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Existence of positive solutions for a fractional elliptic problems with the Hardy-Sobolev-Maz'ya potential and critical nonlinearities

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Abstract
In this paper, we consider the study of a fractional elliptic problem with the Hardy-Sobolev-Maz'ya potential and critical nonlinearities. By means of variational methods and suitable technique, a positive solution to this problem is obtained.

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Keywords: fractional operator; Hardy-Sobolev-Maz'ya potential; critical nonlinearities; positive solution

1 Introduction and main result

In this paper, we consider the existence of the solutions for the following problem:

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|z|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|z|^\alpha} + \lambda f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ with $N \geq 3$ and $2 \leq k < N$, $0 \leq \mu < a_{k,s} := 2^{2s}\Gamma^2(\frac{k+2s}{4})/\Gamma^2(\frac{k-2s}{4})$, $s \in (0,1)$, $\Gamma(t) = \int_0^{+\infty} \tau^{t-1}e^{-\tau}\,d\tau$. A point $x \in \mathbb{R}^N$ is denoted as $x = (z,w) \in \mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$. $\lambda > 0$ is a real parameter, the number $2_s^*(\alpha) = 2(N-\alpha)/(N-2s)$ is a critical Hardy-Sobolev exponent with $s \in (0,1)$ and $\alpha \in [0,2s)$. The nonlinearity term f is continuous function and satisfies suitable hypotheses. Here, $(-\Delta)^s$ is the fractional Laplace operator (see [1, 2]) defined, up to a normalization factor, by

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad x \in \mathbb{R}^N.$$

In recent years, much attention has been focused on the study of the problems involving fractional operators. The fractional operators appear in several applications to some models related to probability, mathematical, finances or fluid mechanics, soft thin films, stratified materials, multiple scattering and minimal surfaces (see [3–6]). When $\mu = 0$ and

$\alpha = 0$, problem (1) reduces to critical fractional equation. Abundant results have been accumulated (see [7–12]).

For a class of fractional elliptic problems with the Hardy potential

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = g(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2}$$

Abdellaoui and Medina *et al.* in [13] gave the solvability of the problem (2) for the linear case $g(x,t) = g(x)$ and the nonlinear case $g(x,t) = \frac{h(x)}{t^\sigma}$, respectively. For critical case, a positive solution was obtained in [14] with by the Lagrange multipliers technique. Moreover, the authors in [15] have studied the solvability of problem (2) for the case $g(x,t)$ involving concave-convex nonlinearities.

Recently, Jiang and Tang in [16] had considered the problem (1) for the case $s = 1$, they supposed the nonlinearity term $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies the following conditions:

(f_1) $f(x,t) = 0$ for $t \leq 0$ uniformly for $x \in \overline{\Omega}$. There exists a nonempty open subset $\Omega_0 \subset \Omega$ with $(0, w^0) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \in \Omega_0$, such that $f(x,t) \geq 0$ for almost everywhere $x \in \Omega$ and all $t > 0$; $f(x,t) > 0$ for almost $x \in \Omega_0$ and all $t > 0$.

(f_2) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$ and $\lim_{t\to +\infty} \frac{f(x,t)}{t^{2_s^*(\alpha)-1}} = 0$ uniformly for $x \in \overline{\Omega}$.

For $\lambda > 0$ large enough, they obtained the existence of positive solutions of problem (1) for $s = 1$ by using variational methods. For the case $s = 1$ and $\lambda = 1$, Ding and Tang in [17] obtained the existence of positive solutions for problem (1) by the variational methods and some analysis techniques with f satisfying the (AR) condition. For related papers on the semilinear elliptic equations with Hardy-Sobolev critical exponents of (1) for $s = 1$, we just mention [18, 19] and the references therein.

To the best of our knowledge, there is no result in the literature on the fractional elliptic problem with Hardy-Sobolev-Maz'ya potential and critical nonlinearities. Motivated by the above papers, our aim is to study the existence of positive solutions for problem (1) and our main result of this paper is as follows.

Theorem 1 *Assume that conditions* (f_1) *and* (f_2) *hold. Then there exists* $\lambda^* > 0$ *such that* $\lambda \geq \lambda^*$, *problem* (1) *admits a positive solution.*

2 Functional setting and useful tools

We will denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{\frac{1}{2}}.$$

We consider the function space

$$X_0^s = \{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \},$$

with the norm

$$\|u\|_{X_0^s} = \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mu\int_\Omega \frac{u^2}{|z|^{2s}}\,dx\right)^{\frac{1}{2}},$$

which is equivalent to its general norm due to the Hardy inequality

$$a_{k,s}\int_{\mathbb{R}^N} \frac{u^2}{|z|^{2s}} \le \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy, \tag{3}$$

where $Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \setminus \Omega$. We can introduce the best fractional critical Hardy-Sobolev constant $S_{\mu,\alpha}$, given by

$$S_{\mu,\alpha} = \inf_{u\in X_0^s(\mathbb{R}^N\setminus(0,w^0)),u\neq 0} \frac{\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mu\int_\Omega \frac{u^2}{|z|^{2s}}\,dx}{(\int_{\mathbb{R}^n} \frac{|u(x)|^{2_s^*(\alpha)}}{|z|^\alpha}\,dx)^{2/2_s^*(\alpha)}}. \tag{4}$$

From [20], we know that $S_{\mu,\alpha}$ is attained by functions

$$U_\varepsilon(x) = \frac{\varepsilon^{\frac{(N-2s)}{2}}}{(\varepsilon^2 + |x-x_0|^2)^{\frac{(N-2s)}{2}}}.$$

Let $u^+ = \max\{u,0\}$, the energy functional $J_\lambda : X_0^s \to \mathbb{R}$ associated to the problem (1) is defined as

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\mu}{2}\int_\Omega \frac{(u^+)^2}{|z|^{2s}}\,dx \\ &\quad - \frac{1}{2_s^*(\alpha)}\int_\Omega \frac{(u^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx - \lambda\int_\Omega F(x,u^+)\,dx, \end{aligned} \tag{5}$$

for all $u \in X_0^s$, where $F(x,t)$ is a primitive function of $f(x,t)$ defined by $F(x,t) = \int_0^t f(x,\tau)\,d\tau$. Obviously, J_λ is a $C^1(X_0^s)$ functional, and it is well known that the solutions of problem (1) are the critical points of the energy functional J_λ. In fact, if u is a weak solution of problem (1), we have

$$\begin{aligned} \langle J'_\lambda(u),\varphi\rangle &= \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy - \mu\int_\Omega \frac{u^+\varphi}{|z|^{2s}}\,dx \\ &\quad - \int_\Omega \frac{(u^+)^{2_s^*(\alpha)-1}\varphi}{|z|^\alpha}\,dx - \lambda\int_\Omega f(x,u^+)\varphi\,dx \\ &= 0, \end{aligned} \tag{6}$$

for all $\varphi \in X_0^s$. Now, we will give some essential lemmas as follows.

Lemma 1 *Let $\lambda > 0$ and f satisfies assumptions (f_1) and (f_2). We can deduce that*:

(i) *there exist $\varsigma, \rho > 0$ such that $J_\lambda(u) \ge \varsigma > 0$ for any $u \in X_0^s$, with $\|u\|_{X_0^s} = \rho$;*

(ii) *there exists $e \in X_0^s$, with $e \ge 0$ in $\mathbb{R}^N$ such that $J_\lambda(e) < 0$ and $\|e\|_{X_0^s} > \rho$.*

Proof of Lemma 1 (i) Fixing $\lambda > 0$, from (f_2), for $\varepsilon > 0$, there exists $C_1 > 0$, one has

$$\bigl|F(x,t)\bigr| \le \varepsilon |t|^2 + C_1|t|^{2_s^*(\alpha)}, \quad \forall (x,t) \in \Omega \times \mathbb{R}^+. \tag{7}$$

It is evident that $X_0^s \hookrightarrow L^q(\Omega)$ for $1 \le q \le 2_s^*(\alpha)$, then there exists $C_2 > 0$ such that

$$\int_\Omega |u|^q \, dx \le C_2 \|u\|_{X_0^s}^q. \tag{8}$$

Take $u \in X_0^s$. Combining (4), (5), (7) and (8), we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|_{X_0^s}^2 - \frac{1}{2^*(s)} \int_\Omega \frac{|u|^{2_s^*(\alpha)}}{|z|^\alpha} \, dx - \lambda \int_\Omega F(x,u) \, dx \\ &\ge \frac{1}{2}\|u\|_{X_0^s}^2 - \frac{1}{2_s^*(\alpha)} \int_\Omega \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha} \, dx - \lambda \int_\Omega \bigl(\varepsilon |u|^2 + C_1 |u|^{2_s^*(\alpha)}\bigr) \, dx \\ &\ge \frac{1}{2}\|u\|_{X_0^s}^2 - \frac{C_3}{2_s^*(\alpha)} \|u\|_{X_0^s}^{2_s^*(\alpha)} - \lambda \varepsilon C_4 \|u\|_{X_0^s}^2 - \lambda C_5 \|u\|_{X_0^s}^{2_s^*(\alpha)}, \end{aligned}$$

where C_i, $i = 3,4,5$, are positive constants. For $\varepsilon > 0$ small and according to the fact $2 < 2^*(s)$, then there exists $\rho > 0$ small enough such that $J_\lambda(u) \ge \varsigma > 0$, for any $\|u\|_{X_0^s} = \rho$.

(ii) Given $\lambda > 0$. Take $v \in X_0^s$, with $v \ge 0$ in $\mathbb{R}^N$ and $\|v\|_{X_0^s} = 1$. From (f_1), we get

$$\begin{aligned} J_\lambda(tv) &= \frac{1}{2}\|tv\|_{X_0^s}^2 - \frac{1}{2_s^*(\alpha)} \int_\Omega \frac{|tv|^{2_s^*(\alpha)}}{|x|^\alpha} \, dx - \lambda \int_\Omega F(x,tv) \, dx \\ &\le \frac{t^2}{2}\|v\|_{X_0^s}^2 - \frac{t^{2_s^*(\alpha)}}{2_s^*(\alpha)} \int_\Omega \frac{|v|^{2_s^*(\alpha)}}{|x|^\alpha} \, dx, \end{aligned}$$

then $J_\lambda(tv) \to -\infty$ as $t \to +\infty$. Choosing $e = t_* v$ with $t_* > 0$ large enough, we get $\|e\|_{X_0^s} > \rho$ and $J_\lambda(e) < 0$. This completes the proof of the Lemma 1. □

We recall that a sequence $\{u_j\}_{j\in\mathbb{N}} \subset X_0^s$ is a Palais-Smale sequence for the functional J_λ at level c_λ if

$$J_\lambda(u_j) \to c_\lambda \quad \text{and} \quad J_\lambda'(u_j) \to 0 \quad \text{in } \bigl(X_0^s\bigr)',$$

as $j \to \infty$. We say that J_λ satisfies the Palais-Smale condition if every Palais-Smale sequence of J_λ has a convergent subsequence in X_0^s. Now put

$$c_\lambda = \inf_{g\in\Gamma} \max_{t\in[0,1]} J_\lambda\bigl(g(t)\bigr),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1], X_0^s\bigr) : g(0) = 0, J_\lambda\bigl(g(1)\bigr) < 0\bigr\}.$$

Obviously, $c_\lambda > 0$ from Lemma 1. Next, we introduce an asymptotic condition for the level c_λ.

Lemma 2 *Under the conditions of Lemma* 1, $\lim_{\lambda\to\infty} c_\lambda = 0$.

Proof of Lemma 2 Fix $\lambda > 0$. Since the functional J_λ satisfies the Mountain pass geometry, there exists $t_\lambda > 0$ verifying $J_\lambda(t_\lambda e) = \max_{t\geq 0} J_\lambda(te)$, where $e \in X_0^s$ is the function given in Lemma 1. Hence, by (6), we have $\langle J'_\lambda(t_\lambda e), t_\lambda e\rangle = 0$, that is,

$$t_\lambda^2 \|e\|^2_{X_0^s} = t_\lambda^{2_s^*(\alpha)} \int_\Omega \frac{(e^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx + \lambda \int_\Omega f\bigl(x, t_\lambda e^+\bigr) t_\lambda e^+\,dx. \tag{9}$$

From (f_1), we have

$$t_\lambda^2 \|e\|^2_{X_0^s} \geq t_\lambda^{2_s^*(\alpha)} \int_\Omega \frac{(e^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx,$$

which implies that $\{t_\lambda\}$ is bounded. Hence, there exist a number $t_0 \geq 0$ and a subsequence of $\{\lambda_j\}_{j\in\mathbb{N}}$, which we still denote by $\{\lambda_j\}_{j\in\mathbb{N}}$, such that $\lambda_j \to +\infty$ and $t_{\lambda_j} \to t_0$ as $j \to \infty$. So by (9) there exists $D > 0$ such that $t_\lambda^2 \|e\|^2_{X_0^s} \leq D$ for any $j \in \mathbb{N}$, then

$$\lambda_j \int_\Omega f\bigl(x, t_{\lambda_j} e^+\bigr) t_{\lambda_j} e^+\,dx + t_{\lambda_j}^{2_s^*(\alpha)} \int_\Omega \frac{(e^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx \leq D. \tag{10}$$

If $t_0 > 0$, by (f_1) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{j\to\infty} \int_\Omega f\bigl(x, t_{\lambda_j} e^+\bigr) t_{\lambda_j} e^+\,dx = \int_\Omega f\bigl(x, t_0 e^+\bigr) t_0 e^+\,dx > 0.$$

Recalling that $\lambda_j \to \infty$, we have

$$\lim_{j\to\infty} \left(\lambda_j \int_\Omega f\bigl(x, t_{\lambda_j} e^+\bigr) t_{\lambda_j} e^+\,dx + t_{\lambda_j}^{p_s^*(\alpha)} \int_\Omega \frac{(e^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx \right) = \infty,$$

which contradicts (10). Thus $t_0 = 0$ for $\lambda_j \to \infty$. Now, let us consider the path $g(t) = te$, for $t \in [0,1]$, which belongs to Γ. By Lemma 1 and (f_1), we get

$$0 < c_\lambda \leq \max_{t\in[0,1]} J_\lambda\bigl(g(t)\bigr) \leq J_\lambda(t_\lambda e) \leq \frac{t_\lambda^2}{2} \|e\|^2_{X_0^s}.$$

Notice that $t_{\lambda_j} \to t_0 = 0$ as $j \to \infty$, one has

$$\lim_{\lambda\to+\infty} \frac{t_\lambda^2}{2} \|e\|^2_{X_0^s} = 0,$$

which leads to $\lim_{\lambda\to\infty} c_\lambda = 0$. This completes the proof of the Lemma 2. □

Lemma 3 *Assume that conditions (f_1) and (f_2) hold. If $\{u_j\} \subset X_0^s$ is a $(PS)_{c_\lambda}$ condition of J_λ, then $\{u_j\}$ is bounded in X_0^s.*

Proof of Lemma 3 By (f_2) and the boundedness of Ω, for any $\varepsilon > 0$, there exists $T > 0$, such that

$$\bigl|F(x,t)\bigr| \leq \varepsilon |t|^{2_s^*(\alpha)}, \quad x \in \Omega, t \geq T; \qquad \bigl|F(x,t)\bigr| \leq C_6(\varepsilon), \quad t \in (0,T];$$
$$\bigl|f(x,t)t\bigr| \leq \varepsilon |t|^{2_s^*(\alpha)}, \quad x \in \Omega, t \geq T; \qquad \bigl|f(x,t)t\bigr| \leq C_7(\varepsilon), \quad t \in (0,T],$$

for $C_i(\varepsilon) > 0$, $i = 6, 7$. Furthermore, for any $(x,t) \in \Omega \times \mathbb{R}^+$, we have

$$\left|F(x,t)\right| \leq C_6(\varepsilon) + \varepsilon|t|^{2_s^*(\alpha)}, \qquad \left|f(x,t)t\right| \leq C_7(\varepsilon) + \varepsilon|t|^{2_s^*(\alpha)}.$$

Then, for $\xi \in (2, 2_s^*(\alpha))$, one has

$$F(x,t) - \frac{1}{2}f(x,t)t \leq F(x,t) - \frac{1}{\xi}f(x,t)t \leq C_8(\varepsilon) + \varepsilon|t|^{2_s^*(\alpha)}, \tag{11}$$

for $C_8(\varepsilon) > 0$ and any $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$. Set $l(x,t) := \frac{|t|^{2_s^*(\alpha)-1}}{|z|^\alpha} + \lambda f(x,t)$, we claim that $l(x,t)$ satisfies the (AR) condition. By (11), one easily gets

$$\begin{aligned}
\xi L(x,t) - l(x,y)t &= \left(\frac{\xi}{2_s^*(\alpha)} - 1\right)\frac{|t|^{2_s^*(\alpha)}}{|z|^\alpha} + \lambda\bigl(\xi F(x,t) - f(x,t)t\bigr) \\
&\leq \left(\frac{\xi}{2_s^*(\alpha)} - 1\right)\frac{|t|^{2_s^*(\alpha)}}{|z|^\alpha} + \lambda\xi C_8(\varepsilon) + \lambda\xi\varepsilon|t|^{2_s^*(\alpha)} \\
&= \left(\left(\frac{\xi}{2_s^*(\alpha)} - 1\right)|z|^{-\alpha} + \lambda\xi\varepsilon\right)|t|^{2_s^*(\alpha)} + \lambda\xi C_8(\varepsilon),
\end{aligned}$$

where $L(x,t) = \int_0^t l(x,\tau)\,d\tau$. Thus, for a fixed $\lambda > 0$ and $\varepsilon > 0$ sufficiently small, there exists $T_\lambda' > 0$, such that

$$0 \leq \xi L(x,t) \leq l(x,t)t, \quad t \geq T_\lambda'.$$

Moreover, by (f_2), we obtain

$$L(x,t) - \frac{1}{\xi}l(x,t)t \leq \max_{x\in\Omega, 0\leq t\leq T_\lambda'}\left(F(x,t) - \frac{1}{\xi}f(x,t)t\right) := T_\lambda,$$

for any $0 \leq t \leq T_\lambda'$. Notice that $\xi < 2_s^*(\alpha)$, we obtain $T_\lambda > 0$. It follows from the above inequalities that

$$L(x,t) - \frac{1}{\xi}l(x,t)t \leq T_\lambda, \quad \text{for all } x \in \overline{\Omega}\backslash\bigl\{(0,w^0)\bigr\}, t \geq 0. \tag{12}$$

Combining (f_2), (6) and (12), it follows that

$$\begin{aligned}
c_\lambda + 1 &\geq J_\lambda(u_j) - \frac{1}{\xi}\langle J_\lambda'(u_j), u_j\rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{\xi}\right)\|u_j\|_{X_0^s}^2 + \left(\frac{1}{\xi} - \frac{1}{2_s^*(\alpha)}\right)\int_\Omega \frac{(u_j^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx \\
&\quad - \lambda\int_\Omega\left(F\bigl(x,u_j^+\bigr) - \frac{1}{\xi}f\bigl(x,u_j^+\bigr)u_j^+\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\xi}\right)\|u_j\|_{X_0^s}^2 - \int_\Omega\left(L\bigl(x,u_j^+\bigr) - \frac{1}{\xi}l\bigl(x,u_j^+\bigr)u_j^+\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\xi}\right)\|u_j\|_{X_0^s}^2 - T_\lambda|\Omega|.
\end{aligned}$$

Hence, we obtain $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0^s. This completes the proof of Lemma 3. □

Lemma 4 *Assume that conditions* (f_1) *and* (f_2) *hold. Then* J_λ *satisfies the* $(PS)_{c_\lambda}$ *condition with* $c_\lambda < \frac{2s-\alpha}{2(N-\alpha)} S_{\mu,\alpha}^{\frac{N-\alpha}{2s-\alpha}} - T_\lambda |\Omega|$, *where* T_λ *is a bounded constant given in Lemma* 3.

Proof of Lemma 4 Let $\{u_j\}_{j\in\mathbb{N}} \subset X_0^s$ be a $(PS)_{c_\lambda}$ sequence of J_λ. From Lemma 3, we know that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. Thus there exist a subsequence (still denoted by $\{u_j\}_{j\in\mathbb{N}}$) and $u_\lambda \in X_0^s$ such that

$$\begin{cases} u_j \rightharpoonup u_\lambda, & \text{weakly in } X_0^s, \\ u_j \to u_\lambda, & \text{strongly in } L^p, 2 \le p < 2_s^*(\alpha), \\ u_j \rightharpoonup u_\lambda, & \text{weakly in } L^{2_s^*(\alpha)}(\Omega, |z|^{-\alpha}), \\ u_j \to u_\lambda, & \text{a.e. in } \mathbb{R}^n. \end{cases} \tag{13}$$

Due to the continuity of embedding $X_0^s \hookrightarrow H^s(\Omega) \hookrightarrow L^\nu(\Omega)$ and the boundedness of $\{u_j\}_{j\in\mathbb{N}}$, there exists a constant C_ν such that $\|u\|_\nu \le \|u\|_{X_0^s} \le C_\nu$ for all $u \in X_0^s$ and $\nu \in [2, 2_s^*(\alpha)]$. Now, according to (f_2), for any $\varepsilon > 0$, there exists a $a(\varepsilon) > 0$ such that

$$\bigl|F(x,t)\bigr| \le \frac{1}{2C_\nu} \varepsilon |t|^{2_s^*(\alpha)} + a(\varepsilon) \quad \text{for } (x,t) \in \Omega \times \mathbb{R}^+.$$

Set $\delta = \frac{\varepsilon}{2a(\varepsilon)} > 0$. When $E \subset \Omega$, meas $E < \delta$, one gets

$$\begin{aligned} \left| \int_E F\bigl(x, u_j^+\bigr)\, dx \right| &\le \int_E a(\varepsilon)\, dx + \frac{1}{2C} \varepsilon \int_E \bigl|u_j^+\bigr|^{2_s^*(\alpha)}\, dx \\ &\le a(\varepsilon) \operatorname{meas} E + \frac{1}{2C_\nu} \varepsilon C_\nu \le \varepsilon. \end{aligned}$$

Obviously, $\{\int_\Omega F(x, u_j^+)\, dx, j \in N\}$ is equi-absolutely continuous. It follows from Vitali's convergence theorem that

$$\int_\Omega F\bigl(x, u_j^+\bigr)\, dx \to \int_\Omega F\bigl(x, u_\lambda^+\bigr)\, dx, \quad \text{as } j \to \infty. \tag{14}$$

Similarly, we get

$$\int_\Omega f\bigl(x, u_j^+\bigr) u_j\, dx \to \int_\Omega f\bigl(x, u_\lambda^+\bigr) u_\lambda\, dx, \quad \text{as } j \to \infty. \tag{15}$$

By (13) and (15) we obtain

$$\begin{aligned} &\lim_{j\to\infty} \bigl\langle J_\lambda'(u_j), \nu \bigr\rangle \\ &\quad = \int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\nu(x) - \nu(y))}{|x-y|^{N+2s}}\, dx\, dy - \mu \int_\Omega \frac{u_\lambda^+ \nu}{|z|^{2s}}\, dx \\ &\qquad - \int_\Omega \frac{(u_\lambda^+)^{2_s^*(\alpha)-1} \nu}{|z|^\alpha}\, dx - \lambda \int_\Omega f\bigl(x, u_\lambda^+\bigr) \nu\, dx \\ &\quad = 0, \end{aligned}$$

for any $v \in X_0^s$. That is, $\langle J_\lambda'(u_\lambda), v\rangle = 0$ for any $v \in X_0^s$. Then u_λ is a critical point of J_λ, thus u_λ is a solution of problem (1). It follows from (6) and (12) that

$$\begin{aligned} J_\lambda(u_\lambda) &= J_\lambda(u_\lambda) - \frac{1}{\xi}\bigl\langle J_\lambda'(u_\lambda), u_\lambda\bigr\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\xi}\right)\|u_\lambda\|_{X_0^s}^2 + \left(\frac{1}{\xi} - \frac{1}{2_s^*(\alpha)}\right)\int_\Omega \frac{|u_\lambda|^{2_s^*(\alpha)}}{|z|^\alpha}\,dx \\ &\quad - \lambda\int_\Omega \left(F(x,u_\lambda) - \frac{1}{\xi}f(x,u_\lambda)u_\lambda\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\xi}\right)\|u_\lambda\|_{X_0^s}^2 - \int_\Omega \left(L(x,u_\lambda) - \frac{1}{\xi}l(x,u_\lambda)u_\lambda\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\xi}\right)\|u_\lambda\|_{X_0^s}^2 - T_\lambda|\Omega| \\ &\geq -T_\lambda|\Omega|, \end{aligned}$$

for $\xi \in (2, 2_s^*(\alpha))$. Now, let $w_j = u_j - u_\lambda$, by the Brezis-Lieb lemma [21], we have

$$\begin{aligned} &\int_Q \frac{|u_j(x) - u_j(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad = \int_Q \frac{|w_j(x) - w_j(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + o(1), \end{aligned} \tag{16}$$

$$\int_\Omega \frac{u_j^2}{|z|^{2s}}\,dx = \int_\Omega \frac{w_j^2}{|z|^{2s}}\,dx + \int_\Omega \frac{u_\lambda^2}{|z|^{2s}}\,dx + o(1), \tag{17}$$

$$\int_\Omega \frac{(u_j^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx = \int_\Omega \frac{(w_j^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx + \int_\Omega \frac{(u_\lambda^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx + o(1). \tag{18}$$

Since $J_\lambda(u_j) = c_\lambda + o(1)$, by (14) and (16)-(18), we obtain

$$J_\lambda(u_j) = J_\lambda(u_\lambda) + \frac{1}{2}\|w_j\|_{X_0^s}^2 - \frac{1}{2_s^*(\alpha)}\int_\Omega \frac{(w_j^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx = c_\lambda + o(1). \tag{19}$$

According to $\langle J_\lambda'(u_j), u_j\rangle = o(1)$, (15) and (16)-(18), we get

$$\|w_j\|_{X_0^s}^2 - \int_\Omega \frac{(w_j^+)^{2_s^*(\alpha)}}{|z|^{2s}}\,dx = o(1). \tag{20}$$

Assume that $\|w_j\|_{X_0^s} \to l$, it follows from (20) that

$$\int_\Omega \frac{(w_j^+)^{2_s^*(\alpha)}}{|z|^{2s}}\,dx \to l^2,$$

as $j \to \infty$. From (4), one has

$$\|w_j\|_{X_0^s}^2 \geq S_{\mu,\alpha}\left(\int_\Omega \frac{(w_j^+)^{2_s^*(\alpha)}}{|z|^{2s}}\,dx\right)^{\frac{2}{2_s^*(\alpha)}}.$$

We get $l \geq S_{\mu,\alpha}^{\frac{N-\alpha}{2s-\alpha}}$. It follows from (19) and (20) that

$$\begin{aligned} c_\lambda + o(1) &= J_\lambda(u_\lambda) + \frac{1}{2}\|w_j\|_{X_0^s}^2 - \frac{1}{2_s^*(\alpha)}\int_\Omega \frac{(w_j^+)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx + o(1) \\ &\geq \frac{2s-\alpha}{2(N-\alpha)} S_{\mu,\alpha}^{\frac{N-\alpha}{2s-\alpha}} - T_\lambda|\Omega| + o(1), \end{aligned}$$

which contradicts $c_\lambda < \frac{2s-\alpha}{2(N-\alpha)} S_{\mu,\alpha}^{\frac{N-\alpha}{2s-\alpha}} - T_\lambda|\Omega|$. Therefore, we have $l = 0$, which implies that $u_j \to u_\lambda$ in X_0^s. This completes the proof of the Lemma 4. □

3 Proof of Theorem 1

Thanks to Lemmas 1, 2, 3 and Lemma 4, the functional J_λ satisfies all the assumptions of the mountain pass theorem for any $\lambda \geq \lambda^*$, with $\lambda^* > 0$. This guarantees the existence of a critical point $u_\lambda \in X_0^s$ for J_λ at level c_λ. Since $J_\lambda(u_\lambda) = c_\lambda > 0 = J_\lambda(0)$ we have $u_\lambda \not\equiv 0$. By [22, Lemma 8], we have $u_\lambda^- \in X_0^s$ and we let $\varphi = u_\lambda^-$ in (6), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y))}{|x-y|^{N+2s}}\,dy - \mu\int_\Omega \frac{(u_\lambda^-)^2}{|z|^{2s}}\,dx \\ &= \int_\Omega \frac{(u_\lambda^-)^{2^*(s)}}{|z|^\alpha}\,dx + \lambda\int_\Omega f(x,u_\lambda)u_\lambda^-\,dx. \end{aligned}$$

Moreover, for a.e. $x, y \in \mathbb{R}^N$, one has

$$\begin{aligned} &\big(u_\lambda(x) - u_\lambda(y)\big)\big(u_\lambda^-(x) - u_\lambda^-(y)\big) \\ &= -u_\lambda^+(x)u_\lambda^-(y) - u_\lambda^-(x)u_\lambda^+(y) - \big(u_\lambda^-(x) - u_\lambda^-(y)\big)^2 \\ &\leq -\big|u_\lambda^-(x) - u_\lambda^-(y)\big|^2. \end{aligned} \tag{21}$$

From (3) and (21), we have

$$\begin{aligned} &\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y))}{|x-y|^{N+2s}}\,dx\,dy - \mu\int_\Omega \frac{(u_\lambda^-)^2}{|z|^{2s}}\,dx \\ &\leq \left(1 - \frac{\mu}{a_{k,s}}\right)\int_{\mathbb{R}^N} \frac{-|u_\lambda^-(x) - u_\lambda^-(y)|^2}{|x-y|^{N+2s}}\,dy \leq 0, \end{aligned}$$

by the fact that $\mu < a_{k,s}$. Hence, according to (f_1), $f(x,u_\lambda(x))u_\lambda^-(x) = 0$ for $x \in \mathbb{R}^N$, we obtain

$$\int_\Omega \frac{(u_\lambda^-)^{2_s^*(\alpha)}}{|z|^\alpha}\,dx \leq 0,$$

which implies that $u_\lambda^- \equiv 0$. Hence, $u_\lambda \geq 0$. It implies $(-\Delta)^s u_\lambda \geq 0$. Then, by the strong maximum principle, we obtain u_λ is a positive solution of problem (1). This completes the proof of Theorem 1.

4 Conclusion

In this paper, we devoted our study to the existence of solutions for a fractional elliptic problems with the Hardy-Sobolev-Maz'ya potential and critical nonlinearities. The approach of this paper is by the well-known mountain pass theorem. The nonlinear term

f satisfies assumptions (f_1), (f_2) without the (AR) conditions. We established a new term $l(x,t)$, which satisfies the (AR) conditions combined with the critical term $\frac{|t|^{2_s^*(\alpha)-1}}{|z|^{\alpha}}$ by using some analysis techniques. Then we overcame the compactness and obtained a positive solution of problem (1). Our results are new and the work established in this paper is of quite a general nature.

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Competing interests
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Authors' contributions
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References
1. Nezza, ED, Palatucci, G, Valdinoci, E: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521-573 (2012)
2. Silvestre, L: Regularity of the obstacle problem for a fractional power of the Laplace operator. Commun. Pure Appl. Math. **60**(1), 67-112 (2007)
3. Servadei, R, Valdinoci, E: Mountain Pass solutions for non-local elliptic operators. J. Math. Anal. Appl. **389**(2), 887-898 (2012)
4. Guliyev, VS, Omarova, MN, Ragusa, MA, Scapellato, A: Commutators and generalized local Morrey spaces. J. Math. Anal. Appl. **457**(2), 1388-1402 (2018)
5. Scapellato, A: On some qualitative results for the solution to a Dirichlet problem in Local Generalized Morrey Spaces. 1798, UNSP 020138 (2017) doi:10.1063/1.4972730
6. Scapellato, A: Some properties of integral operators on generalized Morrey spaces. AIP Conf. Proc. **1863**(21), 510004 (2017). doi:10.1063/1.4992662
7. Luo, H, Tang, X, Li, S: Mountain pass and linking type sign-changing solutions for nonlinear problems involving the fractional Laplacian. Bound. Value Probl. **2017**, 108 (2017).
8. Shang, X, Zhang, J, Yang, Y: Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent. Commun. Pure Appl. Anal. **13**(2), 567-584 (2014)
9. Servadei, R, Valdinoci, E: The Brezis-Nirenberg result for the fractional Laplacian. Trans. Am. Math. Soc. **367**(1), 67-102 (2015)
10. Fiscella, A, Bisci, GM, Servadei, R: Bifurcation and multiplicity results for critical nonlocal fractional Laplacian problems. Bull. Sci. Math. **140**(1), 14-35 (2016)
11. Servadei, R, Valdinoci, E: Fractional Laplacian equations with critical Sobolev exponent. Rev. Mat. Complut. **28**(3), 1-22 (2015)
12. Fiscella, A, Bisci, GM, Servadei, R: Multiplicity results for fractional Laplace problems with critical growth. Manuscr. Math. 1-20 (2016)
13. Abdellaoui, B, Medina, M, Peral, I, Primo, A: The effect of the Hardy potential in some Calderón-Zygmund properties for the fractional Laplacian. J. Differ. Equ. **260**(11), 8160-8206 (2016)
14. Dipierro, S, Montoro, L, Peral, I, Sciunzi, B: Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential. Calc. Var. Partial Differ. Equ. **55**(4), 1-29 (2016)
15. Barrios, B, Medina, M, Peral, I: Some remarks on the solvability of non-local elliptic problems with the Hardy potential. Commun. Contemp. Math. **16**(4), 1350046 (2014)
16. Jiang, RT, Tang, CL: Semilinear elliptic problems involving Hardy-Sobolev-Maz'ya potential and Hardy-Sobolev critical exponents. Electron. J. Differ. Equ. **2016**, 12 (2016)
17. Ding, L, Tang, CL: Existence and multiplicity of solutions for semilinear elliptic equations with Hardy terms and Hardy-Sobolev critical exponents. Appl. Math. Lett. **20**(12), 1175-1183 (2007)
18. Bouchekif, M, Matallah, A: Multiple positive solutions for elliptic equations involving a concave term and critical Sobolev-Hardy exponent. Appl. Math. Lett. **22**(2), 268-275 (2009)
19. Shang, YY: Existence and multiplicity of positive solutions for some Hardy-Sobolev critical elliptic equation with boundary singularities. Nonlinear Anal. **75**(5), 2724-2734 (2012)
20. Cotsiolis, A, Tavoularis, N: Best constants for Sobolev inequalities for higher order fractional derivatives. J. Math. Anal. Appl. **295**, 225-236 (2004)

21. Brézis, H, Lieb, E: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**(3), 486-490 (1983)
22. Fiscella, A, Valdinoci, E: A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Anal. **94**, 156-170 (2014)