Ismaeel *Boundary Value Problems* (2017) 2017:174
DOI 10.1186/s13661-017-0907-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the Wiener criterion in higher dimensions

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Abstract

The Wiener criterion is a sufficient and necessary condition for the solvability of the Dirichlet problem. However, its geometric interpretation is not clear. In the case that the domain satisfies an exterior spine condition, the requirement for the spine is clear in dimension 3. In this note, we intend to obtain the condition that the exterior spine should satisfy in higher dimensions.

MSC: Primary 35A01; 35B65; 35J05; 35J25

Keywords: Wiener criterion; Laplace equation; Dirichlet problem; boundary regularity

1 Introduction

The Laplace equation arises widely in physics and engineering and is a classical prototype of partial differential equations. Consider the following Dirichlet problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ is a bounded domain and g is a continuous function on $\partial\Omega$.

In 1851, Riemann [1] proposed the famous Dirichlet principle, which states that there always exists a harmonic function continuous up to the boundary and coinciding with g on the boundary. However, Lebesgue [2] constructed a bounded domain on which the Dirichlet problem is not always solvable in 1912. By Perron's method [3, 4], there always exists a harmonic function u in Ω with respect to g. If $x_0 \in \partial\Omega$ is a regular point (see [4, p.25] for the definition), then u is continuous up to x_0. Hence, the solvability of (1.1) reduces to the problem whether the boundary points are regular. In 1924, Wiener [5] provided a sufficient and necessary condition for the regularity of x_0. This is the famous Wiener criterion, which solves the Dirichlet problem completely.

However, the geometric interpretation of Wiener criterion is not clear, and it is not easy to verify whether a domain satisfies the Wiener criterion at some boundary point. One of the interesting cases is that the boundary of the domain near some boundary point is constructed by a spine. Precisely, suppose that for a proper coordinate system, $0 \in \partial\Omega$, and there exist $0 < r_0 < 1$ and a continuous nondecreasing function $\varphi : R \to R$ with $\varphi(0) = 0$

such that

$$B_{r_0} \cap \Omega = B_{r_0} \cap \{x \in R^n | |x'| > \varphi(x_n) \text{ if } x_n > 0\}, \tag{1.2}$$

where B_r denotes the open ball in R^n with radius r and center 0, $x = (x_1, \ldots, x_n) \in R^n$ and $x' = (x_1, \ldots, x_{n-1})$. If '=' is replaced by '$\subset$' in (1.2), we call that Ω satisfies the exterior spine condition with φ at 0.

It is natural to ask what condition φ should satisfy to guarantee that 0 is a regular point. It is well known that if $\varphi(r) \geq Cr$ for some positive constant C, i.e., the domain satisfies the exterior cone condition, then 0 is a regular point (see [6] and [4, Problem 2.12]). In this note, we will consider the weaker condition and always assume that

$$\varphi(r)/r \to 0 \quad \text{as } r \to 0 \quad \text{and} \quad \varphi(r) < r, \quad \forall 0 < r < r_0. \tag{1.3}$$

In the dimension $n = 3$, if Ω satisfies (1.2), it is well known that 0 is a regular point if and only if φ satisfies (see Theorem 5.2 and the following example in [7])

$$\int_0^{r_0} \frac{dr}{r|\ln \varphi(r)|} = +\infty. \tag{1.4}$$

What condition φ should satisfy in higher dimensions is not known. Besides, the following boundary regularity results are well known. Suppose that g is identically zero on a portion of the boundary near 0. If $\varphi(r) \geq Cr^{1/2}$, which implies that Ω satisfies the exterior sphere condition, then the solution is Lipschitz continuous at 0. If $\varphi(r) \geq Cr$, i.e., Ω satisfies the exterior cone condition, then the solution is Hölder continuous at 0. A natural question is whether the condition $\varphi(r) \geq Cr^a$ for some constant C and $a > 1$ can guarantee the continuity (even Hölder continuity) of the solution at 0. The geometric meaning of this condition is that Ω satisfies the exterior Hölder spine condition. It should be pointed out that for the dimension $n = 3$, the continuity of the solution in this case is guaranteed (see (1.4)).

This note is devoted to deriving the sufficient and necessary condition for φ in a higher dimension and answer the above question.

First, we recall the Wiener criterion. For any bounded domain $\Omega \subset R^n$, its capacity is defined by (see Section 2.9 in [4] and Section 19 of Chapter XI in [7])

$$\operatorname{cap} \Omega = \inf_{v \in K} \int |Dv|^2,$$

where

$$K = \{v \in C_0^1(R^n) | v = 1 \text{ on } \Omega\}$$

and $C_0^1(R^n)$ denotes the set of functions having continuous derivatives and compact support in R^n.

Suppose that $0 \in \partial\Omega$ and $0 < \lambda < r_0$ is a constant. Let

$$\Omega_j = \{x \notin \Omega | \lambda^{j+1} \leq |x| \leq \lambda^j\} \quad \text{and} \quad C_j = \operatorname{cap} \Omega_j \quad \text{for } j = 1, 2, \ldots.$$

The Wiener criterion states that 0 is a regular point if and only if the series

$$\sum_{j=1}^{\infty} \frac{C_j}{\lambda^{j(n-2)}} \tag{1.5}$$

diverges (see Section 2.9 in [4], Section 19 of Chapter XI in [7] and Theorem 5.2 in [7]).

Our main result is the following theorem.

Theorem 1.1 *Suppose that* $n \geq 4$, $0 \in \partial\Omega$ *and* (1.2) *is satisfied. Then* 0 *is a regular point with respect to* (1.1) *if and only if* φ *satisfies*

$$\int_0^{r_0} \frac{\varphi^{n-3}(r)\,dr}{r^{n-2}} = +\infty. \tag{1.6}$$

An immediate consequence is the following.

Corollary 1.2 *Let* $n \geq 4$ *and* $0 \in \partial\Omega$. *Suppose that* Ω *satisfies the exterior spine condition with* φ *at* 0 *and* (1.6) *holds.*

Then 0 *is a regular point with respect to* (1.1).

Remark 1.3 *From* (1.6), *the special dimensions* $n = 2, 3$ *should be noted. In addition, for* $n \geq 4$ *and* $a > 1$, $\varphi(r) \geq Cr^a$ *is not enough to guarantee that* 0 *is a regular point, which is an essential difference to dimensions* 2 *and* 3.

2 Proof of Theorem 1.1

Now, we give the proof of Theorem 1.1.

Proof We use ellipsoids to approximate Ω_j and hence to estimate C_j. Clearly, Ω_j is contained in an ellipsoid E_j with semi-axes $2\lambda^j$ and $2\varphi(\lambda^j)$ ($n-1$ repeats), and Ω_j contains an ellipsoid $\tilde{E}_j$ with semi-axes λ^{j+1} and $\varphi(\lambda^{j+1})/2$ ($n-1$ repeats). The capacity for this kind ellipsoid E is (see (125) in [8])

$$\operatorname{cap} E = \left(\beta^2 - \gamma^2\right)^{(n-2)/2} \Big/ \int_0^{\arcsin(\sqrt{\mu})} \frac{\sin^{n-3}\theta}{\cos^{n-2}\theta}\,d\theta, \tag{2.1}$$

where β and γ are semi-axes of E with $\beta > \gamma$ and $\mu = 1 - \gamma^2/\beta^2$. Next, let

$$I_{m,k} = \int_0^{\arcsin(\sqrt{\mu})} \frac{\sin^m\theta}{\cos^k\theta}\,d\theta.$$

Then $I_{m,k}$ has the following reduction formula for $k \neq 1$ (see (155) in [8]):

$$I_{m,k} = \frac{\mu^{(m-1)/2}}{(k-1)(1-\mu)^{(k-1)/2}} - \frac{m-1}{k-1} I_{m-2,k-2}.$$

We say that $A \simeq B$ if $A_1 A \leq B \leq A_2 A$, where A_1 and A_2 are constants depending only on the dimension n. If $1 - \mu \ll 1$, then

$$\frac{\mu^{(m-1)/2}}{(k-1)(1-\mu)^{(k-1)/2}} \simeq (1-\mu)^{-(k-1)/2}.$$

Substitute $m = n - 3$ and $k = n - 2$ and, by noting for $k = 1$ (see [9, Section 442.10])

$$I_{0,1} = \frac{1}{2}\ln\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}},$$

we have

$$I_{n-3,n-2} \simeq (1-\mu)^{-(n-3)/2}.$$

Therefore, substituting into (2.1) leads to

$$\operatorname{cap} E \simeq \left(\beta^2 - \gamma^2\right)^{(n-2)/2}\frac{\gamma^{n-3}}{\beta^{n-3}} \simeq \beta\gamma^{n-3}.$$

Let $E = E_j$ with $\beta = 2\lambda^j$ and $\gamma = 2\varphi(\lambda^j)$, then $1 - \mu \ll 1$ for sufficient large j (recall (1.3)) and hence

$$C_j = \operatorname{cap}\Omega_j \le \operatorname{cap} E_j \le A_1\lambda^j\varphi^{n-3}\left(\lambda^j\right).$$

Similarly, let $E = \tilde{E}_j$ with $\beta = \lambda^{j+1}$ and $\gamma = \varphi(\lambda^{j+1})/2$, then

$$C_j = \operatorname{cap}\Omega_j \ge \operatorname{cap}\tilde{E}_j \ge A_2\lambda^{j+1}\varphi^{n-3}\left(\lambda^{j+1}\right).$$

Note that

$$\sum_{j=1}^{\infty}\frac{\lambda^j\varphi^{n-3}(\lambda^j)}{\lambda^{j(n-2)}} = \infty$$

is equivalent to

$$\sum_{j=1}^{\infty}\frac{\lambda^{j+1}\varphi^{n-3}(\lambda^{j+1})}{\lambda^{(j+1)(n-2)}} = \infty.$$

Hence, (1.5) holds if and only if

$$\sum_{j=1}^{\infty}\frac{\lambda^j\varphi^{n-3}(\lambda^j)}{\lambda^{j(n-2)}} = \infty,$$

whose integral representation is exactly (1.6). Therefore, 0 is a regular point if and only if (1.6) holds and hence Theorem 1.1 is completed. □

3 Conclusion

Let Ω be a bounded domain in R^n and $x_0 \in \partial\Omega$. The Wiener criterion provides a necessary and sufficient condition on Ω to solve the Dirichlet problem of the Laplace equation (i.e., (1.1)). However, its geometric meaning is not clear and it is not easy to verify whether a domain satisfies the Wiener criterion at some boundary point. One interesting case is that Ω is constructed by a spine near x_0 (see (1.2)). It is natural to ask what condition the spine should satisfy to guarantee that x_0 is a regular point. When the dimension $n = 2$ or 3, the results are well known. In this paper, we consider $n > 3$ and prove that x_0 is a regular point with respect to the Dirichlet problem if and only if the spine satisfies (1.6).

Funding
This research is supported by NSFC 11671316.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The author declares that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors read and approved the final manuscript.

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Received: 9 July 2017 Accepted: 13 November 2017 Published online: 21 November 2017

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