was studied:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), & t \in [0,T], T > 0, 1 < q \le 2, \\ x(a) = -x(T), \quad x'(a) = -x'(T), & 0 < a < T, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given function. Observe that the interval $[0,T]$ can be replaced with an interval of the form $(-\infty,T]$ with a in it. This means that the anti-periodic phenomena can start from an arbitrary point in $(-\infty,T)$.

3.1 An interesting analogy

Here we describe the relationship between the Green's functions of lower- and higher-order anti-periodic fractional BVPs. Note that the underbraced term in the Green's function of an anti-periodic fractional BVP indicates the additional term to the Green's function of the immediate lower-order anti-periodic fractional BVP.

(a) The Green's function for the problem

$$\begin{cases} {}^cD^q x(t) = f_1(t), & 0 < q \le 1, t \in [0,T], \\ x(0) = -x(T) \end{cases}$$

is

$$G(t,s,q) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{(T-s)^{q-1}}{2\Gamma(q)}, & s \le t, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)}, & t \le s. \end{cases}$$

(b) The Green's function for the problem

$$\begin{cases} {}^cD^q x(t) = f_1(t), & 1 < q \le 2, t \in [0,T], \\ x(0) = -x(T), & x'(0) = -x'(T) \end{cases}$$

is

$$G(t,s,q) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{(T-s)^{q-1}}{2\Gamma(q)} + \underbrace{\frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)}}, & s \le t, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \underbrace{\frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)}}, & t \le s. \end{cases} \tag{3.11}$$

(c) The Green's function for the problem

$$\begin{cases} {}^cD^q x(t) = f_1(t), & 2 < q \le 3, t \in [0,T], \\ x(0) = -x(T), & x'(0) = -x'(T), \quad x''(0) = -x''(T) \end{cases}$$

is

$$G(t,s,q) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)} \\ \quad + \underbrace{\frac{1}{4}(-t^2+Tt)\frac{(T-s)^{q-3}}{\Gamma(q-2)}}, & s \le t, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)} \\ \quad + \underbrace{\frac{1}{4}(-t^2+Tt)\frac{(T-s)^{q-3}}{\Gamma(q-2)}}, & t \le s. \end{cases}$$

(d) The Green's function for the problem

$$\begin{cases} {}^cD^q x(t) = f_1(t), \quad 3 < q \leq 4, t \in [0,T], \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \qquad x'''(0) = -x'''(T) \end{cases}$$

is

$$G(t,s,q) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)} \\ \quad + \frac{1}{4}(-t^2+Tt)\frac{(T-s)^{q-3}}{\Gamma(q-2)} + \underbrace{\frac{1}{48}(-4t^3+6Tt^2-T^3)\frac{(T-s)^{q-4}}{\Gamma(q-3)}}, \\ \quad s \leq t, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)} \\ \quad + \frac{1}{4}(-t^2+Tt)\frac{(T-s)^{q-3}}{\Gamma(q-2)} + \underbrace{\frac{1}{48}(-4t^3+6Tt^2-T^3)\frac{(T-s)^{q-4}}{\Gamma(q-3)}}, \\ \quad t \leq s. \end{cases}$$

(e) For the anti-periodic boundary value problem of fractional differential equations of order $q \in (4,5]$,

$$\begin{cases} {}^cD^q x(t) = f_1(t), \quad 4 < q \leq 5, t \in [0,T], \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \qquad x''(0) = -x''(T), \qquad x'''(0) = -x'''(T), \\ x^{(4)}(0) = -x^{(4)}(T), \end{cases}$$

the Green's function is

$$G(t,s,q) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)} \\ \quad + \frac{1}{4}(-t^2+Tt)\frac{(T-s)^{q-3}}{\Gamma(q-2)} + \frac{1}{48}(-4t^3+6Tt^2-T^3)\frac{(T-s)^{q-4}}{\Gamma(q-3)} \\ \quad + \underbrace{\frac{1}{48}(-t^4+2Tt^3-T^3t)\frac{(T-s)^{q-5}}{\Gamma(q-4)}}, \quad s \leq t \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{1}{4}(-2t+T)\frac{(T-s)^{q-2}}{\Gamma(q-1)} \\ \quad + \frac{1}{4}(-t^2+Tt)\frac{(T-s)^{q-3}}{\Gamma(q-2)} + \frac{1}{48}(-4t^3+6Tt^2-T^3)\frac{(T-s)^{q-4}}{\Gamma(q-3)} \\ \quad + \underbrace{\frac{1}{48}(-t^4+2Tt^3-T^3t)\frac{(T-s)^{q-5}}{\Gamma(q-4)}}, \quad t \leq s. \end{cases} \tag{3.12}$$

3.2 Further generalization of classical anti-periodic problems - new results

Consider a Caputo type fractional differential equation

$${}^cD^q x(t) = f\big(t, x(t)\big), \quad 5 < q \leq 6, t \in [0,T], \tag{3.13}$$

supplemented with the following anti-periodic boundary conditions:

$$\begin{cases} x(0) = -x(T), \qquad x'(0) = -x'(T), \\ x''(0) = -x''(T), \qquad x'''(0) = -x'''(T), \\ x^{(4)}(0) = -x^{(4)}(T), \qquad x^{(5)}(0) = -x^{(5)}(T), \end{cases} \tag{3.14}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function.

By means of standard tools of fractional calculus, we can express the solution of problem (3.13)-(3.14) in terms of the Green's function as follows:

$$x(t) = \int_0^T G(t,s,q) f_1(s)\,ds,$$

where

$$G(t,s,q) = \begin{cases} \underbrace{\begin{aligned} &\tfrac{(t-s)^{q-1}}{\Gamma(q)} - \tfrac{(T-s)^{q-1}}{2\Gamma(q)} + \tfrac{1}{4}(-2t+T)\tfrac{(T-s)^{q-2}}{\Gamma(q-1)} \\ &\quad + \tfrac{1}{4}(-t^2+Tt)\tfrac{(T-s)^{q-3}}{\Gamma(q-2)} + \tfrac{1}{48}(-4t^3+6Tt^2-T^3)\tfrac{(T-s)^{q-4}}{\Gamma(q-3)} \\ &\quad + \tfrac{1}{48}(-t^4+2Tt^3-T^3t)\tfrac{(T-s)^{q-5}}{\Gamma(q-4)} \\ &\quad + \tfrac{1}{240}(-t^5+\tfrac{5}{2}Tt^4-\tfrac{5}{2}T^3t^2-5T^4t+3T^5)\tfrac{(T-s)^{q-6}}{\Gamma(q-5)}, \end{aligned}} \\ \quad s \le t, \\ \underbrace{\begin{aligned} &-\tfrac{(T-s)^{q-1}}{2\Gamma(q)} + \tfrac{1}{4}(-2t+T)\tfrac{(T-s)^{q-2}}{\Gamma(q-1)} \\ &\quad + \tfrac{1}{4}(-t^2+Tt)\tfrac{(T-s)^{q-3}}{\Gamma(q-2)} + \tfrac{1}{48}(-4t^3+6Tt^2-T^3)\tfrac{(T-s)^{q-4}}{\Gamma(q-3)} \\ &\quad + \tfrac{1}{48}(-t^4+2Tt^3-T^3t)\tfrac{(T-s)^{q-5}}{\Gamma(q-4)} \\ &\quad + \tfrac{1}{240}(-t^5+\tfrac{5}{2}Tt^4-\tfrac{5}{2}T^3t^2-5T^4t+3T^5)\tfrac{(T-s)^{q-6}}{\Gamma(q-5)}, \end{aligned}} \\ \quad t \le s. \end{cases} \tag{3.15}$$

Let $\mathcal{A} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$.

Associated with problem (3.13)-(3.14), we define an operator $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ by

$$\begin{aligned} (\mathcal{H}x)(t) = {}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds \\ & + \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f\bigl(s,x(s)\bigr)\,ds + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f\bigl(s,x(s)\bigr)\,ds \\ & + \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f\bigl(s,x(s)\bigr)\,ds + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f\bigl(s,x(s)\bigr)\,ds \\ & + \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)} f\bigl(s,x(s)\bigr)\,ds, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &\zeta_1(t) = \frac{1}{4}(-2t+T), \qquad \zeta_2(t) = \frac{1}{4}\bigl(-t^2+Tt\bigr), \\ &\zeta_3(t) = \frac{1}{48}\bigl(-4t^3+6Tt^2-T^3\bigr), \qquad \zeta_4(t) = \frac{1}{48}\bigl(-t^4+2Tt^3-T^3t\bigr), \\ &\zeta_5(t) = \frac{1}{240}\biggl(-t^5+\frac{5}{2}Tt^4-\frac{5}{2}T^3t^2-5T^4t+3T^5\biggr). \end{aligned} \tag{3.17}$$

Notice that problem (3.13)-(3.14) has solutions if and only if the operator $\mathcal{H}$ has fixed points.

For the sake of computational convenience, we set

$$\Lambda = \max_{t\in[0,T]} \left|\overline{\Lambda}(t)\right|, \tag{3.18}$$

where

$$\begin{aligned}\overline{\Lambda}(t) &= \frac{t^q}{\Gamma(q+1)} - \frac{T^q}{2\Gamma(q+1)} + \frac{1}{4}(-2t+T)\frac{T^{q-1}}{\Gamma(q)} + \frac{1}{4}\left(-t^2+Tt\right)\frac{T^{q-2}}{\Gamma(q-1)} \\ &\quad + \frac{1}{48}\left(-4t^3+6Tt^2-T^3\right)\frac{T^{q-3}}{\Gamma(q-2)} + \frac{1}{48}\left(-t^4+2Tt^3-T^3t\right)\frac{T^{q-4}}{\Gamma(q-3)} \\ &\quad + \frac{1}{240}\left(-t^5+\frac{5}{2}Tt^4-\frac{5}{2}T^3t^2-5T^4t+3T^5\right)\frac{T^{q-5}}{\Gamma(q-4)}.\end{aligned}$$

Now we present existence results for problem (3.13)-(3.14). Our first result is based on Banach's fixed point theorem.

Theorem 3.13 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following condition holds*:

(H_1) $|f(t,x)-f(t,y)|\le \ell|x-y|$, $\forall t\in[0,T]$, $x,y\in\mathbb{R}$, $\ell>0$.

Then problem (3.13)-(3.14) *has a unique solution if* $\ell\Lambda<1$, *where* Λ *is given by* (3.18).

Proof Setting $\sup_{t\in[0,T]}|f(t,0)|=\varrho$, $\varepsilon>\Lambda\varrho(1-\Lambda\ell)^{-1}$, we show that $\mathcal{H}B_\varepsilon\subset B_\varepsilon$, where the operator $\mathcal{H}$ is given by (3.16) and $B_\varepsilon=\{x\in\mathcal{A}:\|x\|\le\varepsilon\}$. Now, for $x\in B_\varepsilon$, $t\in[0,T]$, using

$$\begin{aligned}\left|f\left(t,x(t)\right)\right| &= \left|f\left(t,x(t)\right)-f(t,0)+f(t,0)\right| \\ &\le \left|f\left(t,x(t)\right)-f(t,0)\right|+\left|f(t,0)\right|\le \ell\varepsilon+\varrho,\end{aligned}$$

and (3.18), we get

$$\begin{aligned}\left\|(\mathcal{H}x)\right\| &\le \sup_{t\in[0,T]}\left\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left|f\left(s,x(s)\right)\right|ds+\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}\left|f\left(s,x(s)\right)\right|ds\right. \\ &\quad +\zeta_1(t)\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}\left|f\left(s,x(s)\right)\right|ds+\zeta_2(t)\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}\left|f\left(s,x(s)\right)\right|ds \\ &\quad +\zeta_3(t)\int_0^T\frac{(T-s)^{q-4}}{\Gamma(q-3)}\left|f\left(s,x(s)\right)\right|ds+\zeta_4(t)\int_0^T\frac{(T-s)^{q-5}}{\Gamma(q-4)}\left|f\left(s,x(s)\right)\right|ds \\ &\quad \left.+\zeta_5(t)\int_0^T\frac{(T-s)^{q-6}}{\Gamma(q-5)}\left|f\left(s,x(s)\right)\right|ds\right\} \\ &\le (\ell\varepsilon+\varrho)\sup_{t\in[0,T]}\left\{\frac{t^q}{\Gamma(q+1)}+\frac{T^q}{2\Gamma(q+1)}+\overline{\zeta}_1\frac{T^{q-1}}{\Gamma(q)}+\overline{\zeta}_2\frac{T^{q-2}}{\Gamma(q-1)}\right. \\ &\quad \left.+\overline{\zeta}_3\frac{T^{q-3}}{\Gamma(q-2)}+\overline{\zeta}_4\frac{T^{q-4}}{\Gamma(q-3)}+\overline{\zeta}_5\frac{T^{q-5}}{\Gamma(q-4)}\right\}\le(\ell\varepsilon+\varrho)\Lambda\le\varepsilon,\end{aligned}$$

which implies that $\mathcal{H}B_\varepsilon\subset B_\varepsilon$.

Next, for $x, y \in \mathbb{R}$ and for each $t \in [0, T]$, we obtain

$$
\begin{aligned}
\|\mathcal{H}x - \mathcal{H}y\| \le \sup_{t\in[0,T]} \Big\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\
& + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\
& + \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\
& + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\
& + \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\
& + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \\
& + \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds \Big\} \\
\le \ell\|x-y\| \sup_{t\in[0,T]} \Big\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\,ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\,ds \\
& + \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\,ds + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\,ds \\
& + \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\,ds + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}\,ds \\
& + \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)}\,ds \Big\} \le \ell\Lambda\|x-y\|,
\end{aligned}
$$

where Λ is given by (3.18). Then we deduce from the assumption $\ell\Lambda < 1$ that the operator $\mathcal{H}$ is a contraction. Therefore, it follows by Banach's fixed point theorem that problem (3.13)-(3.14) has a unique solution on $[0, T]$. This completes the proof. □

In the next result, we use Krasnoselskii's fixed point theorem [46].

Lemma 3.14 (Krasnoselskii) *Let $\mathcal{P}$ be a closed, convex, bounded and nonempty subset of a Banach space X. Let ψ_1, ψ_2 be operators such that* (i) *$\psi_1\nu_1 + \psi_2\nu_2 \in \mathcal{P}$ whenever $\nu_1, \nu_2 \in \mathcal{P}$*; (ii) *$\psi_1$ is compact and continuous; and* (iii) *ψ_2 is a contraction mapping. Then there exists $\omega \in \mathcal{P}$ such that $\omega = \psi_1\omega + \psi_2\omega$.*

Theorem 3.15 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (H_1) *and*

(H_2) $|f(t,x)| \le \gamma(t)$, $\forall(t,x) \in [0,T] \times \mathbb{R}$, *and* $\gamma \in C([0,T],\mathbb{R}^+)$.

Then there exists at least one solution for problem (3.13)-(3.14) *on* $[0, T]$ *if*

$$
\begin{aligned}
& \frac{\ell T^q}{\Gamma(q+1)} \bigg[\frac{1}{2} + \overline{\zeta}_1 T^{-1} q + \overline{\zeta}_2 T^{-2} q(q-1) + \overline{\zeta}_3 T^{-3} q(q-1)(q-2) \\
& \quad + \overline{\zeta}_4 T^{-4} q(q-1)(q-2)(q-3) + \overline{\zeta}_5 T^{-5} q(q-1)(q-2)(q-3)(q-4) \bigg] < 1. \qquad (3.19)
\end{aligned}
$$

Proof Letting $\delta \geq \|\gamma\|\Lambda$, ($\|\gamma\| = \max_{t\in[0,T]}|\gamma(t)|$), we consider $B_\delta = \{x \in \mathcal{A} : \|x\| \leq \delta\}$ and define the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on B_δ as

$$\begin{aligned}
(\mathcal{H}_1 x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds, \\
(\mathcal{H}_2 x)(t) &= -\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds \\
&\quad + \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f\bigl(s,x(s)\bigr)\,ds + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f\bigl(s,x(s)\bigr)\,ds \\
&\quad + \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f\bigl(s,x(s)\bigr)\,ds + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f\bigl(s,x(s)\bigr)\,ds \\
&\quad + \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)} f\bigl(s,x(s)\bigr)\,ds.
\end{aligned}$$

It is easy to show that $\|(\mathcal{H}_1\widetilde{x}) + (\mathcal{H}_2\widetilde{y})\| \leq \|\gamma\|\Lambda \leq \delta$ for $\widetilde{x},\widetilde{y} \in B_\delta$, where Λ is given by (3.18). Hence, $\mathcal{H}_1\widetilde{x} + \mathcal{H}_2\widetilde{x} \in B_\delta$.

Next, we will show that the operator $\mathcal{H}_2$ is a contraction. For $x, y \in \mathbb{R}$, $t \in [0,T]$, we can obtain

$$\begin{aligned}
\|\mathcal{H}_2 x - \mathcal{H}_2 y\| &\leq \sup_{t\in[0,T]}\Biggl\{\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad + \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad + \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad + \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds\Biggr\} \\
&\leq \ell\|x-y\|\sup_{t\in[0,T]}\Biggl\{\frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\,ds + \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\,ds \\
&\quad + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\,ds + \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)}\,ds \\
&\quad + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)}\,ds + \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)}\,ds\Biggr\} \\
&\leq \frac{\ell\|x-y\|T^q}{\Gamma(q+1)}\biggl[\frac{1}{2} + \overline{\zeta}_1 T^{-1}q + \overline{\zeta}_2 T^{-2}q(q-1) + \overline{\zeta}_3 T^{-3}q(q-1)(q-2) \\
&\quad + \overline{\zeta}_4 T^{-4}q(q-1)(q-2)(q-3) + \overline{\zeta}_5 T^{-5}q(q-1)(q-2)(q-3)(q-4)\biggr].
\end{aligned}$$

In view of assumption (3.19), the last inequality implies that $\mathcal{H}_2$ is a contraction.

Now, we will show that $\mathcal{H}_1$ is compact and continuous. The operator $\mathcal{H}_1$ is continuous by the continuity of f. Also, $\mathcal{H}_1$ is uniformly bounded on B_δ as $\|\mathcal{H}_1 x\| \leq \frac{\|\gamma\|}{\Gamma(q+1)}$. Moreover, for $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}\big|(\mathcal{H}_1 x)(t_2) - (\mathcal{H}_1 x)(t_1)\big| &\leq \left|\int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds - \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right| \\ &\leq \frac{\|\gamma\|}{\Gamma(q+1)}\big(\big|t_2^q - t_1^q\big| + 2(t_2-t_1)^q\big),\end{aligned}$$

which as $(t_2 - t_1) \to 0$ tends to zero independent of x. So, $\mathcal{H}_1$ is relatively compact on B_δ. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{H}_1$ is compact on B_δ. Therefore, all the conditions of Krasnoselskii's fixed point theorem are satisfied; and consequently, problem (3.13)-(3.14) has at least one solution on $[0, T]$. This completes the proof. □

The next result is based on the Leray-Schauder nonlinear alternative [47].

Lemma 3.16 (Nonlinear alternative for single-valued maps) *Let E be a Banach space, E_1 be a closed, convex subset of E, V be an open subset of E_1 and $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \to E_1$ is a continuous, compact (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of E_1) map. Then either*

(i) *$\mathcal{U}$ has a fixed point in $\overline{V}$, or*

(ii) *there is $x \in \partial V$ (the boundary of V in E_1) and $\kappa \in (0,1)$ with $x = \kappa\mathcal{U}(x)$.*

Theorem 3.17 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_3) *there exist a function $p \in \mathcal{C}([0,T], \mathbb{R}^+)$ and a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \leq p(t)\varphi(\|x\|)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$;*

(H_4) *there exists a constant $M > 0$ such that $\frac{M}{\varphi(M)\|p\|}\Lambda^{-1} > 1$.*

Then problem (3.13)-(3.14) *has at least one solution on* $[0, T]$.

Proof First, we will show that the operator $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ defined by (3.16) maps bounded sets into bounded sets in $\mathcal{A}$. Let $B_\epsilon = \{x \in \mathcal{A} : \|x\| \leq \epsilon\}$ for $\epsilon > 0$ be a bounded set in $\mathcal{A}$. Then, in view of (H_3), we obtain, for $x \in B_\epsilon$,

$$\begin{aligned}\big|(\mathcal{H}x)(t)\big| \leq{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\varphi\big(\|x\|\big)\,ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} p(s)\varphi\big(\|x\|\big)\,ds \\ &+ \zeta_1(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} p(s)\varphi\big(\|x\|\big)\,ds + \zeta_2(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} p(s)\varphi\big(\|x\|\big)\,ds \\ &+ \zeta_3(t)\int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} p(s)\varphi\big(\|x\|\big)\,ds + \zeta_4(t)\int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} p(s)\varphi\big(\|x\|\big)\,ds \\ &+ \zeta_5(t)\int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)} p(s)\varphi\big(\|x\|\big)\,ds \\ \leq{}& \frac{\varphi(\epsilon)\|p\|T^q}{\Gamma(q+1)}\bigg[\frac{3}{2} + \overline{\zeta}_1 T^{-1}q + \overline{\zeta}_2 T^{-2}q(q-1) \\ &+ \overline{\zeta}_3 T^{-3}q(q-1)(q-2) + \overline{\zeta}_4 T^{-4}q(q-1)(q-2)(q-3) \\ &+ \overline{\zeta}_5 T^{-5}q(q-1)(q-2)(q-3)(q-4)\bigg].\end{aligned}$$

Next, it will be shown that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{A}$. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $x \in B_\epsilon$. Then

$$\begin{aligned}
&\big|(\mathcal{H}x)(t_2) - (\mathcal{H}x)(t_1)\big| \\
&\quad\le \frac{\varphi(\epsilon)\|p\|}{\Gamma(q+1)}\bigg[\big|t_2^q - t_1^q\big| + 2(t_2-t_1)^q + \frac{1}{2}(t_2-t_1)T^{q-1}q \\
&\qquad + \frac{1}{4}(t_2-t_1)\big(T-(t_2+t_1)\big)T^{q-2}q(q-1) \\
&\qquad + (t_2-t_1)\bigg(\frac{1}{8}T(t_2+t_1) + \frac{1}{12}\big(t_2^2+t_1t_2+t_1^2\big)\bigg)T^{q-3}q(q-1)(q-2) \\
&\qquad + (t_2-t_1)\bigg(\frac{1}{48}T^3 + \frac{1}{24}T\big(t_2^2+t_1t_2+t_1^2\big) \\
&\qquad + (t_2+t_1)\big(t_2^2+t_1^2\big)\bigg)T^{q-4}q(q-1)(q-2)(q-3) \\
&\qquad + (t_2-t_1)\bigg(\frac{1}{48}T^4 + \frac{1}{96}T^3(t_2+t_1) + \frac{1}{96}T(t_2+t_1)\big(t_2^2+t_1^2\big) \\
&\qquad + \frac{1}{240}\big(t_2^4+t_1^3t_2+t_1^2t_2^2+t_1t_2^3+t_1^4\big)\bigg)T^{q-5}q(q-1)(q-2)(q-3)(q-4)\bigg].
\end{aligned}$$

Obviously, the right-hand side tends to zero independently of $x \in B_\epsilon$ as $(t_2 - t_1) \to 0$. Hence, by the Arzelá-Ascoli theorem, the operator $\mathcal{H}$ is completely continuous.

Let x be a solution of the given problem. Then, for $\rho \in (0, T)$, using the method of computation employed to show the boundedness of the operator $\mathcal{H}$, we obtain

$$\begin{aligned}
\big|x(t)\big| &= \big|\rho(\mathcal{H}x)(t)\big| \\
&\le \frac{\varphi(\|x\|)\|p\|T^q}{\Gamma(q+1)}\bigg\{\frac{3}{2} + \overline{\zeta}_1T^{-1}q + \overline{\zeta}_2T^{-2}q(q-1) \\
&\quad + \overline{\zeta}_3T^{-3}q(q-1)(q-2) + \overline{\zeta}_4T^{-4}q(q-1)(q-2)(q-3) \\
&\quad + \overline{\zeta}_5T^{-5}q(q-1)(q-2)(q-3)(q-4)\bigg\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x\|\bigg[\frac{\varphi(\|x\|)\|p\|T^q}{\Gamma(q+1)}\bigg\{\frac{3}{2} + \overline{\zeta}_1T^{-1}q + \overline{\zeta}_2T^{-2}q(q-1) \\
&\quad + \overline{\zeta}_3T^{-3}q(q-1)(q-2) + \overline{\zeta}_4T^{-4}q(q-1)(q-2)(q-3) \\
&\quad + \overline{\zeta}_5T^{-5}q(q-1)(q-2)(q-3)(q-4)\bigg\}\bigg]^{-1} \le 1.
\end{aligned}$$

In view of condition (H_4), there exists $M > 0$ such that $\|x\| \neq M$. Let us choose $\mathcal{N} = \{x \in \mathcal{A} : \|x\| < M + 1\}$. Observe that the operator $\mathcal{H} : \overline{\mathcal{N}} \to \mathcal{A}$ is continuous and completely continuous. From the choice of $\mathcal{N}$, there is no $x \in \partial\mathcal{N}$ such that $x = \rho\mathcal{H}(x)$ for some $\rho \in (0, T)$. Therefore, by Lemma 3.16, we have that the operator $\mathcal{H}$ has a fixed point $x \in \overline{\mathcal{N}}$ which is a solution of problem (3.13)-(3.14). This completes the proof. □

Our final result is based on the following fixed point theorem.

Theorem 3.18 *Let X be a Banach space. Assume that $K : X \to X$ is a completely continuous operator and the set $W = \{u \in X | u = \lambda K u, 0 < \lambda < 1\}$ is bounded. Then K has a fixed point in X.*

Theorem 3.19 *Assume that there exists a positive constant M_1 such that $|f(t,x)| \leq M_1$ for all $t \in [0,T]$, $x \in \mathcal{A}$. Then there exists at least one solution for problem* (3.13)-(3.14) *on* $[0,T]$.

Proof From the previous result, we have that the operator $\mathcal{H}$ is completely continuous.

Now, we define a set $U = \{x \in \mathcal{A} : x = \chi \mathcal{H} x, 0 < \chi < 1\}$ and show that it is bounded. For $x \in U$, $t \in [0,T]$, we have

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, x(s)\big)\, ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f\big(s, x(s)\big)\, ds \\ & + \zeta_1(t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f\big(s, x(s)\big)\, ds + \zeta_2(t) \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f\big(s, x(s)\big)\, ds \\ & + \zeta_3(t) \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f\big(s, x(s)\big)\, ds + \zeta_4(t) \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f\big(s, x(s)\big)\, ds \\ & + \zeta_5(t) \int_0^T \frac{(T-s)^{q-6}}{\Gamma(q-5)} f\big(s, x(s)\big)\, ds. \end{aligned}$$

We can obtain that $|x(t)| = \chi |(\mathcal{H}x)(t)| \leq M_1 \Lambda = M_2$, then $\|x\| \leq M_2$, $\forall x \in U$, $t \in [0,T]$. Thus, U is bounded. Therefore, by Theorem 3.18, problem (3.13)-(3.14) has at least one solution on $[0,1]$. This completes the proof. □

Example 3.20 Consider a fractional boundary value problem with anti-periodic boundary conditions given by

$$\begin{cases} {}^cD^{\frac{11}{2}} x(t) = \frac{x}{\sqrt{t^2+9}} + \frac{\sin(x)}{t^2+4} + \frac{1}{15}, \quad t \in [0,2], \\ x(0) = -x(2), \qquad x'(0) = -x'(2), \qquad x''(0) = -x''(2), \\ x'''(0) = -x'''(2), \qquad x^{(4)}(0) = -x^{(4)}(2), \qquad x^{(5)}(0) = -x^{(5)}(2). \end{cases} \tag{3.20}$$

Here, $q = 11/2$, $T = 2$ and $\ell = \frac{7}{12}$ as $|f(t,x) - f(t,y)| \leq \frac{7}{12} \|x - y\|$. Using the given data, we get $\Lambda = \max_{t \in [0,2]} |\overline{\Lambda}(t)| \approx 0.49217$. Clearly, $\ell \Lambda \approx 0.28710 < 1$. Hence, all the conditions of Theorem 3.13 are satisfied. Therefore, the conclusion of Theorem 3.13 applies, and problem (3.20) has a unique solution on $[0,2]$.

Example 3.21 Consider the following fractional differential equation:

$${}^cD^{\frac{11}{2}} x(t) = \frac{(t+1)}{9} \left(\frac{2}{\pi} \tan^{-1}(x) + x \right), \quad t \in [0,2], \tag{3.21}$$

subject to the boundary conditions of Example 3.20. In this case, $|f(t,x)| \leq (t+1)(1 + \|x\|)/9$. Let us fix $p(t) = (t+1)/9$, $\varphi(\|x\|) = 1 + \|x\|$ and $\|p\| = 1/3$.

By the assumption $M/\varphi(M)\|p\|\Lambda > 1$, we find that $M > \widetilde{M}$, where $\widetilde{M} \approx 0.19625$. Thus, by Theorem 3.17, there exists at least one solution for problem (3.21).

4 Sequential fractional differential equations

In [48], Aqlan et al. studied some new boundary value problems of Liouville-Caputo type sequential fractional differential equation:

$$\big({}^cD^\alpha + k{}^cD^{\alpha-1}\big)u(t) = f\big(t,u(t)\big), \quad 1 < \alpha \leq 2, 0 < t < T, T > 0, \tag{4.1}$$

subject to anti-periodic type (non-separated) boundary conditions of the form

$$\alpha_1 u(0) + \rho_1 u(T) = \beta_1, \qquad \alpha_2 u'(0) + \rho_2 u'(T) = \beta_2, \tag{4.2}$$

and anti-periodic type (non-separated) nonlocal integral boundary conditions

$$\begin{aligned} \alpha_1 u(0) + \rho_1 u(T) &= \lambda_1 \int_0^\eta u(s)\,ds + \lambda_2, \\ \alpha_2 u'(0) + \rho_2 u'(T) &= \mu_1 \int_\xi^T u(s)\,ds + \mu_2, \end{aligned} \tag{4.3}$$

where ${}^cD^\alpha$ denotes the Liouville-Caputo fractional derivative of order α, $k \in \mathbb{R}^+$, $0 < \eta < \xi < T$, $\alpha_1, \alpha_2, \rho_1, \rho_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \mu_1\mu_2 \in \mathbb{R}$ with $\alpha_1 + \rho_1 \neq 0$, $\alpha_2 + \rho_2 e^{-kT} \neq 0$, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Instead of writing the so-called 'Caputo' derivative, they called it 'Liouville-Caputo' derivative as it was introduced by Liouville many decades ago.

Several existence and uniqueness results were obtained for problem (4.1)-(4.2) by using the operator $\mathcal{H} : \mathcal{E} \to \mathcal{E}$ given by

$$\begin{aligned} (\mathcal{H}u)(t) = \nu_1(t) &+ \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(x,u(x)\big)\,dx \right) ds \\ &+ \nu_2(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(s,u(s)\big)\,ds \\ &+ \nu_3(t) \int_0^T e^{-k(T-s)} \left(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(x,u(x)\big)\,dx \right) ds, \end{aligned} \tag{4.4}$$

where $\mathcal{E} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|, t \in [0,T]\}$ and

$$\begin{aligned} \nu_1(t) &= \frac{\beta_1}{(\alpha_1+\rho_1)} + \frac{((\alpha_1 + \rho_1 e^{-kT}) - (\alpha_1+\rho_1)e^{-kt})\beta_2}{k(\alpha_1+\rho_1)(\alpha_2+\rho_2 e^{-kT})}, \\ \nu_2(t) &= \frac{\rho_2((\alpha_1+\rho_1)e^{-kt} - (\alpha_1+\rho_1 e^{-kT}))}{k(\alpha_1+\rho_1)(\alpha_2+\rho_2 e^{-kT})}, \qquad \nu_3(t) = \frac{\alpha_1\rho_2 - \alpha_2\rho_1 - \rho_2(\alpha_1+\rho_1)e^{-kt}}{(\alpha_1+\rho_1)(\alpha_2+\rho_2 e^{-kT})}. \end{aligned}$$

To study the existence of solutions for problems (4.1) and (4.3), the following fixed point operator $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ (associated with the given problem) was considered:

$$\begin{aligned} (\mathcal{G}u)(t) = B_1(t) \bigg\{ &\lambda_1 \int_0^\eta \left(\int_0^s e^{-k(s-x)} I^{\alpha-1} h(x)\,dx \right) ds \\ &- \rho_1 \int_0^T e^{-k(T-s)} I^{\alpha-1} h(s)\,ds + \lambda_2 \bigg\} \end{aligned}$$

$$
\begin{aligned}
&+ B_2(t)\left\{\mu_1 \int_\xi^T \left(\int_0^s e^{-k(s-x)} I^{\alpha-1}h(x)\,dx\right) ds\right. \\
&\left. + k\rho_2 \int_0^T e^{-k(T-s)} I^{\alpha-1}h(s)\,ds - \rho_2 I^{\alpha-1}h(T) + \mu_2\right\} \\
&+ \int_0^t e^{-k(t-s)} I^{\alpha-1}h(s)\,ds,
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
&B_1(t) = \frac{(\epsilon_2 e^{-kt} + \delta_2)}{\Delta}, \qquad B_2(t) = \frac{(\epsilon_1 e^{-kt} - \delta_1)}{\Delta}, \quad \Delta = \delta_1\epsilon_2 + \delta_2\epsilon_1, \\
&\delta_1 = \alpha_1 + \rho_1 e^{-kT} + \frac{\lambda_1}{k}\big(e^{-k\eta} - 1\big), \qquad \epsilon_1 = (\alpha_1 + \rho_1 - \lambda_1\eta), \\
&\delta_2 = -k\alpha_2 - k\rho_2 e^{-kT} + \frac{\mu_1}{k}\big(e^{-kT} - e^{-k\xi}\big), \qquad \epsilon_2 = \mu_1(T - \xi).
\end{aligned}
\tag{4.6}
$$

In a more recent work [49], the authors presented a novel idea of unification of anti-periodic and multipoint boundary conditions and developed the existence theory for sequential fractional differential equations by applying some standard fixed point theorems due to Banach, Krasnoselskii, Leray-Schauder alternative criterion, and Leray-Schauder degree theory. Precisely, the following problem was investigated:

$$
\begin{cases}
({}^cD^q + k\,{}^cD^{q-1})u(t) = f(t, u(t)), \quad 2 < q \leq 3, 0 < t < T, \\
\alpha_1 u(0) + \sum_{i=1}^m a_i u(\eta_i) + \gamma_1 u(T) = \beta_1, \\
\alpha_2 u'(0) + \sum_{i=1}^m b_i u'(\eta_i) + \gamma_2 u'(T) = \beta_2, \\
\alpha_3 u''(0) + \sum_{i=1}^m c_i u''(\eta_i) + \gamma_3 u''(T) = \beta_3,
\end{cases}
\tag{4.7}
$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}$ $(j = 1, 2, 3)$, $a_i, b_i, c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m)$, $k \in \mathbb{R}^+$ and f is an appropriately chosen continuous function. The new boundary conditions in (4.7) can be interpreted as the values of the unknown function, and its first- and second-order derivatives at the end points of the interval under consideration relate to the linear combination of the values of the unknown function, and its first- and second-order derivatives at interior points $\eta_i \in (0, T)$.

5 Coupled anti-periodic boundary conditions

In [50], Alsulami et al. introduced a new kind of boundary value problems of coupled Caputo type fractional differential equations:

$$
\begin{cases}
{}^cD^\alpha x(t) = f(t, x(t), y(t)), \quad t \in [0, T], 1 < \alpha \leq 2, \\
{}^cD^\beta y(t) = g(t, x(t), y(t)), \quad t \in [0, T], 1 < \beta \leq 2,
\end{cases}
\tag{5.1}
$$

subject to the following non-separated coupled boundary conditions:

$$
\begin{cases}
x(0) = \lambda_1 y(T), \qquad x'(0) = \lambda_2 y'(T), \\
y(0) = \mu_1 x(T), \qquad y'(0) = \mu_2 x'(T),
\end{cases}
\tag{5.2}
$$

where ${}^cD^\alpha$, ${}^cD^\beta$ denote the Caputo fractional derivatives of order α and β, respectively, $f,g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are appropriately chosen functions, and λ_i, μ_i, $i=1,2$, are real constants with $\lambda_i\mu_i\neq 1$, $i=1,2$.

In order to obtain the existence and uniqueness results for problem (5.1)-(5.2), the authors derived the operator $T:X\times X\to X\times X$ defined by

$$T(u,v)(t)=\begin{pmatrix}T_1(u,v)(t)\\ T_2(u,v)(t)\end{pmatrix},$$

where $X=\{u(t)|u(t)\in C([0,T],\mathbb{R})\}$ endowed with the norm $\|u\|=\sup\{|u(t)|,t\in[0,T]\}$ is a Banach space,

$$\begin{aligned}
&T_1(u,v)(t)\\
&\quad=\frac{\mu_2}{1-\lambda_2\mu_2}\left(\frac{\lambda_1T(\mu_1\lambda_2+1)}{1-\lambda_1\mu_1}+\lambda_2t\right)B_{2f}+\frac{\lambda_2}{1-\lambda_2\mu_2}\left(\frac{T(\mu_1+\mu_2)\lambda_1}{1-\lambda_1\mu_1}+t\right)A_{2g}\\
&\qquad+\frac{\lambda_1}{1-\lambda_1\mu_1}(A_{1g}+\mu_1B_{1f})+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(s,x(s),y(s)\bigr)\,ds,\\
&T_2(u,v)(t)\\
&\quad=\frac{\mu_2}{1-\lambda_2\mu_2}\left(\frac{T\mu_1(\lambda_1+\lambda_2)}{1-\lambda_1\mu_1}+t\right)B_{2f}+\frac{\lambda_2}{1-\lambda_2\mu_2}\left(\frac{T\mu_1(\lambda_1\mu_2+1)}{1-\lambda_1\mu_1}+\mu_2t\right)A_{2g}\\
&\qquad+\frac{\mu_1}{1-\lambda_1\mu_1}(\lambda_1A_{1g}+B_{1f})+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g\bigl(s,x(s),y(s)\bigr)\,ds,\\
&A_{1g}=\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g\bigl(s,x(s),y(s)\bigr)\,ds,\qquad B_{1f}=\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(s,x(s),y(s)\bigr)\,ds,\\
&A_{2g}=\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g\bigl(s,x(s),y(s)\bigr)\,ds,\qquad B_{2f}=\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f\bigl(s,x(s),y(s)\bigr)\,ds.
\end{aligned}$$

In the most recent work [51], Ahmad et al. investigated the existence of solutions for the following boundary value problem of nonlinear Caputo sequential fractional differential equations:

$$\begin{cases}({}^cD^\alpha+k_1{}^cD^{\alpha-1})x(t)=f(t,x(t),y(t)), & 1<\alpha\le 2, t\in[0,T],\\ ({}^cD^\beta+k_2{}^cD^{\beta-1})y(t)=g(t,x(t),y(t)), & 1<\beta\le 2, t\in[0,T],\end{cases}\tag{5.3}$$

supplemented with coupled anti-periodic type boundary conditions

$$\begin{cases}x(0)=a_1y(T), & x'(0)=a_2y'(T),\\ y(0)=b_1x(T), & y'(0)=b_2x'(T),\end{cases}\tag{5.4}$$

where ${}^cD^\alpha$, ${}^cD^\beta$ denote the Caputo fractional derivative of order α and β, respectively, $k_1,k_2\in\mathbb{R}^+$, $T>0$ and $f,g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, and a_1, a_2, b_1, b_2 are real constants with $a_1b_1\neq 1$ and $a_2b_2e^{-(k_1T+k_2T)}\neq 1$.

We briefly describe the results obtained for problem (5.3)-(5.4). First of all, they obtained an operator $\mathcal{H} : X \times X \longrightarrow X \times X$ given by

$$\mathcal{H}(u,v)(t) = \begin{pmatrix} \mathcal{H}_1(u,v)(t) \\ \mathcal{H}_2(u,v)(t) \end{pmatrix}, \tag{5.5}$$

where

$$\begin{aligned} \mathcal{H}_1(u,v)(t) = {} & \int_0^t e^{-k_1(t-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau \right) ds \\ & + \rho \biggl[\mu_1(t) \int_0^T e^{-k_2(T-s)} \left(\int_0^s \frac{(s-\tau)^{\beta-2}}{\Gamma(\beta-1)} g\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau \right) ds \\ & + \mu_2(t) \int_0^T e^{-k_1(T-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau \right) ds \\ & + \mu_3(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(s, u(s), v(s)\bigr)\, ds \\ & + \mu_4(t) \int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g\bigl(s, u(s), v(s)\bigr)\, ds \biggr], \end{aligned} \tag{5.6}$$

$$\begin{aligned} \mathcal{H}_2(u,v)(t) = {} & \int_0^t e^{-k_2(t-s)} \left(\int_0^s \frac{(s-\tau)^{\beta-2}}{\Gamma(\beta-1)} g\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau \right) ds \\ & + \rho_2 \biggl[\nu_1(t) \int_0^T e^{-k_2(T-s)} \left(\int_0^s \frac{(s-\tau)^{\beta-2}}{\Gamma(\beta-1)} g\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau \right) ds \\ & + \nu_2(t) \int_0^T e^{-k_1(T-s)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau \right) ds \\ & + \nu_3(t) \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(s, u(s), v(s)\bigr)\, ds \\ & + \nu_4(t) \int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g\bigl(s, u(s), v(s)\bigr)\, ds \biggr]. \end{aligned} \tag{5.7}$$

The product space $X \times X$ equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$ is a Banach space ($X = \{u(t) | u(t) \in C[0,T]\}$ endowed with the usual supremum norm $\|u\| = \max\{|u(t)|, t \in [0,T]\}$ is a Banach space),

$$\begin{aligned} & \mu_1(t) = a_2 k_2^2 e^{-k_1 t} + \delta_1, \qquad \mu_2(t) = k_1 k_2 a_2 b_2 e^{-(k_1 t + k_2 T)} + \delta_2, \\ & \mu_3(t) = \delta_3 - k_2 a_2 b_2 e^{-(k_2 T + k_1 t)}, \qquad \mu_4(t) = \delta_4 - k_2 a_2 e^{-k_1 t}, \\ & \nu_1(t) = k_1 k_2 a_2 b_2 e^{-(k_1 T + k_2 t)} + \sigma_1, \qquad \nu_2(t) = k_1^2 b_2 e^{-k_2 t} + \sigma_2, \\ & \nu_3(t) = \sigma_3 - k_1 b_2 e^{-k_2 t}, \qquad \nu_4(t) = \sigma_4 - k_1 a_2 b_2 e^{-(k_1 T + k_2 t)}, \\ & \delta_1 = k_2 \bigl(c_3 \gamma_1 + a_1 c_2 a_2 \gamma_2 e^{-k_1 T}\bigr), \qquad \delta_2 = k_1 \bigl(c_2 a_1 \gamma_2 + c_3 b_2 \gamma_1 e^{-k_2 T}\bigr), \\ & \delta_3 = c_2 a_1 \gamma_4 - c_3 b_2 \gamma_1 e^{-k_2 T}, \qquad \delta_4 = c_3 \gamma_3 - c_2 a_1 a_2 \gamma_2 e^{-k_1 T}, \\ & \sigma_1 = c_2 k_2 \bigl(b_1 \gamma_1 + a_2 \gamma_2 e^{-k_1 T}\bigr), \qquad \sigma_2 = c_2 k_1 \bigl(\gamma_2 + b_1 b_2 \gamma_1 e^{-k_2 T}\bigr), \\ & \sigma_3 = c_2 \bigl(\gamma_4 - b_1 b_2 \gamma_1 e^{-k_2 T}\bigr), \qquad \sigma_4 = c_2 \bigl(b_1 \gamma_3 - a_2 \gamma_2 e^{-k_1 T}\bigr), \\ & c_1 = a_2 b_2 e^{-(k_1 T + k_2 T)}, \qquad c_2 = \frac{1}{1 - a_1 b_1}, \qquad c_3 = 1 + a_1 b_1 c_2, \qquad \gamma_1 = a_1 k_1 - a_2 k_2, \end{aligned}$$

$$\gamma_2 = b_1k_2 - b_2k_1, \qquad \gamma_3 = a_2k_2 - a_1c_1k_1,$$
$$\gamma_4 = b_2k_1 - b_1c_1k_2, \qquad \rho = \frac{1}{k_1k_2(1-c_1)}. \tag{5.8}$$

To establish the desired results, the following conditions were assumed:

(H_1) There exist real constants $m_i, n_i > 0$ ($i = 1,2$), and $m_0 > 0$, $n_0 > 0$ such that $|f(t,x_1,x_2)| \leq m_0 + m_1|x_1| + m_2|x_2|$, $|g(t,x_1,x_2)| \leq n_0 + n_1|x_1| + n_2|x_2|$, $\forall x_i \in \mathbb{R}$, $i = 1,2$.

(H_2) $f,g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and there exist constants ℓ_i, $\overline{\ell_i}$, $i = 1,2$, such that for all $t \in [0,T]$, $u_i, v_i \in \mathbb{R}$, $i = 1,2$, $|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq \ell_1|u_1 - v_1| + \ell_2|u_2 - v_2|$, $|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq \overline{\ell}_1|u_1 - v_1| + \overline{\ell}_2|u_2 - v_2|$.

For brevity, the following notations were set for computational convenience:

$$S_1 = \max_{t\in[0,T]}\left\{\frac{t^{\alpha-1}(1-e^{-k_1t})}{k_1\Gamma(\alpha)} + \rho\left[\frac{|\mu_2(t)|T^{\alpha-1}(1-e^{-k_1T})}{k_1\Gamma(\alpha)} + \frac{|\mu_3(t)|T^{\alpha-1}}{\Gamma(\alpha)}\right]\right\}, \tag{5.9}$$
$$S_2 = \max_{t\in[0,T]}\left\{\rho\left[\frac{|\mu_1(t)|T^{\beta-1}(1-e^{-k_2T})}{k_2\Gamma(\beta)} + \frac{|\mu_4(t)|T^{\beta-1}}{\Gamma(\beta)}\right]\right\}, \tag{5.10}$$
$$S_3 = \max_{t\in[0,T]}\left\{\frac{t^{\beta-1}(1-e^{-k_2t})}{k_2\Gamma(\beta)} + \rho\left[\frac{|\nu_1(t)|T^{\beta-1}(1-e^{-k_2T})}{k_2\Gamma(\beta)} + \frac{|\nu_4(t)|T^{\beta-1}}{\Gamma(\beta)}\right]\right\}, \tag{5.11}$$
$$S_4 = \max_{t\in[0,T]}\left\{\rho\left[\frac{|\nu_2(t)|T^{\alpha-1}(1-e^{-k_1T})}{k_1\Gamma(\alpha)} + \frac{|\nu_3(t)|T^{\alpha-1}}{\Gamma(\alpha)}\right]\right\}, \tag{5.12}$$

and

$$S_0 = \min\bigl\{1 - \bigl[m_1(S_1+S_4) + n_1(S_2+S_3)\bigr], 1 - \bigl[m_2(S_1+S_4) + n_2(S_2+S_3)\bigr]\bigr\}, \tag{5.13}$$

$m_i, n_i \geq 0$ ($i = 1,2$).

Theorem 5.1 (Existence result via the Leray-Schauder alternative) *Assume that* (H_1) *holds and that*

$$m_1(S_1+S_4) + n_1(S_2+S_3) < 1, \qquad m_2(S_1+S_4) + n_2(S_2+S_3) < 1,$$

where S_1, S_2, S_3 *and* S_4 *are given by* (5.9), (5.10), (5.11) *and* (5.12), *respectively. Then problem* (5.3)-(5.4) *has at least one solution on* $[0,T]$.

Theorem 5.2 (Uniqueness result via Banach's contraction mapping principle) *Let* (H_2) *and the following assumption hold*:

$$(\ell_1+\ell_2)(S_1+S_4) + (\overline{\ell}_1+\overline{\ell}_2)(S_2+S_3) < 1, \tag{5.14}$$

where S_1, S_2, S_3 *and* S_4 *are given by* (5.9), (5.10), (5.11) *and* (5.12) *respectively. Then there exists a unique solution for problem* (5.3)-(5.4) *on* $[0,T]$.

Example 5.3 Consider the following fully coupled fractional boundary value problem:

$$\begin{cases} ({}^cD^{3/2} + \frac{1}{2}{}^cD^{1/2})x(t) = \frac{|x(t)|}{(t+3)^4(1+|x(t)|)} + \frac{1}{27(1+y^2(t))} + \frac{1}{81}, & t \in [0,2], \\ ({}^cD^{3/2} + \frac{3}{2}{}^cD^{1/2})y(t) = \frac{\sin(2\pi x(t))}{40\pi} + \frac{1}{10\sqrt{t+4}} + \frac{|y(t)|}{60(1+|y(t)|)}, & t \in [0,2], \\ x(0) = \frac{1}{2}y(2), \qquad x'(0) = y'(2), \qquad y(0) = \frac{1}{4}x(2), \qquad y'(0) = -x'(2). \end{cases} \tag{5.15}$$

Here $T = 2$, $k_1 = 1/2$, $k_2 = 3/2$, $a_1 = 1/2$, $a_2 = 1$, $b_1 = 1/4$, $b_2 = -1$, $f(t,u,v) = \frac{|u|}{(t+3)^4(1+|u|)} + \frac{1}{27(1+v^2)} + \frac{1}{81}$, $g(t,u,v) = \frac{\sin(2\pi u)}{40\pi} + \frac{1}{10\sqrt{t+4}} + \frac{|v|}{60(1+|v|)}$. Using the given data, it was found that $S_1 \approx 3.415456$, $S_2 \approx 3.430998$, $S_3 \approx 1.868630$, $S_4 \approx 2.474067$ (S_1, S_2, S_3 and S_4 are given by (5.9), (5.10), (5.11) and (5.12), respectively).

(a) Clearly, $m_0 = \frac{1}{81}$, $m_1 = \frac{1}{81}$, $m_2 = \frac{1}{27}$, $n_0 = \frac{1}{20}$, $n_1 = \frac{1}{20}$, $n_2 = \frac{1}{60}$ as $|f(t,u,v)| \leq \frac{1}{81} + \frac{1}{81}|u| + \frac{1}{27}|v|$ and $|g(t,u(t),v(t))| \leq \frac{1}{20} + \frac{1}{20}|u| + \frac{1}{60}|v|$. Also $m_1(S_1 + S_4) + n_1(S_2 + S_3) = 0.337692 < 1$ and $m_2(S_1 + S_4) + n_2(S_2 + S_3) = 0.306458 < 1$. Thus the hypothesis of Theorem 5.1 is satisfied. Hence, by the conclusion of Theorem 5.1, problem (5.15) has at least one solution on $[0,2]$.

(b) Since $|f(t,u_2,v_2) - f(t,u_1,v_1)| \leq \frac{1}{81}|u_2 - u_1| + \frac{1}{27}|v_2 - v_1|$, $|g(t,u_2,v_2) - g(t,u_1,v_1)| \leq \frac{1}{20}|u_2 - u_1| + \frac{1}{60}|v_2 - v_1|$, therefore $\ell_1 = \frac{1}{81}$, $\ell_2 = \frac{1}{27}$, $\overline{\ell}_1 = \frac{1}{20}$, $\overline{\ell}_2 = \frac{1}{60}$, $m_0 = \frac{1}{81}$. Further, $[(\ell_1 + \ell_2)(S_1 + S_4) + (\overline{\ell}_1 + \overline{\ell}_2)(S_2 + S_3)] = 0.644149 < 1$. Thus all the conditions of Theorem 5.2 are satisfied. Therefore, the conclusion of Theorem 5.2 applies and hence problem (5.15) has a unique solution on $[0,2]$.

Remark 5.4 Fixing the parameters involved in conditions (5.4), several new results follow as special cases of the present work. For example, if $x(0) = 0$, $y(0) = 0$ ($a_1 = 0 = b_1$), $x'(0) = a_2y'(T)$, $y'(0) = b_2x'(T)$, our results correspond to a problem with coupled flux type conditions. By selecting $a_1 = 1 = b_1$ and $a_2 \neq 1 \neq b_2$, we obtain the results for a nonlinear fractional-order coupled system with semi-periodic coupled boundary conditions of the form $x(0) = y(T)$, $x'(0) = a_2y'(T)$, $y(0) = x(T)$, $y'(0) = b_2x'(T)$. In case we choose $a_1 = 1 = a_2$ and $b_1 = -1 = b_2$ or vice versa, our results correspond to a boundary value problem of nonlinear coupled fractional differential equations subject to a combination of coupled periodic and anti-periodic boundary conditions of the form $x(0) = y(T)$, $x'(0) = y'(T)$, $y(0) = -x(T)$, $y'(0) = -x'(T)$ or $x(0) = -y(T)$, $x'(0) = -y'(T)$, $y(0) = x(T)$, $y'(0) = x'(T)$.

6 Fractional-order anti-periodic boundary conditions

Ahmad and Nieto [52] introduced fractional-order anti-periodic boundary conditions and investigated the existence and uniqueness of solutions for the following problem:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), & t \in [0,T], 1 < q \leq 2, \\ x(0) = -x(T), \qquad {}^cD^p x(0) = -{}^cD^p x(T), & 0 < p < 1. \end{cases} \tag{6.1}$$

In [53], Wang and Liu studied problem (6.1) with the nonlinearity of the form $f(t,u(t), {}^cD^\alpha u(t))$, $0 < \alpha < 1$.

In [54], the authors applied Schaefer's fixed point theorem to prove the existence of solutions for an anti-periodic boundary value problem of Caputo type fractional differential equations involving a p-Laplacian operator of the form

$$\begin{cases} D^{\beta}_{0^+}\phi_p(D^{\alpha}_{0^+}x(t)) = f(t,x(t)), & t \in [0,1], 1 < \alpha, \beta \leq 2, \\ x(0) = -x(1), \quad D^{\alpha}_{0^+}x(0) = -D^{\alpha}_{0^+}x(1). \end{cases} \tag{6.2}$$

Later, the authors in [55] obtained several existence results for a higher-order Caputo fractional differential equation supplemented with fractional anti-periodic boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), & t \in [0,T], 2 < q \leq 3, \\ x(0) = -x(T), \quad {}^cD^p x(0) = -{}^cD^p x(T), \quad {}^cD^{p+1}x(0) = -{}^cD^{p+1}x(T), & 0 < p < 1. \end{cases}$$

By applying Banach's contraction mapping principle and Leray-Schauder degree theory, Chai [56] obtained the existence results for the problem

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t) = f(t,u(t),{}^cD^{\alpha_1}_{0^+}u(t),{}^cD^{\alpha_2}_{0^+}u(t)), & t \in (0,1), \\ u(0) = -u(1), \quad t^{\beta_1-1}{}^cD^{\beta_1}_{0^+}u(t)|_{t\to 0^+} = -t^{\beta_1-1}{}^cD^{\beta_1}_{0^+}u(t)|_{t=1}, \\ t^{\beta_2-1}{}^cD^{\beta_2}_{0^+}u(t)|_{t\to 0^+} = -t^{\beta_2-1}{}^cD^{\beta_2}_{0^+}u(t)|_{t=1}, \end{cases} \tag{6.3}$$

where ${}^cD^{\gamma}$ denotes the Caputo fractional derivative of order γ, the constants α, α_1, α_2, β_1, β_2 are such that $2 < \alpha \leq 3$, $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2$, $0 < \beta_1 < 1 < \beta_2 < 2$ and f is a given continuous function. The inclusion case of problem (6.3) was discussed in [57].

Ahmad and Nieto [58] obtained some existence results for a problem of Riemann-Liouville fractional differential equations with fractional boundary conditions:

$$\begin{aligned} &D^{\alpha}u(t) = f\big(t,u(t)\big), \quad t \in [0,T], \alpha \in (1,2], \\ &D^{\alpha-2}u\big(0^+\big) = b_0 D^{\alpha-2}u\big(T^-\big), \qquad D^{\alpha-1}u\big(0^+\big) = b_1 D^{\alpha-1}u\big(T^-\big), \end{aligned}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α and $b_0 \neq 1$ and $b_1 \neq 1$. Observe that the fractional boundary conditions in this problem can be regarded as Riemann-Liouville anti-periodic boundary conditions for $b_0 = -1 = b_1$.

Agarwal et al. [59] investigated the existence and uniqueness of solutions for a new kind of q-anti-periodic boundary value problem of sequential q-fractional integro-differential equations given by

$$\begin{cases} {}^cD^{\alpha}_q({}^cD^{\gamma}_q + \lambda)x(t) = Af(t,x(t)) + BI^{\rho}_q g(t,x(t)), & 0 \leq t \leq 1, 0 < q < 1, \\ x(0) = -x(1), \quad (t^{(1-\gamma)}D_q x(t))|_{t=0} = -D_q x(1), \end{cases}$$

where ${}^cD^{\alpha}_q$ and ${}^cD^{\gamma}_q$ denote the fractional q-derivative of the Caputo type, $0 < \alpha, \gamma \leq 1$, $I^{\rho}_q(\cdot)$ denotes the Riemann-Liouville integral with $0 < \rho < 1$, f, g are given continuous functions, $\lambda \in \mathbb{R}$ and A, B are real constants.

Ahmad et al. [60] obtained some existence results for sequential fractional q-integro-difference equations with perturbed anti-periodic boundary conditions given by

$$\begin{cases} {}^cD^{\beta}_q({}^cD^{\gamma}_q + \lambda)x(t) = pf(t,x(t)) + kI^{\xi}_q g(t,x(t)), & 0 \leq t \leq 1, 0 < q < 1, \\ x(a) = -x(1), \quad {}^cD^{\gamma}_q x(a) = -{}^cD^{\gamma}_q x(1), & 0 < a \ll 1, \end{cases}$$

where ${}^cD_q^{\beta}$ and ${}^cD_q^{\gamma}$ denote the fractional q-derivative of Caputo type, $0 < \beta, \gamma \leq 1$, $I_q^{\xi}(\cdot)$ denotes the Riemann-Liouville integral with $0 < \xi < 1$, f, g are given continuous functions, $\lambda \neq 0$ and p, k are real constants.

More details on anti-periodic boundary value problems involving q-difference and fractional q-difference equations can be found in a recent text by Ahmad et al. [61].

7 Conclusions

We have presented an up-to-date review of the results on boundary value problems of nonlinear fractional-order differential equations, inclusions and coupled systems supplemented with a variety of anti-periodic (and anti-periodic type) boundary conditions. In Section 2, we have given some basic definitions of fractional calculus and model equations involving fractional-order derivatives. In Section 3, we have collected a variety of results on classical anti-periodic boundary value problems of nonlinear fractional differential equations, inclusions and impulsive equations. The concept of parametric type anti-periodic boundary conditions is also outlined. The relationship between the Green's functions of lower- and higher-order anti-periodic fractional boundary value problems is also described. Some new results related to further generalization of classical anti-periodic problems are discussed in detail and illustrated with examples. Section 4 contains some recent results on boundary value problems of Liouville-Caputo (Caputo) type sequential fractional differential equations supplemented with anti-periodic type (non-separated) two-point and nonlocal multipoint boundary conditions. In Section 5, some existence results for a new kind of boundary value problem of coupled Caputo type fractional differential equations equipped with non-separated coupled boundary conditions are given. Some results involving fractional order anti-periodic boundary conditions are elaborated in Section 6. We recall that anti-periodic boundary conditions appear in numerous situations such as interpolation problems, anti-periodic wavelets, mathematical problems of ordinary, partial and impulsive differential equations, problems in physics, etc. Keeping in view the importance of anti-periodic type boundary value problems occurring in several disciplines, the present survey provides a detailed description of the work on the topic completed over a period of the last decade and may serve as a platform for the researchers who are interested in exploring more and more insights in this topic.

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Author details
[1]Department of Mathematics, Texas A&M University, Kingsville, TX 78363-8202, USA. [2]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

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