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Sharp well-posedness of the Cauchy problem for a generalized Ostrovsky equation with positive dispersion

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Abstract

The goal of this paper is two-fold. Firstly, by using the Fourier restriction norm method and the fixed point theorem, we prove that the Cauchy problem for a generalized Ostrovsky equation

$$\partial_x\Big(u_t - \beta\partial_x^3 u + \frac{1}{3}\partial_x(u^3)\Big) - \gamma u = 0, \quad \beta > 0, \gamma > 0,$$

is locally well-posed in $H^s(\mathbf{R})$ with $s \geq \frac{1}{4}$. Secondly, we prove that the Cauchy problem for a generalized Ostrovsky equation is not well-posed in $H^s(\mathbf{R})$ with $s < \frac{1}{4}$ in the sense that the solution map is C^3.

MSC: 35G25

Keywords: generalized Ostrovsky equation with positive dispersion; Cauchy problem; sharp well-posedness

1 Introduction

In this paper, we are concerned with the Cauchy problem for a generalized Ostrovsky equation with positive dispersion,

$$\partial_x\Big(u_t - \beta\partial_x^3 u + \frac{1}{3}\partial_x\big(u^3\big)\Big) - \gamma u = 0, \quad \gamma > 0, \beta \in \mathbf{R}. \tag{1.1}$$

Here $u(x,t)$ represents the free surface of the liquid and the parameter $\gamma > 0$ measures the effect of rotation. (1.1) describes the propagation of internal waves of even modes in the ocean; for instance, see the work of Galkin and Stepanyants [1], Leonov [2], and Shrira [3, 4]. The parameter β determines the type of dispersion, more precisely, when $\beta < 0$, (1.1) denotes the generalized Ostrovsky equation with negative dispersion; when $\beta > 0$, (1.1) denotes the generalized Ostrovsky equation with positive dispersion.

When $\gamma = 0$, (1.1) reduces to the modified Korteweg-de Vries equation which has been investigated by many authors; for instance, see [5–11]. Kenig *et al.* [9] proved that the Cauchy problem for the modified KdV equation is locally well-posed in $H^s(\mathbf{R})$ with $s \geq \frac{1}{4}$. Kenig *et al.* [10] proved that the Cauchy problem for the modified KdV equation is ill-posed in $H^s(\mathbf{R})$ with $s < \frac{1}{4}$. Colliander *et al.* [6] proved that the Cauchy problem for the modified KdV equation is globally well-posed in $H^s(\mathbf{R})$ with $s > \frac{1}{4}$ and globally well-posed in $H^s(\mathbf{T})$ with $s \geq \frac{1}{2}$. Guo [7] and Kishimoto [11] proved that the modified KdV equation is globally well-posed in $H^{\frac{1}{4}}(\mathbf{R})$ with the aid of the I method and some new spaces.

Now we give a brief review of the Ostrovsky equation,

$$u_t - \beta\partial_x^3 u + \frac{1}{3}\partial_x\bigl(u^2\bigr) - \gamma\partial_x^{-1}u = 0, \quad \gamma > 0. \tag{1.2}$$

Equation (1.2) was proposed by Ostrovsky in [12] as a model for weakly nonlinear long waves in a rotating liquid, by taking into account the Coriolis force, to describe the propagation of surface waves in the ocean in a rotating frame of reference. The parameter β determines the type of dispersion, more precisely, $\beta < 0$ (negative dispersion) for surface and internal waves in the ocean or surface waves in a shallow channel with an uneven bottom and $\beta > 0$ (positive dispersion) for capillary waves on the surface of liquid or for oblique magneto-acoustic waves in plasma [1, 13–15]. Some authors have investigated the stability of the solitary waves or soliton solutions of (1.2); for instance, see [16–18].

Many people have studied the Cauchy problem for (1.2), for instance, see [17, 19–30]. The result of [23, 25, 31] showed that $s = -\frac{3}{4}$ is the critical regularity index for (1.2). Coclite and di Ruvo [32, 33] have investigated the convergence of the Ostrovsky equation to the Ostrovsky-Hunter one and the dispersive and diffusive limits for Ostrovsky-Hunter type equation. Recently, Li *et al.* [34] proved that the Cauchy problem for the Ostrovsky equation with negative dispersion is locally well-posed in $H^{-\frac{3}{4}}(\mathbf{R})$.

Levandosky and Liu [16] studied the stability of solitary waves of the generalized Ostrovsky equation,

$$\bigl[u_t - \beta u_{xxx} + \bigl(f(u)\bigr)_x\bigr]_x = \gamma u, \quad x \in \mathbf{R}, \tag{1.3}$$

where f is a C^2 function which is homogeneous of degree $p \geq 2$ in the sense that it satisfies $sf'(s) = pf(s)$. Levandosky [18] studied the stability of ground state solitary waves of (1.4) with homogeneous nonlinearities of the form $f(u) = c_1|u|^p + c_2|u|^{p-1}u$, $c_1, c_2 \in \mathbf{R}$, $p \geq 2$.

Equation (1.1) can be written in the following form:

$$u_t - \beta\partial_x^3 u + \frac{1}{3}\partial_x\bigl(u^3\bigr) - \gamma\partial_x^{-1}u = 0. \tag{1.4}$$

Let $w(x,t) = \beta^{-\frac{1}{2}}u(x, \beta^{-1}t)$, then $w(x,t)$ is the solution to

$$w_t - w_{xxx} + \frac{1}{3}\partial_x\bigl(w^3\bigr) - \gamma\beta^{-1}w = 0.$$

Without loss of generality, we can assume that $\beta = \gamma = 1$.

Motivated by [35], firstly, by using the $X_{s,b}$ spaces introduced by [36–40] and developed in [8, 41, 42] and the Strichartz estimates established in [19, 43], we prove that (1.3) with initial data

$$u(x,0) = u_0(x) \tag{1.5}$$

is locally well-posed in $H^s(\mathbf{R})$ with $s \geq \frac{1}{4}$, $\beta > 0$, $\gamma > 0$; secondly, we prove that the problems (1.3), (1.5) are not quantitatively well-posed in $H^s(\mathbf{R})$ with $s < \frac{1}{4}$, $\beta \neq 0$, $\gamma > 0$. Thus, our result is sharp.

We introduce some notations before giving the main result. Throughout this paper, we assume that C is a positive constant which may vary from line to line and $0 < \epsilon < 10^{-4}$. $A \sim B$ means that $|B| \leq |A| \leq 4|B|$. $A \gg B$ means that $|A| > 4|B|$. $\psi(t)$ is a smooth function supported in $[-1, 2]$ and equals 1 in $[-1, 1]$. We assume that $\mathcal{F}u$ is the Fourier transformation of u with respect to both space and time variables and $\mathcal{F}^{-1}u$ is the inverse transformation of u with respect to both space and time variables, while $\mathcal{F}_x u$ denotes the Fourier transformation of u with respect to the space variable and $\mathcal{F}_x^{-1}u$ denotes the inverse transformation of u with respect to the space variable. Let $I \subset \mathbf{R}$, $\chi_I(x) = 1$ if $x \in I$; $\chi_I(x) = 0$ if x does not belong to I. Let

$$\langle \cdot \rangle = 1 + |\cdot|, \qquad \phi(\xi) = \xi^3 + \frac{1}{\xi}, \qquad \sigma = \tau + \phi(\xi), \qquad \sigma_j = \tau_j + \phi(\xi_j) \quad (j = 1, 2, 3).$$

The space $X_{s,b}$ is defined by

$$X_{s,b} = \left\{ u \in \mathscr{S}'\left(\mathbf{R}^2\right) : \|u\|_{X_{s,b}} = \left\| \langle \xi \rangle^s \left\langle \tau + \phi(\xi) \right\rangle^b \mathcal{F}u(\xi, \tau) \right\|_{L^2_{\tau\xi}(\mathbf{R}^2)} < \infty \right\}.$$

The space $X_{s,b}^T$ denotes the restriction of $X_{s,b}$ onto the finite time interval $[-T, T]$ and is equipped with the norm

$$\|u\|_{X_{s,b}^T} = \inf\left\{ \|w\|_{X_{s,b}} : w \in X_{s,b}, u(t) = w(t) \text{ for } -T \leq t \leq T \right\}.$$

The main results of this paper are as follows.

Theorem 1.1 *Let $s \geq \frac{1}{4}$ and $\beta > 0$ and $\gamma > 0$. Then the problems* (1.4), (1.5) *are locally well-posed in $H^s(\mathbf{R})$. More precisely, for $u_0 \in H^s(\mathbf{R})$, there exist a $T > 0$ and a unique solution $u \in C([-T, T]; H^s(\mathbf{R}))$.*

Remark 1 The result of Theorem 1.1 is optimal in the sense of Theorem 1.2.

Theorem 1.2 *Let $s < \frac{1}{4}$ and $\beta > 0$ and $\gamma > 0$. Then the problems* (1.4), (1.5) *are not well-posed in $H^s(\mathbf{R})$ in the sense that the solution map is C^3.*

The rest of the paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we establish a trilinear estimate. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2.

2 Preliminaries

In this section, we give Lemmas 2.1-2.4.

Lemma 2.1 *Let $0 < \epsilon < \frac{1}{10^8}$ and $\mathcal{F}(P^a f)(\xi) = \chi_{\{|\xi| \geq a\}}(\xi)\mathcal{F}f(\xi)$ with $a \geq 2$ and $\mathcal{F}(D_x^b f)(\xi) = |\xi|^b \mathcal{F}f(\xi)$ with $b \in \mathbf{R}$. Then we have*

$$\|u\|_{L^6_{xt}} \leq C\|u\|_{X_{0,\frac{1}{2}+\epsilon}}, \tag{2.1}$$

$$\left\| D_x^{\frac{1}{6}} P^a u \right\|_{L^6_{xt}} \leq C\|u\|_{X_{0,\frac{1}{2}+\epsilon}}, \tag{2.2}$$

$$\|u\|_{L^4_{xt}} \leq C\|u\|_{X_{0,\frac{3}{4}(\frac{1}{2}+\epsilon)}}. \tag{2.3}$$

For the proof of Lemma 2.1, we refer the reader to (2.27) and (2.21) of [19].

Lemma 2.2 *Let* $\phi(\xi) = \xi^3 + \frac{1}{\xi}$ *and*

$$\mathcal{F}\left(I^s(u,v)\right)(\xi,\tau) = \int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \left|\phi'(\xi_1) - \phi'(\xi_2)\right|^s \mathcal{F}u_1(\xi_1,\tau_1)\mathcal{F}u_2(\xi_2,\tau_2)\,d\xi_1\,d\tau_1.$$

Then we have

$$\left\|I^{\frac{1}{2}}(u_1,u_2)\right\|_{L^2_{xt}} \leq C\prod_{j=1}^{2}\|u_j\|_{X_{0,\frac{1}{2}+\epsilon}}. \tag{2.4}$$

For the proof of Lemma 2.2, we refer the reader to Lemma 2.5 of [43].

Lemma 2.3 *Let* $T \in (0,1)$ *and* $b \in (\frac{1}{2}, \frac{3}{2})$. *Then, for* $s \in \mathbf{R}$ *and* $\theta \in [0, \frac{3}{2} - b)$, *we have*

$$\begin{aligned}
&\left\|\eta_T(t)S(t)\phi\right\|_{X_{s,b}(\mathbf{R}^2)} \leq CT^{\frac{1}{2}-b}\|\phi\|_{H^s(\mathbf{R})},\\
&\left\|\eta_T(t)\int_0^t S(t-\tau)F(\tau)\,d\tau\right\|_{X_{s,b}(\mathbf{R}^2)} \leq CT^{\theta}\|F\|_{X_{s,b-1+\theta}(\mathbf{R}^2)}.
\end{aligned}$$

For the proof of Lemma 2.3, we refer the reader to [8, 39, 44].

Lemma 2.4 *Let* $a_j \in \mathbf{R}$ $(j = 1,2,3)$ *and* $\prod_{j=1}^{3} a_j \neq 0$. *Then we have*

$$\begin{aligned}
&\left(\sum_{j=1}^{3} a_j\right)^3 + \frac{1}{\sum_{j=1}^{3} a_j} - \sum_{j=1}^{3}\left(a_j^3 + \frac{1}{a_j}\right)\\
&\quad = 3(a_1+a_2)(a_1+a_3)(a_2+a_3)\left[1 - \frac{1}{3\prod_{j=1}^{3} a_j(\sum_{j=1}^{3} a_j)}\right].
\end{aligned} \tag{2.5}$$

Proof By using the following two identities:

$$\begin{aligned}
&\left(\sum_{j=1}^{3} a_j\right)^3 - \left(\sum_{j=1}^{3} a_j^3\right) = 3(a_1+a_2)(a_1+a_3)(a_2+a_3),\\
&\left(\sum_{j=1}^{3} a_j\right)(a_1a_2 + a_1a_3 + a_2a_3) - \prod_{j=1}^{3} a_j = (a_1+a_2)(a_1+a_3)(a_2+a_3),
\end{aligned}$$

which can be found in [6], we have

$$\begin{aligned}
&\left(\sum_{j=1}^{3} a_j\right)^3 + \frac{1}{\sum_{j=1}^{3} a_j} - \sum_{j=1}^{3}\left(a_j^3 + \frac{1}{a_j}\right) \\
&= \left(\sum_{j=1}^{3} a_j\right)^3 - \sum_{j=1}^{3} a_j^3 - \left[\sum_{j=1}^{3}\frac{1}{a_j} - \frac{1}{\sum_{j=1}^{3} a_j}\right] \\
&= 3(a_1+a_2)(a_1+a_3)(a_2+a_3) - \left[\frac{(\sum_{j=1}^{3} a_j)(a_1a_2+a_1a_3+a_2a_3) + \prod_{j=1}^{3} a_j}{\prod_{j=1}^{3} a_j(\sum_{j=1}^{3} a_j)}\right] \\
&= 3(a_1+a_2)(a_1+a_3)(a_2+a_3) - \left[\frac{(a_1+a_2)(a_1+a_3)(a_2+a_3)}{\prod_{j=1}^{3} a_j(\sum_{j=1}^{3} a_j)}\right] \\
&= 3(a_1+a_2)(a_1+a_3)(a_2+a_3)\left[1 - \frac{1}{3\prod_{j=1}^{3} a_j(\sum_{j=1}^{3} a_j)}\right].
\end{aligned}$$

Thus, (2.5) is valid.

This ends the proof of Lemma 2.4. □

3 The trilinear estimate

In this section, by using Lemmas 2.1-2.2, we give the proof of Lemma 3.1.

Lemma 3.1 *Let $u_j \in X_{s,\frac{1}{2}+\epsilon}$ with $s \geq \frac{1}{4}$ and $j = 1,2,3$. Then we have*

$$\left\|\partial_x\left(\prod_{j=1}^{3} u_j\right)\right\|_{X_{s,-\frac{1}{2}+2\epsilon}} \leq C\prod_{j=1}^{3}\|u_j\|_{X_{s,\frac{1}{2}+\epsilon}}. \tag{3.1}$$

Proof To prove (3.1), by duality, it suffices to prove that

$$\int_{\mathbf{R}^2} \bar{u}(x,t)\partial_x\left(\prod_{j=1}^{3} u_j\right) dx\,dt \leq C\left[\prod_{j=1}^{3}\|u_j\|_{X_{s,\frac{1}{2}+\epsilon}}\right]\|u\|_{X_{-s,\frac{1}{2}-2\epsilon}}. \tag{3.2}$$

Let

$$\begin{aligned}
&F(\xi,\tau) = \langle\xi\rangle^{-s}\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\mathscr{F}u(\xi,\tau), \\
&F_j(\xi_j,\tau_j) = \langle\xi_j\rangle^{s}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}\mathscr{F}u_j(\xi_j,\tau_j) \quad (j = 1,2,3).
\end{aligned} \tag{3.3}$$

To obtain (3.2), from (3.3), it suffices to prove that

$$\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2+\xi_3 \\ \tau=\tau_1+\tau_2+\tau_3}} \frac{|\xi|\langle\xi\rangle^s F(\xi,\tau)\prod_{j=1}^{3} F_j(\xi_j,\tau_j)}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\xi_j\rangle^s\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\,d\xi\,d\tau \\
&\quad\leq C\|F\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^{3}\|F_j\|_{L^2_{\xi\tau}}\right).
\end{aligned} \tag{3.4}$$

Without loss of generality, by using the symmetry, we assume that $|\xi_1| \geq |\xi_2| \geq |\xi_3|$ and $F(\xi,\tau) \geq 0$, $F_j(\xi_j,\tau_j) \geq 0$ $(j = 1,2)$. We define

$$\Omega_1 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \in \mathrm{R}^6, \xi = \sum_{j=1}^{3}\xi_j, \tau = \sum_{j=1}^{3}\tau_j, |\xi_3| \leq |\xi_2| \leq |\xi_1| \leq 64\right\},$$

$$\Omega_2 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \in \mathrm{R}^6, \xi = \sum_{j=1}^{3}\xi_j, \tau = \sum_{j=1}^{3}\tau_j, |\xi_1| \geq 64, |\xi_1| \geq 4|\xi_2|\right\},$$

$$\Omega_3 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \in \mathrm{R}^6, \xi = \sum_{j=1}^{3}\xi_j, \tau = \sum_{j=1}^{3}\tau_j, |\xi_1| \geq 64, |\xi_1| \sim |\xi_2|, |\xi_2| \gg |\xi_3|\right\},$$

$$\Omega_4 = \left\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \in \mathrm{R}^6, \xi = \sum_{j=1}^{3}\xi_j, \tau = \sum_{j=1}^{3}\tau_j, |\xi_1| \geq 64, |\xi_1| \sim |\xi_2| \sim |\xi_3|\right\}.$$

Obviously, $\{(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \in \mathrm{R}^6, \xi = \sum_{j=1}^{3}\xi_j, \tau = \sum_{j=1}^{3}\tau_j, |\xi_3| \leq |\xi_2| \leq |\xi_1|\} \subset \bigcup_{j=1}^{4}\Omega_j$. Let

$$K(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) = \frac{|\xi|\langle\xi\rangle^s}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}} \tag{3.5}$$

and

$$I = \int_{\mathbf{R}^2}\int_{\substack{\xi=\sum_{j=1}^{3}\xi_j \\ \tau=\sum_{j=1}^{3}\tau_j}} K(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau)F(\xi,\tau)\prod_{j=1}^{3}F_j(\xi_j,\tau_j)\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\,d\xi\,d\tau.$$

(1) Ω_1. In this subregion, we have

$$K(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \leq \frac{C}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}. \tag{3.6}$$

By using (3.6) and the Cauchy-Schwartz inequality and the Plancherel identity and the Hölder inequality as well as (2.1), we have

$$\begin{aligned} I &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\sum_{j=1}^{3}\xi_j \\ \tau=\sum_{j=1}^{3}\tau_j}} \frac{F(\xi,\tau)\prod_{j=1}^{3}F_j(\xi_j,\tau_j)}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\,d\xi\,d\tau \\ &\leq C\left\|\frac{F(\xi,\tau)}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}}\right\|_{L^2_{\xi\tau}}\left\|\int_{\substack{\xi=\sum_{j=1}^{3}\xi_j \\ \tau=\sum_{j=1}^{3}\tau_j}}\frac{\prod_{j=1}^{3}F_j(\xi_j,\tau_j)}{\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\right\|_{L^2_{\xi\tau}} \\ &\leq C\|F\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^{3}\left\|\mathscr{F}^{-1}\left(\frac{F_j}{\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\right)\right\|_{L^6_{xt}}\right) \\ &\leq C\|F\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^{3}\|F_j\|_{L^2_{\xi\tau}}\right). \end{aligned}$$

(2) Ω_2. In this subregion, since $|\phi'(\xi_1)-\phi'(\xi_2)| = 3|\xi_1^2-\xi_2^2||1+\frac{1}{3\xi_1^2\xi_2^2}| \geq 3|\xi_1^2-\xi_2^2| \geq C|\xi|^2$ and $|\xi| \sim |\xi_1|$, we have

$$\begin{aligned} K(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) &\leq \frac{C|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}} \\ &\leq C\frac{C|\xi_1^2-\xi_2^2|^{\frac{1}{2}}|1+\frac{1}{3\xi_1^2\xi_2^2}|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}} = \frac{C|\phi'(\xi_1)-\phi'(\xi_2)|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}. \end{aligned} \tag{3.7}$$

By using (3.7) and the Cauchy-Schwartz inequality and the Plancherel identity and the Hölder inequality as well as (2.3)-(2.4), since $\frac{3}{4}(\frac{1}{2}+\epsilon) < \frac{1}{2}-2\epsilon$, we have

$$\begin{aligned} I &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\sum_{j=1}^3\xi_j \\ \tau=\sum_{j=1}^3\tau_j}} \frac{|\phi'(\xi_1)-\phi'(\xi_2)|^{\frac{1}{2}}F(\xi,\tau)\prod_{j=1}^3F_j(\xi_j,\tau_j)}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\,d\xi\,d\tau \\ &\leq C\left\|\mathscr{F}^{-1}\left(\frac{F}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}}\right)\right\|_{L^4_{xt}}\left\|I^{\frac{1}{2}}\left(\mathscr{F}^{-1}\left(\frac{F_1}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon}}\right),\mathscr{F}^{-1}\left(\frac{F_1}{\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon}}\right)\right)\right\|_{L^2_{xt}} \\ &\quad\times\left\|\mathscr{F}^{-1}\left(\frac{F_3}{\langle\sigma_3\rangle^{\frac{1}{2}+\epsilon}}\right)\right\|_{L^4_{xt}} \\ &\leq C\|F\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^3\|F_j\|_{L^2_{\xi\tau}}\right). \end{aligned}$$

(3) Ω_3. In this subregion, since $|\phi'(\xi_2)-\phi'(\xi_3)| = 3|\xi_2^2-\xi_3^2||1+\frac{1}{3\xi_2^2\xi_3^2}| \geq 3|\xi_2^2-\xi_3^2| \geq C|\xi_1|^2$, we have

$$\begin{aligned} K(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) &\leq \frac{C|\xi_1|}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}} \\ &\leq CC\frac{C|\xi_2^2-\xi_3^2|^{\frac{1}{2}}|1+\frac{1}{3\xi_2^2\xi_3^2}|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}} \leq \frac{C|\phi'(\xi_2)-\phi'(\xi_3)|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}. \end{aligned} \tag{3.8}$$

By using (3.8) and the Cauchy-Schwartz inequality and the Plancherel identity and the Hölder inequality as well as (2.3)-(2.4), since $\frac{3}{4}(\frac{1}{2}+\epsilon) < \frac{1}{2}-2\epsilon$, we have

$$\begin{aligned} I &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\sum_{j=1}^3\xi_j \\ \tau=\sum_{j=1}^3\tau_j}} \frac{|\phi'(\xi_2)-\phi'(\xi_3)|^{\frac{1}{2}}F(\xi,\tau)\prod_{j=1}^3F_j(\xi_j,\tau_j)}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^3\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\,d\xi\,d\tau \\ &\leq C\left\|\mathscr{F}^{-1}\left(\frac{F}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}}\right)\right\|_{L^4_{xt}}\left\|I^{\frac{1}{2}}\left(\mathscr{F}^{-1}\left(\frac{F_2}{\langle\sigma_2\rangle^{\frac{1}{2}+\epsilon}}\right),\mathscr{F}^{-1}\left(\frac{F_3}{\langle\sigma_3\rangle^{\frac{1}{2}+\epsilon}}\right)\right)\right\|_{L^2_{xt}} \\ &\quad\times\left\|\mathscr{F}^{-1}\left(\frac{F_1}{\langle\sigma_1\rangle^{\frac{1}{2}+\epsilon}}\right)\right\|_{L^4_{xt}} \\ &\leq C\|F\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^3\|F_j\|_{L^2_{\xi\tau}}\right). \end{aligned}$$

(4) Ω_4. In this subregion, since $s \geq \frac{1}{4}$ and $|\xi_1| \sim |\xi_2| \sim |\xi_3|$, we have

$$K(\xi_1,\tau_1,\xi_2,\tau_2,\xi,\tau) \leq \frac{C|\xi_1|^{1-2s}}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}} \leq \frac{C\prod_{j=1}^{3}|\xi_j|^{\frac{1}{6}}}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}. \tag{3.9}$$

By using (3.9) and the Cauchy-Schwartz inequality and the Plancherel identity and the Hölder inequality as well as (2.2), since $\frac{3}{4}(\frac{1}{2}+\epsilon) < \frac{1}{2} - 2\epsilon$, we have

$$\begin{aligned} I &\leq C\int_{\mathbf{R}^2}\int_{\substack{\xi=\sum_{j=1}^{3}\xi_j\\ \tau=\sum_{j=1}^{3}\tau_j}} \frac{F(\xi,\tau)\prod_{j=1}^{3}|\xi_j|^{\frac{1}{6}}F_j(\xi_j,\tau_j)}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}\prod_{j=1}^{3}\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\,d\xi\,d\tau \\ &\leq C\left\|\frac{F}{\langle\sigma\rangle^{\frac{1}{2}-2\epsilon}}\right\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^{3}\left\|D_x^{\frac{1}{6}}P^2\mathcal{F}^{-1}\left(\frac{F_j}{\langle\sigma_j\rangle^{\frac{1}{2}+\epsilon}}\right)\right\|_{L^6_{xt}}\right) \\ &\leq C\|F\|_{L^2_{\xi\tau}}\left(\prod_{j=1}^{3}\|F_j\|_{L^2_{\xi\tau}}\right). \end{aligned}$$

This completes the proof of Lemma 3.1. □

4 Proof of Theorem 1.1

In this section, we use Lemmas 2.3, 3.1 to prove Theorem 1.1.

The solution to (1.3), (1.5) can be formally rewritten as follows:

$$u(t) = e^{-t(-\partial_x^3-\partial_x^{-1})}u_0 + \frac{1}{3}\int_0^t e^{-(t-s)(-\partial_x^3-\partial_x^{-1})}\partial_x\bigl(u^3\bigr)\,ds. \tag{4.1}$$

We define

$$\Phi(u) = \psi(t)e^{-t(-\partial_x^3-\partial_x^{-1})}u_0 + \frac{1}{3}\psi\left(\frac{t}{T}\right)\int_0^t e^{-(t-s)(-\partial_x^3-\partial_x^{-1})}\partial_x\bigl(u^3\bigr)\,ds. \tag{4.2}$$

By taking advantaging of Lemmas 2.3, 3.1, we derive that

$$\begin{aligned} \bigl\|\Phi(u)\bigr\|_{X_{s,\frac{1}{2}+\epsilon}} &\leq C\|u_0\|_{H^s(\mathbf{R})} + C\left\|\psi\left(\frac{t}{T}\right)\int_0^t e^{-(t-s)(-\partial_x^3-\partial_x^{-1})}\partial_x\bigl(u^3\bigr)\,ds\right\|_{X_{s,\frac{1}{2}+\epsilon}} \\ &\leq C\|u_0\|_{H^s(\mathbf{R})} + CT^{\epsilon}\bigl\|\partial_x\bigl(u^3\bigr)\,ds\bigr\|_{X_{s,-\frac{1}{2}+2\epsilon}} \\ &\leq C\|u_0\|_{H^s(\mathbf{R})} + CT^{\epsilon}\|u\|^3_{X_{s,\frac{1}{2}+\epsilon}}. \end{aligned} \tag{4.3}$$

We define $B = \{u \in X_{s,\frac{1}{2}+\epsilon} : \|u\|_{X_{s,\frac{1}{2}+\epsilon}} \leq 2C\|u_0\|_{H^s(\mathbf{R})}\}$. By using (4.3), by choosing T sufficiently small such that $24C^3T^{\epsilon}\|u_0\|^2_{H^s} < 1$, we have

$$\bigl\|\Phi(u)\bigr\|_{X_{s,\frac{1}{2}+\epsilon}} \leq C\|u_0\|_{H^s(\mathbf{R})} + CT^{\epsilon}\bigl(2C\|u_0\|_{H^s(\mathbf{R})}\bigr)^3 \leq 2C\|u_0\|_{H^s(\mathbf{R})}, \tag{4.4}$$

thus, $\Phi(u)$ is a mapping on B. By using a proof similar to (4.4), by choosing T sufficiently small such that $24C^3T^\epsilon\|u_0\|_{H^s}^2 < 1$, we obtain

$$\begin{aligned}
&\big\|\Phi(u_1)-\Phi(u_2)\big\|_{X_{s,\frac{1}{2}+\epsilon}} \\
&\quad\le CT^\epsilon\big[\|u_1\|_{X_{s,\frac{1}{2}+\epsilon}}^2+\|u_1\|_{X_{s,\frac{1}{2}+\epsilon}}\|u_2\|_{X_{s,\frac{1}{2}+\epsilon}}+\|u_2\|_{X_{s,\frac{1}{2}+\epsilon}}^2\big]\|u_1-u_2\|_{X_{s,\frac{1}{2}+\epsilon}} \\
&\quad\le\frac{1}{2}\|u_1-u_2\|_{X_{s,\frac{1}{2}+\epsilon}},
\end{aligned}\tag{4.5}$$

thus, $\Phi(u)$ is a contraction mapping on the closed ball B. Consequently, Φ have a fixed point u and the Cauchy problem for (1.1) possesses a local solution on $[-T,T]$. The uniqueness of the solution is obvious.

This completes the proof of Theorem 1.1.

5 Proof of Theorem 1.2

In this section, inspired by [5, 35, 45], we present the proof of Theorem 1.2. We will prove Theorem 1.2 by contradiction.

We assume that the solution map of (1.4), (1.5) is C^3 in $H^s(\mathbf{R})$ with $s<\frac{1}{4}$. Then, from Theorem 3 of [35], we have

$$\sup_{t\in[0,T]}\big\|B_3(u_0)\big\|_{H^s}\le C\|u_0\|_{H^s}^3\tag{5.1}$$

for $u_0\in H^s(\mathbf{R})$. Here

$$B_1(u_0)=e^{-t(-\partial_x^3-\partial_x^{-1})}u_0,\tag{5.2}$$

$$B_3(u_0)=\frac{1}{3}\int_0^t e^{-(t-\tau)(-\partial_x^3-\partial_x^{-1})}\partial_x\big(\big(B_1(u_0)\big)^3\big)\,d\tau.\tag{5.3}$$

We consider the initial data

$$u_0(x)=r^{-\frac{1}{2}}N^{-s}\left\{e^{iNx}\int_0^r e^{ix\xi}\,d\xi+e^{-iNx}\int_r^{2r}e^{ix\xi}\,d\xi\right\},\quad r^2N=O(1),N\ge 2.\tag{5.4}$$

By using a direct computation, we have

$$\mathcal{F}_xu_0(\xi)=Cr^{-\frac{1}{2}}N^{-s}\big\{\chi_{[-N,-N+r]}(\xi)+\chi_{[N+r,N+2r]}(\xi)\big\}.$$

Here χ_I denotes the characteristic function of a set $I\subset\mathbf{R}$. Obviously,

$$\|u_0\|_{H^s(\mathbf{R})}\sim 1.\tag{5.5}$$

We define $I_1:=[-N,-N+r]$ and $I_2:=[N+r,N+2r]$ and $\Omega_1:=I_1\cup I_2$. By using a direct computation, we have

$$\mathcal{F}_xB_1u_0(\xi)=Ce^{it\phi(\xi)}\mathcal{F}_xu_0(\xi).\tag{5.6}$$

Combining (5.6) with the definition of $B_3(u_0)$, we have

$$B_3(u_0)(x,t) = Cg. \tag{5.7}$$

Here

$$g = Cr^{-\frac{3}{2}}N^{-3s}\int_{\xi_1\in\Omega_1}\int_{\xi_2\in\Omega_1}\int_{\xi_3\in\Omega_1}\left(\sum_{j=1}^{3}\xi_j\right)e^{ix\sum_{j=1}^{3}\xi_j}H(\xi_1,\xi_2,\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3, \tag{5.8}$$

where

$$H(\xi_1,\xi_2,\xi_3) = \frac{e^{it(\phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3))} - e^{it\phi(\sum_{j=1}^{3}\xi_j)}}{\phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3)-\phi(\sum_{j=1}^{3}\xi_j)}. \tag{5.9}$$

We define

$$\theta_1 := \phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3)-\phi\left(\sum_{j=1}^{3}\xi_j\right). \tag{5.10}$$

From Lemma 2.4, we have

$$\theta_1 = -3\big[(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)\big]\left[1-\frac{1}{3\prod_{j=1}^{3}\xi_j(\sum_{j=1}^{3}\xi_j)}\right]. \tag{5.11}$$

To estimate $\|g\|_{H^s(\mathbf{R})}$, we need to consider the following three cases:

$$\begin{aligned}
&\text{Case 1:}\quad \xi_j\in I_1\quad (j=1,2,3),\\
&\text{Case 2:}\quad \xi_j\in I_1\quad (j=1,2,3),\\
&\text{Case 3:}\quad \xi_j\in I_1\quad (j=1,2),\qquad \xi_3\in I_2\quad\text{or}\quad \xi_1\in I_1,\qquad \xi_j\in I_2\quad (j=2,3)\\
&\quad\ \text{or}\quad \xi_j\in I_2\quad (j=1,2),\qquad \xi_3\in I_1\quad\text{or}\quad \xi_1\in I_2,\qquad \xi_j\in I_1\quad (j=2,3).
\end{aligned}$$

We assume that $\|g\|_{H^s(\mathbf{R})}$ corresponding to cases 1, 2, 3 are denoted by L_1, L_2, L_3, respectively.

Case 1. In this case, we have $|\theta_1| \sim N^3$ and $|\xi_1+\xi_2+\xi_3| \sim N$. Since $r^2N = O(1)$, we have

$$L_1 \le Cr^{-\frac{3}{2}}N^{-3s}N^s r^{\frac{5}{2}}N^{-2} \le CN^{-2s-\frac{5}{2}}. \tag{5.12}$$

Case 2. In this case, we have $|\theta_1| \sim N^3$ and $|\xi_1+\xi_2+\xi_3| \sim N$. Since $r^2N = O(1)$, we have

$$L_2 \le Cr^{-\frac{3}{2}}N^{-3s}N^s r^{\frac{5}{2}}N^{-2} \le CN^{-2s-\frac{5}{2}}. \tag{5.13}$$

Case 3. In this case, we have $|\theta_1| \sim r^2N$ and $|\xi_1+\xi_2+\xi_3| \sim N$ as well as $H \le |t|$. Since $r^2N = O(1)$, we have

$$L_3 \ge C|t|r^{-\frac{3}{2}}N^{-3s}N^s r^{\frac{5}{2}}N \ge C|t|N^{-2s+\frac{1}{2}}. \tag{5.14}$$

Combining (5.1), (5.5) with (5.12)-(5.14), we have

$$|t|N^{-2s+\frac{1}{2}} \leq L_3 - L_1 - L_2 \leq \sup_{t\in[0,T]} \left\|B_3(u_0)\right\|_{H^s} \leq C\|u_0\|_{H^s}^3 \sim C. \tag{5.15}$$

For fixed $t > 0$, when $s < \frac{1}{4}$, let $N \longrightarrow \infty$, we have $|t|N^{-2s+\frac{1}{2}} \longrightarrow +\infty$, and this contradicts (5.15).

This ends the proof of Theorem 1.2.

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The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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