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q-Lidstone polynomials and existence results for q-boundary value problems

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Abstract

In this paper, we study some properties of q-Lidstone polynomials by using Green's function of certain q-differential systems. The q-Fourier series expansions of these polynomials are given. As an application, we prove the existence of solutions for the linear q-difference equations

$$(-1)^n D_{q^{-1}}^{2n} y(x) = \phi\big(x, y(x), D_{q^{-1}} y(x), D_{q^{-1}}^2 y(x), \dots, D_{q^{-1}}^k y(x)\big),$$

subject to the boundary conditions

$$D_{q^{-1}}^{2j} y(0) = \beta_j, \qquad D_{q^{-1}}^{2j} y(1) = \gamma_j \quad (\beta_j, \gamma_j \in \mathbb{C}, j = 0, 1, \dots, n-1),$$

where $n \in \mathbb{N}$ and $0 \le k \le 2n-1$. These results are a q-analogue of work by Agarwal and Wong of 1989.

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1 Introduction

In the classical Lidstone expansion theorem [1], an entire function $f(x)$ may be expanded with respect to the points 0 and 1 in the form

$$f(x) = \sum_{n=0}^{\infty} \big(f^{(2n)}(1) A_n(x) - f^{(2n)}(0) A_n(x-1)\big),$$

where A_n is a polynomial of degree $2n+1$ that satisfies

(i) $A_0(x) = x$,
(ii) $A_n(0) = A_n(1) = 0$ for $n \in \mathbb{N}$,
(iii) $A_n''(x) = A_{n-1}(x)$.

The polynomial A_n is called Lidstone polynomial.

Ismail and Mansour [2] introduced a q-analogue of Lidstone's theorem where the two points are 0 and 1. They expanded the function in q-analogues of Lidstone polynomials which are in fact q-Bernoulli polynomials as in the classical case (see Section 2).

It is the object of this paper to give a q-analogue of the results of [3] using the terminology and results given in [2].

This article is organized as follows. In the next section, we state the q-definitions and present some preliminaries of q-calculus which will play an important role in our main results. In Section 3, we define the Green's functions of certain q-differential systems which are related to q-Lidstone polynomials, and Section 4 gives q-Fourier expansions of these functions and for q-Lidstone polynomials. Some interesting results and relationships are obtained. In Section 5, we are interested in the existence of solutions to the following boundary value problem:

$$(-1)^n D_{q^{-1}}^{2n} y(x) = \phi\big(x, y(x), D_{q^{-1}} y(x), D_{q^{-1}}^2 y(x), \dots, D_{q^{-1}}^k y(x)\big), \tag{1.1}$$

$n \in \mathbb{N}$ and $0 \leq k \leq 2n-1$, subject to the boundary conditions

$$D_{q^{-1}}^{2j} y(0) = \beta_j, D_{q^{-1}}^{2j} y(1) = \gamma_j \quad (\beta_j, \gamma_j \in \mathbb{C}, j = 0, 1, \dots, n-1), \tag{1.2}$$

with some conditions imposed on y.

2 Preliminaries

In this paper, we assume that q is a positive number less than one with

$$[x] = \frac{1-q^x}{1-q}.$$

For $t > 0$, the sets $A_{q,t}$, $A_{q,t}^*$ are defined by

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \qquad A_{q,t}^* := A_{q,t} \cup \{0\},$$

where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. Notice, if $t = 1$, we simply use A_q and A_q^* to denote $A_{q,1}$ and $A_{q,1}^*$, respectively.

In the following, we state some of the needed q-notations and results (see [4] and [5]).

The q-shifted fractional is defined by

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j) \quad \text{and} \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \text{for } n \in \mathbb{Z}, a \in \mathbb{C}.$$

The q-gamma function is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z} \quad \text{for } z \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}_0\}.$$

Let f be a function defined on a q-geometric set A, *i.e.*, $qx \in A$ for all $x \in A$. The q-difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x} \quad \text{if } x \in A - \{0\}.$$

The q-integration by parts rule (see [4]) is

$$\int_0^a f(qt)D_q g(t)\,d_q t = (fg)(a) - \lim_{n\to\infty}(fg)\big(aq^n\big) - \int_0^a D_q f(t)g(t)\,d_q t.$$

If X is the set $A_{q,t}$ or $A^*_{q,t}$, then for $n > 1$, $C_q^n(X)$ is the space of all continuous functions with continuous q-derivatives up to order $n-1$ on X. The space $C_q^n(X)$ associated with the norm function

$$\|f\| := \sum_{k=0}^{n-1} \max_{x\in X} \big|D_q^k f(t)\big| \quad \big(f \in C_q^n(X)\big)$$

is a Banach space (see [4]).

Ismail and Mansour [2] defined a q-analogue of the Bernoulli polynomials $B_n(z;q)$, $z \in \mathbb{C}$ by the generating function

$$\frac{tE_q(zt)}{E_q(t/2)e_q(t/2) - 1} = \sum_{n=0}^{\infty} B_n(z;q)\frac{t^n}{[n]!},$$

where the functions $E_q(z)$ and $e_q(z)$ have the series representation

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k+1)}; \quad |z| < 1 \quad \text{and} \quad E_q(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{\Gamma_q(k+1)}; \quad z \in \mathbb{C}.$$

The q-Bernoulli numbers are defined by

$$\beta_n := B_n(0;q).$$

Hence, in terms of the generating function,

$$\frac{t}{E_q(t/2)e_q(t/2) - 1} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]!}. \tag{2.1}$$

Also, they defined two q-analogues of the Euler polynomials through the generating functions

$$\frac{2E_q(xt)}{E_q(t/2)e_q(t/2) + 1} = \sum_{n=0}^{\infty} E_n(x;q)\frac{t^n}{[n]!}, \tag{2.2}$$

$$\frac{2e_q(xt)}{E_q(t/2)e_q(t/2) + 1} = \sum_{n=0}^{\infty} e_n(x;q)\frac{t^n}{[n]!}. \tag{2.3}$$

Notice, $E_0(x;q) = e_0(x;q) = 1$, and $\tilde{E}_n := E_n(0;q) = e_n(0;q)$ for all $n \in \mathbb{N}_0$.

Proposition 2.1 *For $n \in \mathbb{N}$, the q-Bernoulli and q-Euler polynomials satisfy the following q-difference equations*:

$$D_{q^{-1}} B_n(x;q) = [n]B_{n-1}(x;q);$$

$$D_{q^{-1}} E_n(x;q) = [n]E_{n-1}(x;q) \quad \textit{and} \quad D_q e_n(x;q) = [n]e_{n-1}(x;q).$$

Proposition 2.2 *The q-Euler polynomials $E_n(x;q)$ and $e_n(x;q)$ are given by*

$$E_0(x;q) = e_0(x;q) = 1,$$

and for $n \in \mathbb{N}$,

$$E_n(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \tilde{E}_{n-k} x^k, \qquad e_n(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{E}_{n-k} x^k.$$

Recall that (see [6]) an entire function f has a p-exponential growth of order k and a finite type ($p, k \in \mathbb{R} - \{0\}$ with $p > 1$) if there exists a real number $K > 0$, α such that

$$\left|f(x)\right| < K p^{\frac{k}{2}(\frac{\log|x|}{\log p})^2} |x|^{\alpha}.$$

The following results from [2] will be needed in the sequel.

Theorem 2.3 *Let $0 < q < 1$ and f be a function of q^{-1}-exponential growth of order less than or equal to* 1. *Then*

$$f(z) = \sum_{n=0}^{\infty} \left(D_{q^{-1}}^{2n} f(1) A_n(z) - D_{q^{-1}}^{2n} f(0) B_n(z)\right),$$

where A_n and B_n are polynomials of degree $2n+1$ defined by

$$A_n(z) = \frac{2^{2n+1}}{[2n+1]!} \sum_{j=0}^{2n+1} \begin{bmatrix} 2n+1 \\ j \end{bmatrix}_q (-z;q)_j 2^{-j} \beta_{2n+1-j},$$

$$B_n(z) = \frac{2^{2n+1}}{[2n+1]!} B_{2n+1}(z/2;q).$$

Furthermore, the polynomials A_n are defined recursively by $A_0(z) = z$ and, for $n \in \mathbb{N}$, A_n satisfies the second order q-difference equation

$$D_{q^{-1}}^2 A_n(z) = A_{n-1}(z), \qquad A_n(0) = A_n(1) = 0 \quad (n \in \mathbb{N}). \tag{2.4}$$

The polynomials B_n are defined recursively by $B_0(z) = 1 - z$ and, for $n \in \mathbb{N}$, B_n satisfies the second order q-difference equation

$$D_{q^{-1}}^2 B_n(z) = B_{n-1}(z), \qquad B_n(0) = B_n(1) = 0 \quad (n \in \mathbb{N}). \tag{2.5}$$

Lemma 2.4 *Let $z \in \mathbb{C}$. Then*

$$A_n(z) := \varepsilon_{q^{-1}}^{1} B_n(z),$$

where $\varepsilon_{q^{-1}}^{y}$ is a q-translation operator defined by

$$\varepsilon_{q^{-1}}^{y} x^n = x^n \left(-y/x; q^{-1}\right)_n = q^{-\frac{n(n-1)}{2}} y^n (-x/y; q)_n.$$

3 The Green's function of a certain q-differential system

In this section, we consider certain boundary value problems which are related to q-Lidstone polynomials, and then we define these polynomials by using Green's function.

Consider the following q-differential equation:

$$D^2_{q^{-1}}y(x) - f(x) = 0 \quad (x \in A^*_q), \tag{3.1}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 0. \tag{3.2}$$

Theorem 3.1 *The boundary value problem* (3.1)-(3.2) *is equivalent to the basic Fredholm q-integral equation*

$$y(x) = \int_0^1 G(x, q^2 t) f(q^2 t)\, d_q t, \tag{3.3}$$

where

$$G(x,t) = \begin{cases} -t(1-x), & 0 \le t < x \le 1; \\ -x(q-t), & 0 \le x < t \le 1. \end{cases} \tag{3.4}$$

Proof Since $D^2_{q^{-1}}y(x) = \frac{1}{q}(D^2_q y)(\frac{x}{q^2})$, Equation (3.1) can be written as

$$D^2_q y(x) - qf(q^2 x) = 0 \quad (x \in A^*_q). \tag{3.5}$$

By taking double q-integral for (3.5), we obtain

$$y(x) = q\int_0^x (x - qt) f(q^2 t)\, d_q t + c_1 x + c_2, \tag{3.6}$$

where c_1 and c_2 are arbitrary constants. Now, using the boundary conditions, we get

$$c_1 = -q\int_0^1 (1 - qt) f(q^2 t)\, d_q t \quad \text{and} \quad c_2 = 0.$$

Substituting in (3.6), we obtain the required result. □

Now, consider the following equations:

$$\begin{aligned} &G_1(x, q^2 t) := G(x, q^2 t), \\ &G_n(x, q^2 t) = \int_0^1 G(x, q^2 y) G_{n-1}(q^2 y, q^2 t)\, d_q y \quad (n = 2, 3, \ldots) \\ &\qquad = \int_0^1 \cdots \int_0^1 G(x, q^2 t_1) G(q^2 t_1, q^2 t_2) \cdots G(q^2 t_{n-1}, q^2 t)\, d_q t_1\, d_q t_2 \cdots d_q t_{n-1}. \end{aligned} \tag{3.7}$$

Corollary 3.2 *The q-Lidstone polynomials A_m and B_m are given by*

$$\begin{aligned} &A_0(x) = x, \\ &A_m(x) = \int_0^1 G(x, q^2 t) A_{m-1}(q^2 t)\, d_q t = q^2 \int_0^1 t G_m(x, q^2 t)\, d_q t, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &B_0(x) = 1 - x, \\ &B_m(x) = \int_0^1 G(x, q^2 t) B_{m-1}(q^2 t)\, d_q t = \int_0^1 G_m(x, q^2 t)(1 - q^2 t)\, d_q t. \end{aligned} \tag{3.9}$$

Proof The proof follows immediately from Theorem 3.1, Equation (3.7), Equation (2.4) and Equation (2.5). □

Theorem 3.3 *Let $0 < q < 1$ and $g \in C^{2n}(A_q^*)$. Then*

$$g(x) = \sum_{m=0}^{n-1} \bigl[D_{q^{-1}}^{2m} g(1) A_m(x) - D_{q^{-1}}^{2m} g(0) B_m(x) \bigr] + \int_0^1 G_n(x, q^2 t) D_{q^{-1}}^{2n} g(q^2 t)\, d_q t,$$

where A_m and B_m are q-Lidstone polynomials of degree $2m + 1$.

Proof From Theorem 3.1 we can verify that, for $q \in (0, 1)$ and $g \in C^{2n}(A_q^*)$, the q-integral equation

$$g(x) = \int_0^1 G_n(x, q^2 t) f(q^2 t)\, d_q t$$

is the solution of the q-differential system

$$\begin{cases} D_{q^{-1}}^{2n} g(x) - f(x) = 0 \quad (x \in A_q^*), \\ D_{q^{-1}}^{2k} g(0) = D_{q^{-1}}^{2k} g(1) = 0 \quad (k = 0, 1, \ldots, n-1). \end{cases}$$

Furthermore, the unique solution of the system

$$\begin{cases} D_{q^{-1}}^{2n} g(x) - f(x) = 0 \quad (x \in A_q^*), \\ D_{q^{-1}}^{2k} g(0) = a_k, \qquad D_{q^{-1}}^{2k} g(1) = b_k \quad (k = 0, 1, \ldots, n-1) \end{cases} \tag{3.10}$$

is

$$\begin{aligned} g(x) = {}& a_0(x - 1) + b_0 x + \sum_{k=1}^{n-1} a_k \int_0^1 (q^2 t - 1) G_k(x, q^2 t)\, d_q t \\ &+ \sum_{k=1}^{n-1} b_k \int_0^1 q^2 t G_k(x, q^2 t)\, d_q t + \int_0^1 G_n(x, q^2 t) f(q^2 t)\, d_q t. \end{aligned}$$

Replacing a_k, b_k and $f(x)$ by their values in terms of $g(x)$ as given by the q-differential system (3.10), we get

$$\begin{aligned} g(x) &= g(0)(x-1) + g(1)x + \sum_{k=1}^{n-1} D_{q^{-1}}^{2k} g(0) \int_0^1 \bigl(q^2t-1\bigr) G_k\bigl(x,q^2t\bigr)\,d_qt \\ &\quad + \sum_{k=1}^{n-1} D_{q^{-1}}^{2k} g(1) \int_0^1 q^2 t G_k\bigl(x,q^2t\bigr)\,d_qt + \int_0^1 G_n\bigl(x,q^2t\bigr) D_{q^{-1}}^{2n} g\bigl(q^2t\bigr)\,d_qt. \end{aligned}$$

Therefore, according to Equations (3.8) and (3.9), we obtain the required result. □

Remark 3.4 By using Theorem 3.3, and from Equations (2.4) and (2.5), we have

$$\begin{aligned} D_{q^{-1}}^{2j} g(x) &= \sum_{m=j}^{n-1} \bigl[D_{q^{-1}}^{2m} g(1) D_{q^{-1}}^{2j} A_m(x) + D_{q^{-1}}^{2m} g(0) D_{q^{-1}}^{2j} B_m(x)\bigr] \\ &\quad + \int_0^1 G_{n-j}\bigl(x,q^2t\bigr) D_{q^{-1}}^{2n} g(t)\,d_qt \\ &= \sum_{m=j}^{n-1} \bigl[D_{q^{-1}}^{2m} g(1) A_{m-j}(x) + D_{q^{-1}}^{2m} g(0) B_{m-j}(x)\bigr] \\ &\quad + \int_0^1 G_{n-j}\bigl(x,q^2t\bigr) D_{q^{-1}}^{2n} g(t)\,d_qt \\ &= \sum_{m=0}^{n-j-1} \bigl[D_{q^{-1}}^{2(m+j)} g(1) A_m(x) + D_{q^{-1}}^{2(m+j)} g(0) B_m(x)\bigr] \\ &\quad + \int_0^1 G_{n-j}\bigl(x,q^2t\bigr) D_{q^{-1}}^{2n} g(t)\,d_qt, \\ D_{q^{-1}}^{(2j+1)} g(x) &= \sum_{m=0}^{n-j-1} \bigl[D_{q^{-1}}^{2(m+j)} g(1) D_{q^{-1}} A_m(x) + D_{q^{-1}}^{2(m+j)} g(0) D_{q^{-1}} B_m(x)\bigr] \\ &\quad + \int_0^1 D_{q^{-1},x} G_{n-j}\bigl(x,q^2t\bigr) D_{q^{-1}}^{2n} g(t)\,d_qt. \end{aligned}$$

4 Certain q-Fourier expansions

The purpose of this section is to obtain the q-Fourier series expansions of the following q-integrals:

$$\int_0^1 \bigl(q^2t\bigr)^k G_n\bigl(x,q^2t\bigr)\,d_qt, \quad k = 0,1, n \le 4,$$

and then to compute the series expansions of some of q-Lidstone polynomials which will be used to solve the boundary value problem (1.1)-(1.2).

First, recall that the q-trigonometric functions $C_q(z)$ and $S_q(z)$ are defined for $z \in \mathbb{C}$ by

$$C_q(z) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1/2)} z^{2n}}{(q;q)_{2n}} = \frac{z}{1-q} {}_1\phi_1\bigl(0; q; q^2, q^{1/2} z^2\bigr),$$
$$S_q(z) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1/2)} z^{2n+1}}{(q;q)_{2n+1}} = \frac{z}{1-q} {}_1\phi_1\bigl(0; q^3; q^2, q^{3/2} z^2\bigr).$$

The Fourier series expansion for any function defined on the q-linear grid $\mathcal{A}_q$ is the following (see [7, 8]):

$$S_q(f) := \frac{a_0}{2} + \sum_{k=1}^{\infty} \bigl[a_k C_q\bigl(q^{1/2} w_k z\bigr) + b_k S_q(q w_k z)\bigr],$$

where $a_0 = \int_{-1}^{1} f(t)\, d_q t$ and, for $k = 1, 2, \ldots,$

$$a_k = \frac{1}{\mu_k} \int_{-1}^{1} f(t) C_q\bigl(q^{1/2} w_k t\bigr)\, d_q t, \qquad b_k = \frac{\sqrt{q}}{\mu_k} \int_{-1}^{1} f(t) S_q(q w_k t)\, d_q t,$$
$$\mu_k = (1-q) C_q\bigl(q^{1/2} w_k\bigr) S_q'(w_k)$$

on the q-linear grid $\mathcal{A}_q$, where $\{w_k : k \in \mathbb{N}\}$ is the set of positive zeroes of $S_q(z)$.

One can verify that

$$D_{q,z} C_q(wz) = -\frac{w}{1-q} S_q(wz\sqrt{q}) \quad \text{and} \quad D_{q,z} S_q(wz) = \frac{w}{1-q} C_q(wz\sqrt{q}).$$

Lemma 4.1 *Let $x \in A_q^*$ and $n \in \mathbb{N}$. Then*

$$\int_0^1 G\bigl(x, q^2 y\bigr) S_q\bigl(q^n w_k y\bigr)\, d_q y = \frac{(1-q)^2}{q^{2n-5/2} w_k^2} \bigl(x S_q\bigl(q^{n-1} w_k\bigr) - S_q\bigl(q^{n-1} w_k x\bigr)\bigr).$$

Proof By using Equations (3.1) and (3.3), the q-integral

$$y(x) = \int_0^1 G_1\bigl(x, q^2 y\bigr) S_q\bigl(q^n w_k y\bigr)\, d_q y$$

is the solution of the q-differential system

$$\begin{cases} D_{q^{-1}}^2 y(x) - S_q(q^{n-2} w_k x) = 0 & (x \in A_q^*), \\ y(0) = 0, \qquad y(1) = 0. \end{cases} \tag{4.1}$$

Therefore,

$$\begin{aligned} D_q y(x) &= \frac{-(1-q)}{q^{n-\frac{3}{2}} w_k} C_q\bigl(q^{n-\frac{1}{2}} w_k x\bigr) + c_1, \\ y(x) &= \frac{-(1-q)^2}{q^{2n-\frac{5}{2}} w_k^2} S_q\bigl(q^{n-1} w_k x\bigr) + c_1 x + c_2. \end{aligned} \tag{4.2}$$

From the boundary conditions, we get

$$c_1 = \frac{(1-q)^2}{q^{2n-\frac{5}{2}} w_k^2} S_q\big(q^{n-1}w_k\big) \quad \text{and} \quad c_2 = 0.$$

Substituting the values of c_1 and c_2 into Equation (4.2), we obtain the required result. □

Lemma 4.2 *For $x \in A_q^*$, the following q-Fourier series expansion holds*:

$$\int_0^1 G\big(x, q^2 t\big)\, d_q t = -2\sqrt{q}(1-q)^2 \sum_{k=1}^{\infty} \frac{L_k}{w_k^2} S_q(w_k x), \tag{4.3}$$

where

$$L_k := \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}.$$

Proof By computing the q-Fourier series expansion of the function $f(x) = 1$ for $0 < x < 1$, we get

$$1 = 2\sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)} S_q(q w_k t), \quad t \in A_q^*. \tag{4.4}$$

Multiplying (4.4) by $G_1(x, q^2 t)$ and integrating with respect to t from zero to unity, we get

$$\int_0^1 G_1\big(x, q^2 t\big)\, d_q t = 2\sum_{k=1}^{\infty} L_k \int_0^1 G_1\big(x, q^2 t\big) S_q(w_k q t)\, d_q t, \tag{4.5}$$

where

$$L_k := \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)}, \quad x \in A_q^*.$$

By using Lemma 4.1, we get

$$\int_0^1 G_1\big(x, q^2 t\big) S_q(w_k q t)\, d_q t = \frac{-\sqrt{q}(1-q)^2}{w_k^2} S_q(w_k x). \tag{4.6}$$

Substituting from (4.6) into (4.5), we obtain the required series. □

Lemma 4.3 *For $x \in A_q^*$, the following q-Fourier series expansion holds*:

$$\int_0^1 G\big(x, q^2 t\big)\big(q^2 t\big)\, d_q t = -2q^{5/2}(1-q)^2 \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^2} S_q(w_k x),$$

where

$$\widetilde{L_k} := \frac{q w_k C_q(q^{1/2}w_k) - (1-q) S_q(q w_k)}{q^2 w_k^2 C_q(q^{1/2}w_k) S_q'(w_k)}.$$

Proof Considering the function $g(t) = t$ for $0 < t < 1$ and computing the q-Fourier series of the extension of g as an odd function on $[-1,1]$, we get

$$t = 2\sum_{k=1}^{\infty} \widetilde{L}_k S_q(qw_k t) \quad \text{for all } 0 < t < 1, \tag{4.7}$$

where

$$\widetilde{L}_k := \frac{qw_k C_q(q^{1/2}w_k) - (1-q)S_q(qw_k)}{q^2 w_k^2 C_q(q^{1/2}w_k)S_q'(w_k)}. \tag{4.8}$$

Hence, the proof can be performed by using (4.7) similar to the proof of Lemma 4.2. So, we will omit it. □

Throughout the following results, we define the constants L_k and $\widetilde{L}_k$ as in Lemma 4.3 and Lemma 4.3, respectively.

Note that, by using Equation (3.7), we get

$$G_2(x,q^2t) = \int_0^1 G(x,q^2y)G(q^2y,q^2t)\,d_qy. \tag{4.9}$$

Integrating (4.9) with respect to t from 0 to unity and using Lemma 4.2, we obtain

$$\int_0^1 G_2(x,q^2t)\,d_qt = -2\sqrt{q}(1-q)^2\sum_{k=1}^{\infty}\frac{L_k}{w_k^2}\int_0^1 G(x,q^2y)S_q(q^2w_ky)\,d_qy.$$

Again, using Lemma 4.1, we get

$$\int_0^1 G(x,q^2y)S_q(q^2w_ky)\,d_qy = \frac{(1-q)^2}{q^{3/2}w_k^2}\bigl(xS_q(qw_k) - S_q(qw_kx)\bigr).$$

Hence,

$$\int_0^1 G_2(x,q^2t)\,d_qt = -2\frac{(1-q)^4}{q}\sum_{k=1}^{\infty}\frac{L_k}{w_k^4}\bigl(xS_q(qw_k) - S_q(qw_kx)\bigr).$$

Repeating the process for $n = 3$ and $n = 4$, we obtain the following result.

Theorem 4.4 *For $x \in A_q^*$ and $n \leq 4$, the following expansion holds*:

$$\begin{aligned} q^2\int_0^1 G_n(x,q^2t)\,d_qt = \frac{(-1)^{n-1}(1-q)^{2n}}{q^{n(n-3/2)}}\Biggl[&\sum_{k=1}^{\infty}\frac{L_k}{w_k^{2n}}\bigl(xS_q(w_kq^{n-1}) - S_q(w_kq^{n-1}x)\bigr) \\ &+ 2\sum_{i=1}^{n-2}(-1)^i q^{2(n+i-1)}\sum_{k=1}^{\infty}\frac{L_k}{w_k^{2(n-i)}}S_q(q^{n-i-1}w_k) \\ &\times\sum_{k=1}^{\infty}\frac{\widetilde{L}_k}{w_k^{2i}}\bigl(xS_q(q^{i-1}w_k) - S_q(q^{i-1}w_kx)\bigr)\Biggr]. \end{aligned} \tag{4.10}$$

Remark 4.5 In the classical case, Widder [9] concluded a general formula for a Fourier series of the integral of Green's functions G_n for all $n \in \mathbb{N}$. Theorem 4.4 gives a formula for the q-Fourier series of $\int_0^1 G_n(x,q^2t)\,d_qt$ for $n \le 4$, we could not put it in a closed form for all $n \in \mathbb{N}$. However, we can verify that

$$\int_0^1 G_n(x,q^2t)\,d_qt = \frac{(-1)^{n-1}(1-q)^{2n}}{q^{n(n-3/2)}} S_{k,n} \quad (n \in \mathbb{N}),$$

where $S_{k,n}$ denotes a sum of q-series which converge uniformly on A_q^* and depend on the q-trigonometric function S_q and the constants L_k and $\widetilde{L_k}$.

Theorem 4.6 *For $x \in A_q^*$ and $n \le 4$, the following expansion holds*:

$$\begin{aligned}\int_0^1 G_n(x,q^2t)t\,d_qt = \frac{(-1)^{n-1}(1-q)^{2n}}{q^{n(n-3/2)}} \Bigg[&\sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2n}} \big(xS_q(w_kq^{n-1}) - S_q(w_kq^{n-1}x)\big) \\ &+ 2\sum_{i=1}^{n-2} (-1)^i q^{2(n+i-1)} \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2(n-i)}} S_q(q^{n-i-1}w_k) \\ &\times \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2i}} \big(xS_q(q^{i-1}w_k) - S_q(q^{i-1}w_kx)\big)\Bigg].\end{aligned}$$

Proof The proof is similar to the proof of Theorem 4.4 and is omitted. □

The following corollary follows immediately from Theorems 4.4 and 4.6.

Corollary 4.7 *For $x \in A_q^*$ and $n \le 4$, the following expansion holds*:

$$\begin{aligned}&\int_0^1 G_n(x,q^2t)(1-q^2t)\,d_qt \\ &= \frac{(-1)^{n-1}(1-q)^{2n}}{q^{n(n-3/2)}} \Bigg[\sum_{k=1}^{\infty} \frac{1}{w_k^{2n}} (L_k - q^2\widetilde{L_k}) \big(xS_q(w_kq^{n-1}) - S_q(w_kq^{n-1}x)\big) \\ &+ 2\sum_{i=1}^{n-2} (-1)^i q^{2(n+i-1)} \sum_{k=1}^{\infty} \frac{S_q(q^{n-i-1}w_k)}{w_k^{2(n-i)}} (L_k - q^2\widetilde{L_k}) \\ &\times \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2i}} \big(xS_q(q^{i-1}w_k) - S_q(q^{i-1}w_kx)\big)\Bigg].\end{aligned}$$

Corollary 4.8 *For $x \in A_q^*$ and $n \le 4$, the q-Fourier series for the q-Lidstone polynomials $A_n(x)$ and $B_n(x)$ are given by*

$$\begin{aligned}A_n(x) = \frac{(-1)^{n-1}(1-q)^{2n}}{q^{n(n-3/2)}} \Bigg[&\sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2n}} \big(xS_q(w_kq^{n-1}) - S_q(w_kq^{n-1}x)\big) \\ &+ 2\sum_{i=1}^{n-2} (-1)^i q^{2(n+i-1)} \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2(n-i)}} S_q(q^{n-i-1}w_k) \\ &\times \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k^{2i}} \big(xS_q(q^{i-1}w_k) - S_q(q^{i-1}w_kx)\big)\Bigg],\end{aligned}$$

$$\begin{aligned}
B_n(x) &= \frac{(-1)^{n-1}(1-q)^{2n}}{q^{n(n-3/2)}}\Bigg[\sum_{k=1}^{\infty}\frac{1}{w_k^{2n}}\big(L_k - q^2\widetilde{L_k}\big)\big(xS_q\big(w_kq^{n-1}\big) - S_q\big(w_kq^{n-1}x\big)\big)\\
&\quad + 2\sum_{i=1}^{n-2}(-1)^i q^{2(n+i-1)}\sum_{k=1}^{\infty}\frac{S_q(q^{n-i-1}w_k)}{w_k^{2(n-i)}}\big(L_k - q^2\widetilde{L_k}\big)\\
&\quad \times \sum_{k=1}^{\infty}\frac{\widetilde{L_k}}{w_k^{2i}}\big(xS_q\big(q^{i-1}w_k\big) - S_q\big(q^{i-1}w_kx\big)\big)\Bigg].
\end{aligned}$$

Proof It follows immediately from Theorem 4.6, Corollary 4.7, Equations (3.8) and (3.9). □

Proposition 4.9 *There exists a constant C such that*

$$0 \le (-1)^n\int_0^1 G_n\big(x,q^2t\big)\,d_qt \le \frac{(1-q)^{2n}}{q^{n(n-3/2)}}C.$$

Proof By using Equations (3.4) and (3.7), we get

$$(-1)^n\int_0^1 G_n\big(x,tq^{-1}\big)\,d_qt \ge 0.$$

Another inequality follows from Theorem 4.4 together with the result that the series in (4.10) converges uniformly at each fixed point $x \in A_q^*$. □

Proposition 4.10 *There exists a constant $\widetilde{C}$ such that*

$$\int_0^1\big|D_{q^{-1},x}G_n\big(x,q^2t\big)\big|\,d_qt \le \frac{(1-q)^{2(n-1)}}{q^{(n-1)(n-5/2)}}\widetilde{C}.$$

Proof By using (3.7), we have

$$\begin{aligned}
\int_0^1\big|D_{q^{-1},x}G_n\big(x,q^2t\big)\big|\,d_qt &= \int_0^1\Big[D_{q^{-1},x}\int_0^1 G\big(x,q^2y\big)(-1)^{n-1}G_{n-1}\big(q^2y,q^2t\big)\,d_qy\Big]\,d_qt\\
&= \int_0^1\int_0^x(-1)^{n-1}\big(q^2y\big)G_{n-1}\big(q^2y,q^2t\big)\,d_qy\,d_qt\\
&\quad - \int_0^1\int_x^1(-1)^{n-1}\big(q-q^2y\big)G_{n-1}\big(q^2y,q^2t\big)\,d_qy\,d_qt.
\end{aligned}$$

Interchanging the order of the double q-integrations and using Proposition 4.9, we get

$$\begin{aligned}
\int_0^1\big|D_{q^{-1},x}G_n\big(x,q^2t\big)\big|\,d_qt &= \int_0^x\big(q^2y\big)\Big[\int_0^1\big|G_{n-1}\big(q^2y,q^2t\big)\big|\,d_qt\Big]\,d_qy\\
&\quad - \int_x^1\big(q-q^2y\big)\Big[\int_0^1\big|G_{n-1}\big(q^2y,q^2t\big)\big|\,d_qt\Big]\,d_qy\\
&\le \frac{(1-q)^{2(n-1)}}{q^{(n-1)(n-5/2)}}C\Big[\int_0^x\big(q^2y\big)\,d_qy - \int_x^1\big(q-q^2y\big)\,d_qy\Big]\\
&= \frac{(1-q)^{2(n-1)}}{q^{(n-1)(n-5/2)}}C\Big[q(1-x) + \frac{q^2}{(q+1)}\Big]\\
&\le \frac{(1-q)^{2(n-1)}}{q^{(n-1)(n-5/2)}}C\Big[q + \frac{q^2}{(1+q)}\Big].
\end{aligned}$$

Hence, if we define the constant $\widetilde{C}$ as

$$\widetilde{C} := \left(q + \frac{q^2}{(1+q)}\right)C,$$

we get the required result. □

We end this section by computing the q-Fourier expansion of the q-Euler polynomials of degree 2. We start by the following lemma.

Lemma 4.11

$$\sum_{k=1}^{\infty} \frac{L_k}{w_k} = \frac{\sqrt{q}}{2(1-q^2)} \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k} = -\frac{\sqrt{q}}{2[3]!(1-q)}.$$

Proof By computing the q-Fourier series for the function $f(x) = |x|$, we obtain

$$f(x) = \frac{1}{1+q} - \frac{2(1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k^2 C_q(q^{1/2}w_k) S'_q(w_k)} C_q\bigl(q^{1/2}w_k x\bigr).$$

In particular, when $x = 0$, this implies

$$0 = \frac{1}{1+q} - \frac{2(1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k^2 C_q(q^{1/2}w_k) S'_q(w_k)}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{L_k}{w_k} = \frac{\sqrt{q}}{2(1-q^2)}.$$

Similarly, computing the q-Fourier series for the function $g(x) = |x|^2$, we obtain

$$|x|^2 = \frac{1}{[3]} + \frac{2[2](1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k} C_q\bigl(q^{1/2}w_k x\bigr).$$

At $x = 0$, we have

$$\sum_{k=1}^{\infty} \frac{\widetilde{L_k}}{w_k} = -\frac{\sqrt{q}}{2[3]!(1-q)}.$$ □

Theorem 4.12 *For $x \in A_q^*$, the q-Fourier series for q-Euler polynomials $e_2(x;q)$ is given by*

$$e_2(x;q) = \frac{[2]}{q}\left[-2\sqrt{q}(1-q)^2 \sum_{k=1}^{\infty} \frac{L_k}{w_k^2} S_q(w_k x) + \left(\frac{q}{1+q} - \frac{q}{2}\right)x\right].$$

Proof By using Proposition 2.1, we have

$$1 = e_0(x;q) = D_q e_1(x;q).$$

Therefore, for $x \in A_q^*$, the q-Fourier expansion of the function $D_q e_1(x;q)$ is

$$D_q e_1(x;q) = 2\sum_{k=1}^{\infty} \frac{1 - C_q(q^{1/2}w_k)}{w_k C_q(q^{1/2}w_k) S_q'(w_k)} S_q(q w_k x). \tag{4.11}$$

Integrating (4.11) from 0 to x, we obtain

$$e_1(x;q) = \frac{-2(1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{L_k}{w_k} C_q\bigl(q^{1/2} w_k x\bigr) + C_1, \tag{4.12}$$

where C is a constant of integration. This constant is obtained by putting $x = 0$ in Equation (4.12) and then using Lemma 4.11 and the result $e_1(0;q) = \widetilde{E}_1(0) = -\frac{1}{2}$. We get $C_1 = -\frac{1}{2} + \frac{1}{1+q}$. Hence,

$$e_1(x;q) = \frac{-2(1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{L_k}{w_k} C_q\bigl(q^{1/2} w_k x\bigr) + \frac{1}{1+q} - \frac{1}{2}. \tag{4.13}$$

Again, using Proposition 2.1 with $n = 2$, we get

$$e_2(x;q) = [2] \int e_1(x,q)\, d_q x + C_2. \tag{4.14}$$

Substituting Equation (4.13) into Equation (4.14) gives us

$$e_2(x;q) = [2]\left[\frac{-2(1-q)}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{L_k}{w_k} \int C_q\bigl(q^{1/2} w_k x\bigr)\, d_q x + \int \left(\frac{1}{1+q} - \frac{1}{2}\right) d_q x\right] + C_2.$$

This implies

$$e_2(x;q) = [2]\left[\frac{-2(1-q)^2}{\sqrt{q}} \sum_{k=1}^{\infty} \frac{L_k}{w_k^2} S_q(w_k x) + \left(\frac{q}{1+q} - \frac{q}{2}\right) x\right] + C_2.$$

In the last equation putting $x = 0$, we get $C_2 = 0$, and hence the theorem. □

Corollary 4.13 *For $x \in A_q^*$, the following holds*:

$$e_2(x;q) = \frac{[2]}{q}\left[\int_0^1 G\bigl(x, q^2 t\bigr)\, d_q t + \left(\frac{q}{1+q} - \frac{q}{2}\right) x\right].$$

Proof The proof follows immediately from Lemma 4.2 and Theorem 4.12. □

Remark 4.14 From Equation (3.9), we have

$$B_n(x) = \int_0^1 G_n\bigl(x, q^2 t\bigr)\, d_q t - q^2 \int_0^1 t G_n\bigl(x, q^2 t\bigr)\, d_q t.$$

Thus, by using Corollary 4.13 and Equation (3.8), we obtain the following relation:

$$B_1(x) + q^2 A_1(x) = q\left[\frac{e_2(x;q)}{[2]} + \left(\frac{1}{2} - \frac{1}{1+q}\right) x\right]. \tag{4.15}$$

If $q \to 1$, Equation (4.15) coincides with the result which is given by Agarwal and Wong [3] in the classical case.

5 An application: q-boundary value problems

The q-difference equations are important in q-calculus. This subject initiated in the first quarter of the twentieth century [10–13], and it has been developed over the years. Recently, many authors have studied the existence and uniqueness of solutions for some problems of q-difference equations, for instance, see [7, 14–20].

The goal of this section is to solve the boundary value problem (1.1)-(1.2) by using the q-Lidstone expansion theorem. The results here attained are the q-analogue of those given by Agarwal and Wong [3], where they studied the existence of solutions for

$$\begin{cases} (-1)^n x^{(2m)}(t) = f(t, x(t), x'(t), \dots, x^{(k)}(t)), \\ x^{(2i)}(0) = a_i, \\ x^{(2i)}(1) = b_i, \end{cases}$$

where $0 \le k \le 2m-1$ and $i = 0, 1, \dots, m-1$ with some conditions imposed on f and x.

For our purpose, let us define two constants C and $\widetilde{C}$ as in Proposition 4.9 and Proposition 4.10, respectively, and we introduce the following assumptions:

H_1: K_j, $0 \le j \le k$ are given real numbers, and define the nonzero constant M to be the maximum of $|\phi(x, y_0, y_1, y_2, \dots, y_k)|$ on the compact set $A_q^* \times E$, where

$$E = \{(y_0, y_1, y_2, \dots, y_k), |y_j| \le 2K_j, 0 \le j \le k\}.$$

$$H_2 : \frac{(1-q)^{2(n-j)}}{q^{(n-j)(n-j-3/2)}} MC \le K_{2j}, \quad j = 0, 1, 2, \dots, \frac{k}{2};$$

$$H_3 : \frac{(1-q)^{2(n-j-1)}}{q^{(n-j-1)(n-j-5/2)}} M\widetilde{C} \le K_{2j+1}, \quad j = 0, 1, 2, \dots, \frac{k-1}{2};$$

$$H_4 : \max\{|\gamma_j|, |\beta_j|\} + \sum_{i=1}^{n-j-1} \max\{|\gamma_{i+j}|, |\beta_{i+j}|\} \frac{(1-q)^{2i}}{q^{i(i-3/2)}} C \le K_{2j};$$

$$H_5 : |\gamma_j + \beta_j| + \widetilde{C} \sum_{i=1}^{n-j-1} \max\{|\gamma_{i+j}|, |\beta_{i+j}|\} \frac{(1-q)^{2(i-1)}}{q^{(i-1)(i-5/2)}} \le K_{2j+1}.$$

The proof of the existence results for boundary value problem (1.1)-(1.2) depends on q-Lidstone polynomials and the Arzela-Ascoli theorem [21].

Theorem 5.1 *Let $q \in (0,1)$ and $y \in C^n_{q^{-1}}(A_q^*)$ be a real or complex-valued function. Assume that assumptions H_1, H_2, H_3 and H_4 hold. Then the boundary value problem* (1.1)-(1.2) *has a solution in E.*

Proof By using Theorem 3.3, we conclude that the boundary value problem (1.1)-(1.2) is equivalent to the following Fredholm q-integral equation:

$$y(x) = \sum_{i=0}^{n-1} [\gamma_i A_i(x) + \beta_i B_i(x)] + \int_0^1 G_n(x, q^2 t) \phi(t, y(t), \dots, D^k_{q^{-1}} y(t)) \, d_q t. \tag{5.1}$$

Hence, this problem can be interpreted as a fixed point for the mapping $T : C^k_{q^{-1}}(A^*_q) \to C^{2n}_{q^{-1}}(A^*_q)$ which is defined by

$$(Ty)(x) = \sum_{i=0}^{n-1}\bigl[\gamma_i A_i(x) + \beta_i B_i(x)\bigr] + \int_0^1 \bigl|G_n\bigl(x,q^2t\bigr)\bigr|\phi\bigl(t,y(t),\dots,D^k_{q^{-1}}y(t)\bigr)\,d_qt. \tag{5.2}$$

We define the set

$$J\bigl(A^*_q\bigr) := \Bigl\{y(x) \in C^k_{q^{-1}}\bigl(A^*_q\bigr) : \bigl\|D^j_{q^{-1}}y\bigr\| = \max_{0\le x\le 1}\bigl|D^j_{q^{-1}}y(x)\bigr| \le 2K_j,\ 0 \le j \le k\Bigr\}.$$

Notice that $J(A^*_q)$ is a closed subset of the space $C^k_{q^{-1}}(A^*_q)$. We prove that T maps $J(A^*_q)$ into itself.

Let $y(x) \in J(A^*_q)$. Then, from Equation (5.2), Remark 3.4, Proposition 4.9 and hypotheses H_1, H_2 and H_4, we get

$$\begin{aligned}
\bigl|D^{(2j)}_{q^{-1}}(Ty)(x)\bigr| &\le \sum_{i=0}^{n-j-1}\bigl|\gamma_{i+j}A_i(x) + \beta_{i+j}B_i(x)\bigr| + M\int_0^1\bigl|G_{n-j}\bigl(x,q^2t\bigr)\bigr|\,d_qt \\
&\le |\gamma_j x| + \bigl|\beta_j(1-x)\bigr| + \sum_{i=1}^{n-j-1}\biggl|\gamma_{i+j}\int_0^1\bigl(q^2t\bigr)G_i\bigl(x,q^2t\bigr)\,d_qt + \beta_{i+j} \\
&\quad\times\int_0^1\bigl(1-q^2t\bigr)G_i\bigl(x,q^2t\bigr)\,d_qt\biggr| + M\int_0^1\bigl|G_{n-j}\bigl(x,q^2t\bigr)\bigr|\,d_qt \\
&\le \sup_{x\in A^*_q}\bigl[|\gamma_j x| + \bigl|\beta_j(1-x)\bigr|\bigr] + \sum_{i=1}^{n-j-1}\max\bigl\{|\gamma_{i+j}|,|\beta_{i+j}|\bigr\} \\
&\quad\times\int_0^1\bigl|G_i\bigl(x,q^2t\bigr)\bigr|\,d_qt + M\int_0^1\bigl|G_{n-j}\bigl(x,q^2t\bigr)\bigr|\,d_qt \\
&\le \max\bigl\{|\gamma_j|,|\beta_j|\bigr\} + \sum_{i=1}^{n-j-1}\max\bigl\{|\gamma_{i+j}|,|\beta_{i+j}|\bigr\}\frac{(1-q)^{2i}}{q^{i(i-3/2)}}C \\
&\quad+ \frac{(1-q)^{2(n-j)}}{q^{(n-j)(n-j-3/2)}}MC \le 2K_{2j}, \quad j = 0,1,2,\dots,\frac{k}{2}.
\end{aligned} \tag{5.3}$$

Similarly, from Equation (5.2), Remark 3.4, Proposition 4.10 and hypotheses H_3 and H_5, we get

$$\begin{aligned}
\bigl|D^{(2j+1)}_{q^{-1}}(Ty)(x)\bigr| &\le |\gamma_j + \beta_j| + \widetilde{C}\sum_{i=1}^{n-j-1}\max\bigl\{|\gamma_{i+j}|,|\beta_{i+j}|\bigr\}\frac{(1-q)^{2(i-1)}}{q^{(i-1)(i-5/2)}} \\
&\quad+ \frac{(1-q)^{2(n-j-1)}}{q^{(n-j-1)(n-j-5/2)}}M\widetilde{C} \\
&\le K_{2j+1} + K_{2j+1} = 2K_{2j+1}, \quad j = 0,1,2,\dots,\frac{k-1}{2}.
\end{aligned} \tag{5.4}$$

This completes the proof of $T(J(A^*_q)) \subseteq J(A^*_q)$. Furthermore, from the inequalities (5.3) and (5.4) we conclude that the set

$$\bigl\{D^j_{q^{-1}}(T)y(x) : y(x) \in J\bigl(A^*_q\bigr), 0 \le j \le k\bigr\}$$

is uniformly bounded and equicontinuous on $J(A_q^*)$. Therefore, from the Arzela-Ascoli theorem $\overline{T(J(A_q^*))}$ is compact. It means that we can find a fixed point of T in E which satisfies the boundary value problem (1.1)-(1.2). □

Corollary 5.2 *Assume that the function* $\phi(x, y_0, y_1, \ldots, y_k)$ *satisfies the following condition on* $A_q^* \times \mathbb{R}^{k+1}$:

$$\left|\phi(x, y_0, y_1, \ldots, y_k)\right| \leq L + \sum_{j=0}^{k} L_j |y_j|^{\alpha_j}, \tag{5.5}$$

where L, L_j *are nonnegative constants, and* $0 \leq \alpha_j < 1$. *Then the boundary value problem* (1.1)-(1.2) *has a solution.*

Proof By using (5.5), for $y(x) \in J(A_q^*)$, we get

$$\left|\phi\bigl(x, y(x), D_{q^{-1}} y(x), D_{q^{-1}}^2 y(x), \ldots, D_{q^{-1}}^k y(x)\bigr)\right| \leq N,$$

where $N := L + \sum_{j=0}^{k} L_j (2K_j)^{\alpha_j}$. Hence, the result follows by observing that the hypotheses of Theorem 5.1 are satisfied and replacing M by N such that K_j $(0 \leq j \leq k)$ are sufficiently large. □

6 Conclusion

The goal of this paper is to study some properties of q-Lidstone polynomials by using Green's function of certain q-differential systems and then to solve the following boundary value problem:

$$\begin{aligned} &(-1)^n D_{q^{-1}}^{2n} y(x) = \phi\bigl(x, y(x), D_{q^{-1}} y(x), D_{q^{-1}}^2 y(x), \ldots, D_{q^{-1}}^k y(x)\bigr), \\ &D_{q^{-1}}^{2j} y(0) = \beta_j, \qquad D_{q^{-1}}^{2j} y(1) = \gamma_j \quad (\beta_j, \gamma_j \in \mathbb{C}, j = 0, 1, \ldots, n-1), \end{aligned}$$

where $n \in \mathbb{N}$ and $0 \leq k \leq 2n - 1$.

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